Sequences; Induction; the Binomial Theorem

11

The Future of the World Population

WASHINGTON—World population is projected to increase 46 percent by 2050, with most of the growth occurring in the less industrialized areas of the globe.

Projected growth in population by continent, 2003–2050: North America 41.8%; Latin America and the Caribbean 46.2%; Oceania 55.6%; Europe −8.8%; Asia 39.8%; Africa 118.8%.

Note: The United Nations classifies the countries of Latin America and the Caribbean, Asia, Oceania, and Africa as less industrialized with the exception of Australia, New Zealand, and Japan.

Africa's population could soar by more than 1 billion over the next half-century, further straining food and water supplies and social services in areas already struggling, according to a new report.

The latest edition of the "World Population Data Sheet" estimates the global population will rise 46 percent between now and 2050 to about 9 billion, a level also predicted by the United Nations and other groups.

European nations, more industrialized and prosperous, are expected to lose population because of falling birth rates and low immigration.

The U.S. population is expected to grow 45 percent to 422 million in 2050, paced by a stable birth rate and high levels of immigration.

But most of the world's growth will be in developing nations. India's population is estimated to grow 52 percent to 1.6 billion by 2050, when it will surpass China as the world's largest country.

Africa is predicted to more than double in population to 1.9 billion by midcentury.

SOURCE: *The Houston Chronicle* (Houston, TX), July 23, 2003, p. 12.

—*See Chapter Project 1.*

A LOOK BACK, A LOOK AHEAD This chapter may be divided into three independent parts: Sections 11.1–11.3, Section 11.4, and Section 11.5.

In Chapter 2, we defined a function and its domain, which was usually some set of real numbers. In Sections 11.1–11.3, we discuss sequences, which are functions whose domain is the set of positive integers.

Throughout this text, where it seemed appropriate, we have given proofs of many of the results. In Section 11.4, a technique for proving theorems involving natural numbers is discussed.

In the Appendix, Section A.3, we gave formulas for expanding $(x + a)^2$ and $(x + a)^3$. In Section 11.5, we discuss the Binomial Theorem, a formula for the expansion of $(x + a)^n$, where n is a positive integer.

The topics introduced in this chapter are covered in more detail in courses titled *Discrete Mathematics*. Applications of these topics can be found in the fields of computer science, engineering, business and economics, the social sciences, and the physical and biological sciences.

OUTLINE

11.1 Sequences

PREPARING FOR THIS SECTION *Before getting started, review the following concept:*

- Functions (Section 2.1, pp. 56–63)
- Compound Interest (Section 4.7, pp. 315–322)

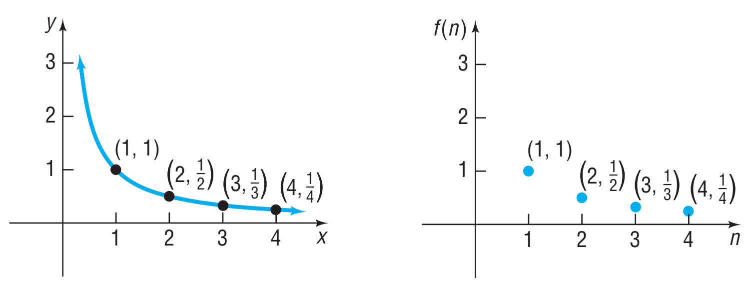
Now work the 'Are You Prepared?' problems on page 841.

OBJECTIVES
1. Write the First Several Terms of a Sequence
2. Write the Terms of a Sequence Defined by a Recursive Formula
3. Use Summation Notation
4. Find the Sum of a Sequence Algebraically and Using a Graphing Utility
5. Solve Annuity and Amortization Problems

A **sequence** is a function whose domain is the set of positive integers.

Because a sequence is a function, it will have a graph. In Figure 1(a), we have the graph of the function $f(x) = \frac{1}{x}$, $x > 0$. If all the points on this graph were removed except those whose x-coordinates are positive integers, that is, if all points were removed except $(1, 1)$, $\left(2, \frac{1}{2}\right)$, $\left(3, \frac{1}{3}\right)$, and so on, the remaining points would be the graph of the sequence $f(n) = \frac{1}{n}$, as shown in Figure 1(b). Notice that we use n to represent the independent variable in a sequence. This serves to remind us that n is a positive integer.

Figure 1



(a) $f(x) = \frac{1}{x}, x > 0$ **(b)** $f(n) = \frac{1}{n}$, n a positive integer

1 Write the First Several Terms of a Sequence

A sequence is usually represented by listing its values in order. For example, the sequence whose graph is given in Figure 1(b) might be represented as

$$f(1), f(2), f(3), f(4), \ldots \quad \text{or} \quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

The list never ends, as the ellipsis indicates. The numbers in this ordered list are called the **terms** of the sequence.

In dealing with sequences, we usually use subscripted letters, such as a_1, to represent the first term, a_2 for the second term, a_3 for the third term, and so on. For the sequence $f(n) = \frac{1}{n}$, we write

$$a_1 = f(1) = 1, \quad a_2 = f(2) = \frac{1}{2}, \quad a_3 = f(3) = \frac{1}{3}, \quad a_4 = f(4) = \frac{1}{4}, \ldots, \quad a_n = f(n) = \frac{1}{n}, \ldots$$

In other words, we usually do not use the traditional function notation $f(n)$ for sequences. For this particular sequence, we have a rule for the nth term, which is $a_n = \frac{1}{n}$, so it is easy to find any term of the sequence.

When a formula for the nth term (sometimes called the **general term**) of a sequence is known, rather than write out the terms of the sequence, we usually represent the entire sequence by placing braces around the formula for the nth term. For example, the sequence whose nth term is $b_n = \left(\frac{1}{2}\right)^n$ may be represented as

$$\{b_n\} = \left\{\left(\frac{1}{2}\right)^n\right\}$$

or by

$$b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{4}, \quad b_3 = \frac{1}{8}, \ldots, \quad b_n = \left(\frac{1}{2}\right)^n, \ldots$$

EXAMPLE 1 Writing the First Several Terms of a Sequence

Write down the first six terms of the following sequence and graph it.

$$\{a_n\} = \left\{\frac{n-1}{n}\right\}$$

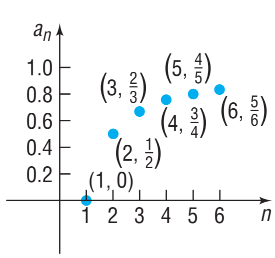
Algebraic Solution

The first six terms of the sequence are

$$a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{2}{3},$$
$$a_4 = \frac{3}{4}, \quad a_5 = \frac{4}{5}, \quad a_6 = \frac{5}{6}$$

See Figure 2 for the graph.

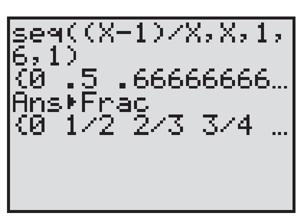
Figure 2



Graphing Solution

Figure 3 shows the sequence generated on a TI-84 Plus graphing calculator. We can see the first few terms of the sequence on the screen. You need to press the right arrow key to scroll right to see the remaining terms of the sequence.

Figure 3


```
seq((X-1)/X,X,1,
6,1)
{0 .5 .66666666…
Ans▸Frac
{0 1/2 2/3 3/4 …
```

We could also obtain the terms of the sequence using the TABLE feature. First, put the graphing utility in SEQuence mode. Using $Y=$, enter the formula for the sequence into the graphing utility. See Figure 4. Set up the table with TblStart = 1 and ΔTbl = 1. See Table 1. Finally, we can graph the sequence. See Figure 5. Notice that the first term of the sequence is not visible since it lies on the x-axis. TRACEing the graph will allow you to determine the terms of the sequence.

Figure 4

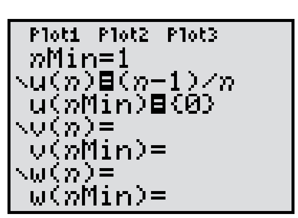
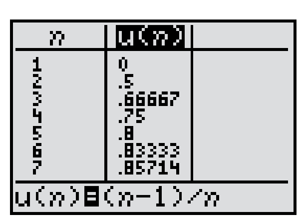

```
Plot1 Plot2 Plot3
nMin=1
\u(n)=(n-1)/n
 u(nMin)={0}
\v(n)=
 v(nMin)=
\w(n)=
 w(nMin)=
```

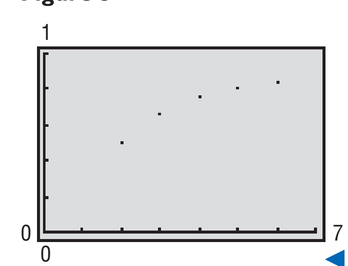
Table 1



n	u(n)
1	0
2	.5
3	.66667
4	.75
5	.8
6	.83333
7	.85714

u(n)=(n-1)/n

Figure 5

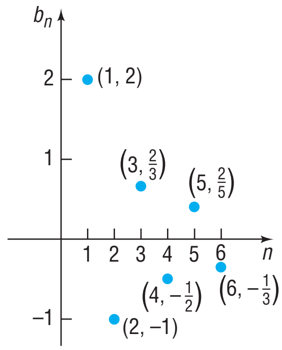


NOW WORK PROBLEM 19.

We will usually provide solutions done by hand. The reader is encouraged to verify solutions using a graphing utility.

EXAMPLE 2 Writing the First Several Terms of a Sequence

Figure 6



Write down the first six terms of the following sequence and graph it.

$$\{b_n\} = \left\{(-1)^{n+1}\left(\frac{2}{n}\right)\right\}$$

Solution The first six terms of the sequence are

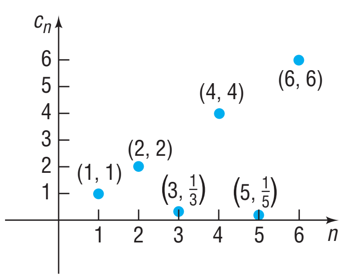
$$b_1 = 2, \quad b_2 = -1, \quad b_3 = \frac{2}{3}, \quad b_4 = -\frac{1}{2}, \quad b_5 = \frac{2}{5}, \quad b_6 = -\frac{1}{3}$$

See Figure 6 for the graph. ◀

Notice in the sequence $\{b_n\}$ in Example 2 that the signs of the terms **alternate**. When this occurs, we use factors such as $(-1)^{n+1}$, which equals 1 if n is odd and -1 if n is even, or $(-1)^n$, which equals -1 if n is odd and 1 if n is even.

EXAMPLE 3 Writing the First Several Terms of a Sequence

Figure 7



Write down the first six terms of the following sequence and graph it.

$$\{c_n\} = \begin{Bmatrix} n & \text{if } n \text{ is even} \\ \dfrac{1}{n} & \text{if } n \text{ is odd} \end{Bmatrix}$$

Solution The first six terms of the sequence are

$$c_1 = 1, \quad c_2 = 2, \quad c_3 = \frac{1}{3}, \quad c_4 = 4, \quad c_5 = \frac{1}{5}, \quad c_6 = 6$$

See Figure 7 for the graph. ◀

NOW WORK PROBLEM 21.

Sometimes a sequence is indicated by an observed pattern in the first few terms that makes it possible to infer the makeup of the nth term. In the example that follows, a sufficient number of terms of the sequence is given so that a natural choice for the nth term is suggested.

EXAMPLE 4 Determining a Sequence from a Pattern

(a) $e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \ldots$ $\qquad a_n = \frac{e^n}{n}$

(b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ $\qquad b_n = \frac{1}{3^{n-1}}$

(c) $1, 3, 5, 7, \ldots$ $\qquad c_n = 2n - 1$

(d) $1, 4, 9, 16, 25, \ldots$ $\qquad d_n = n^2$

(e) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$ $\qquad e_n = (-1)^{n+1}\left(\frac{1}{n}\right)$ ◀

NOW WORK PROBLEM 29.

The Factorial Symbol

If $n \geq 0$ is an integer, the **factorial symbol $n!$** is defined as follows:

$$0! = 1 \qquad 1! = 1$$
$$n! = n(n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \qquad \text{if } n \geq 2$$

Table 2

n	0	1	2	3	4	5	6
$n!$	1	1	2	6	24	120	720

For example, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and so on. Table 2 lists the values of $n!$ for $0 \le n \le 6$.

Because

$$n! = n\underbrace{(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1}_{(n-1)!}$$

we can use the formula

$$\boxed{n! = n(n-1)!}$$

to find successive factorials. For example, because $6! = 720$, we have

$$7! = 7 \cdot 6! = 7(720) = 5040$$

and

$$8! = 8 \cdot 7! = 8(5040) = 40{,}320$$

NOTE Your calculator has a factorial key. Use it to see how fast factorials increase in value. Find the value of 69!. What happens when you try to find 70!? In fact, 70! is larger than 10^{100} (a ***googol***), the largest number most calculators can display. ■

2 Write the Terms of a Sequence Defined by a Recursive Formula

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the nth term by a formula or equation that involves one or more of the terms preceding it. Sequences defined this way are said to be defined **recursively**, and the rule or formula is called a **recursive formula**.

EXAMPLE 5 Writing the Terms of a Recursively Defined Sequence

Write down the first five terms of the following recursively defined sequence.

$$s_1 = 1, \qquad s_n = ns_{n-1}$$

Algebraic Solution

The first term is given as $s_1 = 1$. To get the second term, we use $n = 2$ in the formula $s_n = ns_{n-1}$ to get $s_2 = 2s_1 = 2 \cdot 1 = 2$. To get the third term, we use $n = 3$ in the formula to get $s_3 = 3s_2 = 3 \cdot 2 = 6$. To get a new term requires that we know the value of the preceding term. The first five terms are

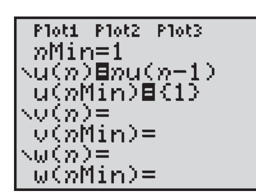
$$s_1 = 1$$
$$s_2 = 2 \cdot 1 = 2$$
$$s_3 = 3 \cdot 2 = 6$$
$$s_4 = 4 \cdot 6 = 24$$
$$s_5 = 5 \cdot 24 = 120$$

Do you recognize this sequence? $s_n = n!$ ◄

Graphing Solution

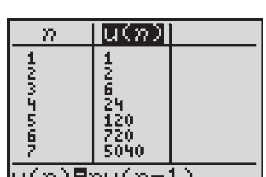
First, put the graphing utility into SEQuence mode. Using $Y =$, enter the recursive formula into the graphing utility. See Figure 8(a). Next, set up the viewing window to generate the desired sequence. Finally, graph the recursion relation and use TRACE to determine the terms in the sequence. See Figure 8(b). For example, we see that the fourth term of the sequence is 24. Table 3 also shows the terms of the sequence.

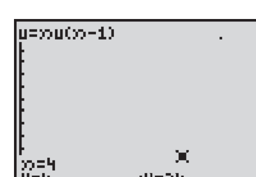
Figure 8



(a)

Table 3





(b) ◄

EXAMPLE 6 **Writing the Terms of a Recursively Defined Sequence**

Write down the first five terms of the following recursively defined sequence.

$$u_1 = 1, \qquad u_2 = 1, \qquad u_{n+2} = u_n + u_{n+1}$$

Solution We are given the first two terms. To get the third term requires that we know each of the previous two terms. That is,

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 \\ u_3 &= u_1 + u_2 = 1 + 1 = 2 \\ u_4 &= u_2 + u_3 = 1 + 2 = 3 \\ u_5 &= u_3 + u_4 = 2 + 3 = 5 \end{aligned}$$

The sequence defined in Example 6 is called a **Fibonacci sequence**, and the terms of this sequence are called **Fibonacci numbers**. These numbers appear in a wide variety of applications (see Problems 91–94).

NOW WORK PROBLEMS 37 AND 45.

3 Use Summation Notation

It is often important to be able to find the sum of the first n terms of a sequence $\{a_n\}$, that is,

$$a_1 + a_2 + a_3 + \cdots + a_n$$

Rather than write down all these terms, we introduce a more concise way to express the sum, called **summation notation**. Using summation notation, we would write the sum as

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

The symbol Σ (the Greek letter sigma, which is an S in our alphabet) is simply an instruction to sum, or add up, the terms. The integer k is called the **index** of the sum; it tells you where to start the sum and where to end it. The expression

$$\sum_{k=1}^{n} a_k$$

is an instruction to add the terms a_k of the sequence $\{a_n\}$ starting with $k = 1$ and ending with $k = n$. We read the expression as "the sum of a_k from $k = 1$ to $k = n$."

EXAMPLE 7 **Expanding Summation Notation**

Write out each sum.

(a) $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$ (b) $\displaystyle\sum_{k=1}^{n} k!$

Solution (a) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ (b) $\displaystyle\sum_{k=1}^{n} k! = 1! + 2! + \cdots + n!$

EXAMPLE 8 **Writing a Sum in Summation Notation**

Express each sum using summation notation.

(a) $1^2 + 2^2 + 3^2 + \cdots + 9^2$ (b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$

Solution (a) The sum $1^2 + 2^2 + 3^2 + \cdots + 9^2$ has 9 terms, each of the form k^2, and starts at $k = 1$ and ends at $k = 9$:

$$1^2 + 2^2 + 3^2 + \cdots + 9^2 = \sum_{k=1}^{9} k^2$$

(b) The sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$$

has n terms, each of the form $\frac{1}{2^{k-1}}$, and starts at $k = 1$ and ends at $k = n$:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = \sum_{k=1}^{n} \frac{1}{2^{k-1}}$$ ◀

The index of summation need not always begin at 1 or end at n; for example, we could have expressed the sum in Example 8(b) as

$$\sum_{k=0}^{n-1} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

Letters other than k may be used as the index. For example,

$$\sum_{j=1}^{n} j! \quad \text{and} \quad \sum_{i=1}^{n} i!$$

each represent the same sum as the one given in Example 7(b).

NOW WORK PROBLEMS 53 AND 63.

4 Find the Sum of a Sequence Algebraically and Using a Graphing Utility

Next we list some properties of sequences using summation notation. These properties are useful for adding the terms of a sequence algebraically.

Theorem

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences and c is a real number, then:

$$\sum_{k=1}^{n} c = \underbrace{c + c + \cdots + c}_{n \text{ terms}} = cn \tag{1}$$

$$\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k \tag{2}$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \tag{3}$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \tag{4}$$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{n} a_k, \quad \text{where } 0 < j < n \tag{5}$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{8}$$

We shall not prove these properties. The proofs of (1) through (5) are based on properties of real numbers; the proofs of (7) and (8) require mathematical induction, which is discussed in Section 11.4. See Problem 97 for a derivation of (6).

EXAMPLE 9

Finding the Sum of a Sequence

Find the sum of each sequence.

(a) $\sum_{k=1}^{5}(3k)$ (b) $\sum_{k=1}^{3}(k^3+1)$ (c) $\sum_{k=1}^{4}(k^2-7k+2)$

Algebraic Solution

(a)
$$\begin{aligned}\sum_{k=1}^{5}(3k) &= 3\sum_{k=1}^{5}k && \text{Property (2)}\\ &= 3\left(\frac{5(5+1)}{2}\right) && \text{Property (6)}\\ &= 3(15)\\ &= 45\end{aligned}$$

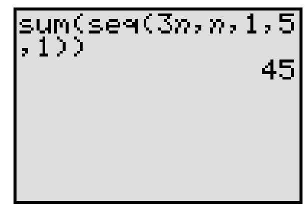
(b)
$$\begin{aligned}\sum_{k=1}^{3}(k^3+1) &= \sum_{k=1}^{3}k^3 + \sum_{k=1}^{3}1 && \text{Property (3)}\\ &= \left(\frac{3(3+1)}{2}\right)^2 + 1(3) && \text{Properties (1) and (8)}\\ &= 36+3\\ &= 39\end{aligned}$$

(c)
$$\begin{aligned}\sum_{k=1}^{4}(k^2-7k+2) &= \sum_{k=1}^{4}k^2 - \sum_{k=1}^{4}(7k) + \sum_{k=1}^{4}2 && \text{Properties (3), (4)}\\ &= \sum_{k=1}^{4}k^2 - 7\sum_{k=1}^{4}k + \sum_{k=1}^{4}2 && \text{Property (2)}\\ &= \frac{4(4+1)(2\cdot 4+1)}{6} - 7\left(\frac{4(4+1)}{2}\right) + 2(4) && \text{Properties (1), (6), (7)}\\ &= 30-70+8\\ &= -32\end{aligned}$$

Graphing Solution

(a) Figure 9 shows the solution using a TI-84 Plus graphing calculator.

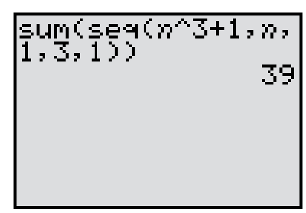
Figure 9



So, $\sum_{k=1}^{5}(3k) = 45$

(b) Figure 10 shows the solution using a TI-84 Plus graphing calculator.

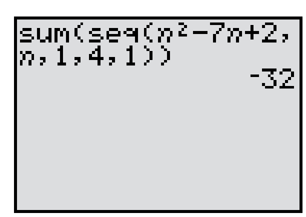
Figure 10



So, $\sum_{k=1}^{3}(k^3+1) = 39$

(c) Figure 11 shows the solution using a TI-84 Plus graphing calculator.

Figure 11



So, $\sum_{k=1}^{4}(k^2-7k+2) = -32$

NOW WORK PROBLEM 75.

5 Solve Annuity and Amortization Problems

In Section 4.7 we developed the compound interest formula, which gives the future value when a fixed amount of money is deposited in an account that pays interest compounded periodically. Often, though, money is invested in small amounts at periodic intervals. An **annuity** is a sequence of equal periodic deposits. The periodic deposits may be made annually, quarterly, monthly, or daily.

When deposits are made at the same time that the interest is credited, the annuity is called **ordinary**. We will only deal with ordinary annuities here. The **amount of an annuity** is the sum of all deposits made plus all interest paid.

Suppose that the initial amount deposited in an annuity is $\$M$, the periodic deposit is $\$P$, and the per annum rate of interest is $r\%$ (expressed as a decimal) compounded N times per year. The periodic deposit is made at the same time that the interest is credited, so N deposits are made per year. The amount A_n of the annuity after n deposits will equal A_{n-1}, the amount of the annuity after $n - 1$ deposits, plus the interest earned on this amount, plus P, the periodic deposit. That is,

$$\underbrace{A_n}_{\text{Amount after } n \text{ deposits}} = \underbrace{A_{n-1}}_{\text{Amount in previous period}} + \underbrace{\frac{r}{N}A_{n-1}}_{\text{Interest earned}} + \underbrace{P}_{\text{Periodic deposit}} = \left(1 + \frac{r}{N}\right)A_{n-1} + P$$

We have established the following result:

Theorem

Annuity Formula

If $A_0 = M$ represents the initial amount deposited in an annuity that earns $r\%$ per annum compounded N times per year, and if P is the periodic deposit made at each payment period, then the amount A_n of the annuity after n deposits is given by the recursive sequence

$$\boxed{A_0 = M, \qquad A_n = \left(1 + \frac{r}{N}\right)A_{n-1} + P, \qquad n \geq 1 \qquad \textbf{(9)}}$$

Formula (9) may be explained as follows: the money in the account initially, A_0, is $\$M$; the money in the account after $n - 1$ payments, A_{n-1}, earns interest $\frac{r}{N}$ during the nth period; so when the periodic payment of P dollars is added, the amount after n payments, A_n, is obtained.

EXAMPLE 10 Saving for Spring Break

A trip to Cancun during spring break will cost \$450 and full payment is due March 2. To have the money, a student, on September 1, deposits \$100 in a savings account that pays 4% per annum compounded monthly. On the first of each month, the student deposits \$50 in this account.

(a) Find a recursive sequence that explains how much is in the account after n months.

(b) Use the TABLE feature to list the amounts of the annuity for the first 6 months.

(c) After the deposit on March 1 is made, is there enough in the account to pay for the Cancun trip?

(d) If the student deposits \$60 each month, will there be enough for the trip after the March 1 deposit?

Solution

(a) The initial amount deposited in the account is $A_0 = \$100$. The monthly deposit is $P = \$50$, and the per annum rate of interest is $r = 0.04$ compounded $N = 12$

Table 4

n	$u(n)$
0	100
1	150.33
2	200.83
3	251.5
4	302.34
5	353.35
6	404.53

u(n)=(1+.04/12)…

Table 5

n	$u(n)$
0	100
1	160.33
2	220.87
3	281.6
4	342.54
5	403.68
6	465.03

u(n)=(1+.04/12)…

times per year. The amount A_n in the account after n monthly deposits is given by the recursive sequence

$$A_0 = 100, \qquad A_n = \left(1 + \frac{r}{N}\right)A_{n-1} + P = \left(1 + \frac{0.04}{12}\right)A_{n-1} + 50$$

(b) In SEQuence mode on a TI-84 Plus, enter the sequence $\{A_n\}$ and create Table 4. On September 1 ($n = 0$), there is \$100 in the account. After the first payment on October 1, the value of the account is \$150.33. After the second payment on November 1, the value of the account is \$200.83. After the third payment on December 1, the value of the account is \$251.50, and so on.

(c) On March 1 ($n = 6$), there is only \$404.53, not enough to pay for the trip to Cancun.

(d) If the periodic deposit, P, is \$60, then on March 1, there is \$465.03 in the account, enough for the trip. See Table 5. ◀

Recursive sequences can also be used to compute information about loans. When equal periodic payments are made to pay off a loan, the loan is said to be **amortized**.

Theorem

Amortization Formula

If \$$B$ is borrowed at an interest rate of r% (expressed as a decimal) per annum compounded monthly, the balance A_n due after n monthly payments of \$$P$ is given by the recursive sequence

$$A_0 = B, \qquad A_n = \left(1 + \frac{r}{12}\right)A_{n-1} - P, \qquad n \geq 1 \qquad \textbf{(10)}$$

Formula (10) may be explained as follows: The initial loan balance is \$$B$. The balance due A_n after n payments will equal the balance due previously, A_{n-1}, plus the interest charged on that amount, reduced by the periodic payment P.

EXAMPLE 11 Mortgage Payments

Table 6

n	$u(n)$
0	180000
1	179852
2	179704
3	179555
4	179405
5	179254
6	179102

u(n)=(1+.07/12)…

John and Wanda borrowed \$180,000 at 7% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be \$1197.54.

(a) Find a recursive formula that represents their balance after each payment of \$1197.54 has been made.

(b) Determine their balance after the first payment is made.

(c) When will their balance be below \$170,000?

Solution

Table 7

n	$u(n)$
52	171067
53	170868
54	170667
55	170465
56	170262
57	170057
58	169852

u(n)=(1+.07/12)…

(a) We use formula (10) with $A_0 = 180{,}000$, $r = 0.07$, and $P = \$1197.54$. Then

$$A_0 = 180{,}000 \qquad A_n = \left(1 + \frac{0.07}{12}\right)A_{n-1} - 1197.54$$

(b) In SEQuence mode on a TI-84 Plus, enter the sequence $\{A_n\}$ and create Table 6. After the first payment is made, the balance is $A_1 = \$179{,}852$.

(c) Scroll down until the balance is below \$170,000. See Table 7. After the fifty-eighth payment is made ($n = 58$), the balance is below \$170,000. ◀

NOW WORK PROBLEM 83.

11.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. For the function $f(x) = \dfrac{x-1}{x}$, find $f(2)$ and $f(3)$. (pp. 61–63)
2. *True or False:* A function is a relation between two sets D and R so that each element x in the first set D is related to exactly one element y in the second set R. (pp. 56–61)
3. If \$1000 is invested at 4% per annum compounded semi-annually, how much is in the account after 2 years? (pp. 315–322)
4. How much do you need to invest now at 5% per annum compounded monthly so that in 1 year you will have \$10,000? (pp. 315–322)

Concepts and Vocabulary

5. A(n) _____ is a function whose domain is the set of positive integers.
6. For the sequence $\{s_n\} = \{4n - 1\}$, the first term is $s_1 =$ _____ and the fourth term is $s_4 =$ _____.
7. $\sum_{k=1}^{4} (2k) =$ _____.
8. *True or False:* Sequences are sometimes defined recursively.
9. *True or False:* A sequence is a function.
10. *True or False:* $\sum_{k=1}^{2} k = 3$

Skill Building

In Problems 11–16, evaluate each factorial expression. Verify your results using a graphing utility.

11. $10!$ **12.** $9!$ **13.** $\dfrac{9!}{6!}$ **14.** $\dfrac{12!}{10!}$ **15.** $\dfrac{3!7!}{4!}$ **16.** $\dfrac{5!8!}{3!}$

In Problems 17–28, write down the first five terms of each sequence.

17. $\{n\}$ **18.** $\{n^2 + 1\}$ **19.** $\left\{\dfrac{n}{n+2}\right\}$ **20.** $\left\{\dfrac{2n+1}{2n}\right\}$

21. $\{(-1)^{n+1}n^2\}$ **22.** $\left\{(-1)^{n-1}\left(\dfrac{n}{2n-1}\right)\right\}$ **23.** $\left\{\dfrac{2^n}{3^n+1}\right\}$ **24.** $\left\{\left(\dfrac{4}{3}\right)^n\right\}$

25. $\left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$ **26.** $\left\{\dfrac{3^n}{n}\right\}$ **27.** $\left\{\dfrac{n}{e^n}\right\}$ **28.** $\left\{\dfrac{n^2}{2^n}\right\}$

In Problems 29–36, the given pattern continues. Write down the nth term of each sequence suggested by the pattern.

29. $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$ **30.** $\dfrac{1}{1\cdot 2}, \dfrac{1}{2\cdot 3}, \dfrac{1}{3\cdot 4}, \dfrac{1}{4\cdot 5}, \ldots$ **31.** $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \ldots$ **32.** $\dfrac{2}{3}, \dfrac{4}{9}, \dfrac{8}{27}, \dfrac{16}{81}, \ldots$

33. $1, -1, 1, -1, 1, -1, \ldots$ **34.** $1, \dfrac{1}{2}, 3, \dfrac{1}{4}, 5, \dfrac{1}{6}, 7, \dfrac{1}{8}, \ldots$ **35.** $1, -2, 3, -4, 5, -6, \ldots$ **36.** $2, -4, 6, -8, 10, \ldots$

In Problems 37–50, a sequence is defined recursively. Write the first five terms.

37. $a_1 = 2;\quad a_n = 3 + a_{n-1}$ **38.** $a_1 = 3;\quad a_n = 4 - a_{n-1}$ **39.** $a_1 = -2;\quad a_n = n + a_{n-1}$

40. $a_1 = 1;\quad a_n = n - a_{n-1}$ **41.** $a_1 = 5;\quad a_n = 2a_{n-1}$ **42.** $a_1 = 2;\quad a_n = -a_{n-1}$

43. $a_1 = 3;\quad a_n = \dfrac{a_{n-1}}{n}$ **44.** $a_1 = -2;\quad a_n = n + 3a_{n-1}$ **45.** $a_1 = 1;\quad a_2 = 2;\quad a_n = a_{n-1}\cdot a_{n-2}$

46. $a_1 = -1;\quad a_2 = 1;\quad a_n = a_{n-2} + na_{n-1}$ **47.** $a_1 = A;\quad a_n = a_{n-1} + d$ **48.** $a_1 = A;\quad a_n = ra_{n-1},\quad r \neq 0$

49. $a_1 = \sqrt{2};\quad a_n = \sqrt{2 + a_{n-1}}$ **50.** $a_1 = \sqrt{2};\quad a_n = \sqrt{\dfrac{a_{n-1}}{2}}$

In Problems 51–60, write out each sum.

51. $\sum_{k=1}^{n} (k+2)$ **52.** $\sum_{k=1}^{n} (2k+1)$ **53.** $\sum_{k=1}^{n} \dfrac{k^2}{2}$ **54.** $\sum_{k=1}^{n} (k+1)^2$ **55.** $\sum_{k=0}^{n} \dfrac{1}{3^k}$

56. $\sum_{k=0}^{n}\left(\frac{3}{2}\right)^k$ **57.** $\sum_{k=0}^{n-1}\frac{1}{3^{k+1}}$ **58.** $\sum_{k=0}^{n-1}(2k+1)$ **59.** $\sum_{k=2}^{n}(-1)^k \ln k$ **60.** $\sum_{k=3}^{n}(-1)^{k+1}2^k$

In Problems 61–70, express each sum using summation notation.

61. $1 + 2 + 3 + \cdots + 20$

62. $1^3 + 2^3 + 3^3 + \cdots + 8^3$

63. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{13}{13+1}$

64. $1 + 3 + 5 + 7 + \cdots + [2(12) - 1]$

65. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + (-1)^6\left(\frac{1}{3^6}\right)$

66. $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \cdots + (-1)^{11+1}\left(\frac{2}{3}\right)^{11}$

67. $3 + \frac{3^2}{2} + \frac{3^3}{3} + \cdots + \frac{3^n}{n}$

68. $\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots + \frac{n}{e^n}$

69. $a + (a + d) + (a + 2d) + \cdots + (a + nd)$

70. $a + ar + ar^2 + \cdots + ar^{n-1}$

In Problems 71–82, find the sum of each sequence (a) algebraically and (b) using a graphing utility.

71. $\sum_{k=1}^{10} 5$ **72.** $\sum_{k=1}^{20} 8$ **73.** $\sum_{k=1}^{6} k$ **74.** $\sum_{k=1}^{4}(-k)$

75. $\sum_{k=1}^{5}(5k + 3)$ **76.** $\sum_{k=1}^{6}(3k - 7)$ **77.** $\sum_{k=1}^{3}(k^2 + 4)$ **78.** $\sum_{k=0}^{4}(k^2 - 4)$

79. $\sum_{k=1}^{6}(-1)^k 2^k$ **80.** $\sum_{k=1}^{4}(-1)^k 3^k$ **81.** $\sum_{k=1}^{4}(k^3 - 1)$ **82.** $\sum_{k=0}^{3}(k^3 + 2)$

Applications and Extensions

83. Credit Card Debt John has a balance of \$3000 on his Discover card that charges 1% interest per month on any unpaid balance. John can afford to pay \$100 toward the balance each month. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0 = \$3000, \qquad B_n = 1.01B_{n-1} - 100$$

(a) Determine John's balance after making the first payment. That is, determine B_1.
(b) Using a graphing utility, determine when John's balance will be below \$2000. How many payments of \$100 have been made?
(c) Using a graphing utility, determine when John will pay off the balance. What is the total of all the payments?
(d) What was John's interest expense?

84. Car Loans Phil bought a car by taking out a loan for \$18,500 at 0.5% interest per month. Phil's normal monthly payment is \$434.47 per month, but he decides that he can afford to pay \$100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18{,}500, \qquad B_n = 1.005B_{n-1} - 534.47$$

(a) Determine Phil's balance after making the first payment. That is, determine B_1.
(b) Using a graphing utility, determine when Phil's balance will be below \$10,000. How many payments of \$534.47 have been made?
(c) Using a graphing utility, determine when Phil will pay off the balance. What is the total of all the payments?
(d) What was Phil's interest expense?

85. Trout Population A pond currently has 2000 trout in it. A fish hatchery decides to add an additional 20 trout each month. In addition, it is known that the trout population is growing 3% per month. The size of the population after n months is given by the recursively defined sequence

$$p_0 = 2000, \qquad p_n = 1.03p_{n-1} + 20$$

(a) How many trout are in the pond at the end of the second month? That is, what is p_2?
(b) Using a graphing utility, determine how long it will be before the trout population reaches 5000.

86. Environmental Control The Environmental Protection Agency (EPA) determines that Maple Lake has 250 tons of pollutants as a result of industrial waste and that 10% of the pollutant present is neutralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 tons of new pollutant entering the lake each year. The amount of pollutant in the lake at the end of each year is given by the recursively defined sequence

$$p_0 = 250, \qquad p_n = 0.9p_{n-1} + 15$$

(a) Determine the amount of pollutant in the lake at the end of the second year. That is, determine p_2.
(b) Using a graphing utility, provide pollutant amounts for the next 20 years.
(c) What is the equilibrium level of pollution in Maple Lake? That is, what is $\lim_{n\to\infty} p_n$?

87. Roth IRA On January 1, 1999, Bob decided to place \$500 at the end of each quarter into a Roth Individual Retirement Account.

(a) Find a recursive formula that represents Bob's balance at the end of each quarter if the rate of return is assumed to be 8% per annum compounded quarterly.
(b) How long will it be before the value of the account exceeds \$100,000?
(c) What will be the value of the account in 25 years when Bob retires?

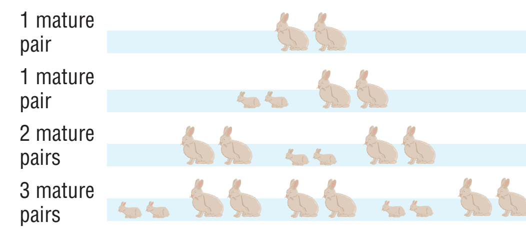
88. Education IRA On January 1, 1999, John's parents decided to place \$45 at the end of each month into an Education IRA.
(a) Find a recursive formula that represents the balance at the end of each month if the rate of return is assumed to be 6% per annum compounded monthly.
(b) How long will it be before the value of the account exceeds \$4000?
(c) What will be the value of the account in 16 years when John goes to college?

89. Home Loan Bill and Laura borrowed \$150,000 at 6% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be \$899.33.
(a) Find a recursive formula for their balance after each monthly payment has been made.
(b) Determine Bill and Laura's balance after the first payment.
(c) Using a graphing utility, create a table showing Bill and Laura's balance after each monthly payment.
(d) Using a graphing utility, determine when Bill and Laura's balance will be below \$140,000.
(e) Using a graphing utility, determine when Bill and Laura will pay off the balance.
(f) Determine Bill and Laura's interest expense when the loan is paid.
(g) Suppose that Bill and Laura decide to pay an additional \$100 each month on their loan. Answer parts (a) to (f) under this scenario.
(h) Is it worthwhile for Bill and Laura to pay the additional \$100? Explain.

90. Home Loan Jodi and Jeff borrowed \$120,000 at 6.5% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be \$758.48.
(a) Find a recursive formula for their balance after each monthly payment has been made.
(b) Determine Jodi and Jeff's balance after the first payment.
(c) Using a graphing utility, create a table showing Jodi and Jeff's balance after each monthly payment.
(d) Using a graphing utility, determine when Jodi and Jeff's balance will be below \$100,000.
(e) Using a graphing utility, determine when Jodi and Jeff will pay off the balance.
(f) Determine Jodi and Jeff's interest expense when the loan is paid.
(g) Suppose that Jodi and Jeff decide to pay an additional \$100 each month on their loan. Answer parts (a) to (f) under this scenario.
(h) Is it worthwhile for Jodi and Jeff to pay the additional \$100? Explain.

91. Growth of a Rabbit Colony A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring (one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?

[**Hint:** A Fibonacci sequence models this colony. Do you see why?]

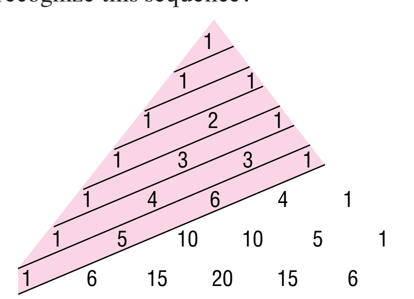


92. Fibonacci Sequence Let

$$u_n = \frac{\left(1 + \sqrt{5}\right)^n - \left(1 - \sqrt{5}\right)^n}{2^n\sqrt{5}}$$

define the nth term of a sequence.
(a) Show that $u_1 = 1$ and $u_2 = 1$.
(b) Show that $u_{n+2} = u_{n+1} + u_n$.
(c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence.

93. Pascal's Triangle Divide the triangular array shown (called Pascal's triangle) using diagonal lines as indicated. Find the sum of the numbers in each of these diagonal rows. Do you recognize this sequence?



94. Fibonacci Sequence Use the result of Problem 92 to do the following problems:
(a) Write the first 10 terms of the Fibonacci sequence.
(b) Compute the ratio $\frac{u_{n+1}}{u_n}$ for the first 10 terms.
(c) As n gets large, what number does the ratio approach? This number is referred to as the **golden ratio**. Rectangles whose sides are in this ratio were considered pleasing to the eye by the Greeks. For example, the facade of the Parthenon was constructed using the golden ratio.
(d) Compute the ratio $\frac{u_n}{u_{n+1}}$ for the first 10 terms.
(e) As n gets large, what number does the ratio approach? This number is also referred to as the **golden ratio**. This ratio is believed to have been used in the construction of the Great Pyramid in Egypt. The ratio equals the sum of the areas of the four face triangles divided by the total surface area of the Great Pyramid.

95. Approximating e^x In calculus, it can be shown that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We can approximate the value of e^x for any x using the following sum

$$e^x \approx \sum_{k=0}^{n} \frac{x^k}{k!}$$

for some n.

(a) Approximate $e^{1.3}$ with $n = 4$.
(b) Approximate $e^{1.3}$ with $n = 7$.
(c) Use a calculator to approximate the value of $e^{1.3}$.
(d) Using trial and error along with a graphing utility's SEQuence mode, determine the value of n required to approximate $e^{1.3}$ correct to eight decimal places.

96. Approximating e^x Refer to Problem 95.

(a) Approximate $e^{-2.4}$ with $n = 3$.
(b) Approximate $e^{-2.4}$ with $n = 6$.
(c) Use a calculator to approximate the value of $e^{-2.4}$.
(d) Using trial and error along with a graphing utility's SEQuence mode, determine the value of n required to approximate $e^{-2.4}$ correct to eight decimal places.

97. Show that

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

[Hint: Let

$$S = 1 + 2 + \cdots + (n-1) + n$$
$$S = n + (n-1) + (n-2) + \cdots + 1$$

Add these equations. Then

$$2S = \underbrace{[1 + n] + [2 + (n-1)] + \cdots + [n + 1]}_{n \text{ terms in brackets}}$$

Now complete the derivation.]

Discussion and Writing

98. Investigate various applications that lead to a Fibonacci sequence, such as art, architecture, or financial markets. Write an essay on these applications.

'Are You Prepared?' Answers

1. $f(2) = \frac{1}{2}; f(3) = \frac{2}{3}$ **2.** True **3.** \$1082.43 **4.** \$9513.28

11.2 Arithmetic Sequences

OBJECTIVES
1. Determine If a Sequence Is Arithmetic
2. Find a Formula for an Arithmetic Sequence
3. Find the Sum of an Arithmetic Sequence

1 Determine If a Sequence Is Arithmetic

When the difference between successive terms of a sequence is always the same number, the sequence is called **arithmetic**. An **arithmetic sequence*** may be defined recursively as $a_1 = a, \quad a_n - a_{n-1} = d,$ or as

$$a_1 = a, \qquad a_n = a_{n-1} + d \qquad \textbf{(1)}$$

where $a = a_1$ and d are real numbers. The number a is the first term, and the number d is called the **common difference**.

The terms of an arithmetic sequence with first term a and common difference d follow the pattern

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \ldots$$

*Sometimes called an **arithmetic progression**.

EXAMPLE 1 **Determining If a Sequence Is Arithmetic**

The sequence

$$4, \quad 7, \quad 10, \quad 13, \ldots$$

is arithmetic since the difference of successive terms is 3. The first term is 4, and the common difference is 3. ◀

EXAMPLE 2 **Determining If a Sequence Is Arithmetic**

Show that the following sequence is arithmetic. Find the first term and the common difference.

$$\{s_n\} = \{3n + 5\}$$

Solution The first term is $s_1 = 3 \cdot 1 + 5 = 8$. The nth and $(n - 1)$st terms of the sequence $\{s_n\}$ are

$$s_n = 3n + 5 \quad \text{and} \quad s_{n-1} = 3(n - 1) + 5 = 3n + 2$$

Their difference is

$$s_n - s_{n-1} = (3n + 5) - (3n + 2) = 5 - 2 = 3$$

Since the difference of two successive terms is constant, the sequence is arithmetic and the common difference is 3. ◀

EXAMPLE 3 **Determining If a Sequence Is Arithmetic**

Show that the sequence $\{t_n\} = \{4 - n\}$ is arithmetic. Find the first term and the common difference.

Solution The first term is $t_1 = 4 - 1 = 3$. The nth and $(n - 1)$st terms are

$$t_n = 4 - n \quad \text{and} \quad t_{n-1} = 4 - (n - 1) = 5 - n$$

Their difference is

$$t_n - t_{n-1} = (4 - n) - (5 - n) = 4 - 5 = -1$$

Since the difference of two successive terms is constant, $\{t_n\}$ is an arithmetic sequence whose common difference is -1. ◀

NOW WORK PROBLEM 5.

2 Find a Formula for an Arithmetic Sequence

Suppose that a is the first term of an arithmetic sequence whose common difference is d. We seek a formula for the nth term, a_n. To see the pattern, we write down the first few terms.

$$\begin{aligned}
a_1 &= a \\
a_2 &= a_1 + d = a + 1 \cdot d \\
a_3 &= a_2 + d = (a + d) + d = a + 2 \cdot d \\
a_4 &= a_3 + d = (a + 2 \cdot d) + d = a + 3 \cdot d \\
a_5 &= a_4 + d = (a + 3 \cdot d) + d = a + 4 \cdot d \\
&\vdots \\
a_n &= a_{n-1} + d = [a + (n - 2)d] + d = a + (n - 1)d
\end{aligned}$$

We are led to the following result:

Theorem

nth Term of an Arithmetic Sequence

For an arithmetic sequence $\{a_n\}$ whose first term is a and whose common difference is d, the nth term is determined by the formula

$$a_n = a + (n - 1)d \tag{2}$$

EXAMPLE 4 Finding a Particular Term of an Arithmetic Sequence

Find the thirteenth term of the arithmetic sequence: $2, 6, 10, 14, 18, \ldots$

Solution The first term of this arithmetic sequence is $a = 2$, and the common difference is 4. By formula (2), the nth term is

$$a_n = 2 + (n - 1)4$$

Hence, the thirteenth term is

$$a_{13} = 2 + 12 \cdot 4 = 50$$

Exploration

Use a graphing utility to find the thirteenth term of the sequence given in Example 4. Use it to find the twentieth and fiftieth terms.

EXAMPLE 5 Finding a Recursive Formula for an Arithmetic Sequence

The eighth term of an arithmetic sequence is 75, and the twentieth term is 39. Find the first term and the common difference. Give a recursive formula for the sequence. What is the nth term of the sequence?

Solution By formula (2), we know that $a_n = a + (n - 1)d$. As a result,

$$\begin{cases} a_8 = a + 7d = 75 \\ a_{20} = a + 19d = 39 \end{cases}$$

This is a system of two linear equations containing two variables, a and d, which we can solve by elimination. Subtracting the second equation from the first equation, we get

$$-12d = 36$$
$$d = -3$$

With $d = -3$, we use $a + 7d = 75$ and find that $a = 75 - 7d = 75 - 7(-3) = 96$. The first term is $a = 96$, and the common difference is $d = -3$.

Using formula (1), a recursive formula for this sequence is

$$a_1 = 96, \qquad a_n = a_{n-1} - 3$$

Using formula (2), a formula for the nth term of the sequence $\{a_n\}$ is

$$a_n = a + (n - 1)d = 96 + (n - 1)(-3) = 99 - 3n$$

Exploration

Graph the recursive formula from Example 5, $a_1 = 96$, $a_n = a_{n-1} - 3$, using a graphing utility. Conclude that the graph of the recursive formula behaves like the graph of a linear function. How is d, the common difference, related to m, the slope of a line?

NOW WORK PROBLEMS 21 AND 27.

3 Find the Sum of an Arithmetic Sequence

The next result gives a formula for finding the sum of the first n terms of an arithmetic sequence.

Theorem

Sum of n Terms of an Arithmetic Sequence

Let $\{a_n\}$ be an arithmetic sequence with first term a and common difference d. The sum S_n of the first n terms of $\{a_n\}$ is

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + a_n) \qquad \textbf{(3)}}$$

Proof

$$
\begin{aligned}
S_n &= a_1 + a_2 + a_3 + \cdots + a_n && \text{Sum of first } n \text{ terms}\\
&= a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] && \text{Formula (2)}\\
&= \underbrace{(a + a + \cdots + a)}_{n \text{ terms}} + [d + 2d + \cdots + (n-1)d] && \text{Rearrange terms}\\
&= na + d[1 + 2 + \cdots + (n-1)]\\
&= na + d\left[\frac{(n-1)n}{2}\right] && \text{Property 6, Section 11.1}\\
&= na + \frac{n}{2}(n-1)d\\
&= \frac{n}{2}[2a + (n-1)d] && \text{Factor out } \frac{n}{2} \qquad \textbf{(4)}\\
&= \frac{n}{2}[a + a + (n-1)d]\\
&= \frac{n}{2}(a + a_n) && \text{Formula (2)} \qquad \textbf{(5)}
\end{aligned}
$$

■

Formula (3) provides two ways to find the sum of the first n terms of an arithmetic sequence. Notice that formula (4) involves the first term and common difference, whereas formula (5) involves the first term and the nth term. Use whichever form is easier.

EXAMPLE 6 **Finding the Sum of n Terms of an Arithmetic Sequence**

Find the sum S_n of the first n terms of the sequence $\{3n + 5\}$; that is, find

$$8 + 11 + 14 + \cdots + (3n + 5)$$

Solution The sequence $\{3n + 5\}$ is an arithmetic sequence with first term $a = 8$ and the nth term $(3n + 5)$. To find the sum S_n, we use formula (5).

$$S_n = \frac{n}{2}(a + a_n) = \frac{n}{2}[8 + (3n + 5)] = \frac{n}{2}(3n + 13)$$

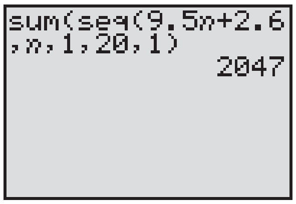
◀

NOW WORK PROBLEM 35.

EXAMPLE 7 Using a Graphing Utility to Find the Sum of 20 Terms of an Arithmetic Sequence

Use a graphing utility to find the sum of the first 20 terms of the sequence $\{9.5n + 2.6\}$.

Figure 12



Solution Figure 12 shows the results obtained using a TI-84 Plus graphing calculator.

The sum of the first 20 terms of the sequence $\{9.5n + 2.6\}$ is 2047. ◀

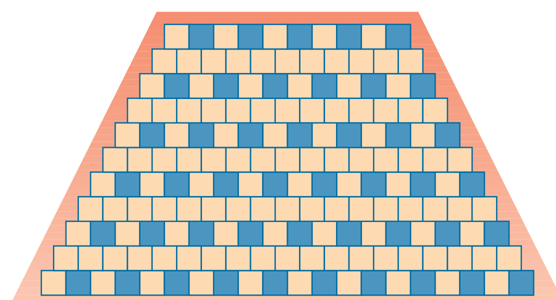
WORK EXAMPLE 7 USING FORMULA (3).

NOW WORK PROBLEM 43.

EXAMPLE 8 Creating a Floor Design

A ceramic tile floor is designed in the shape of a trapezoid 20 feet wide at the base and 10 feet wide at the top. See Figure 13. The tiles, 12 inches by 12 inches, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?

Figure 13



Solution The bottom row requires 20 tiles and the top row, 10 tiles. Since each successive row requires one less tile, the total number of tiles required is

$$S = 20 + 19 + 18 + \cdots + 11 + 10$$

This is the sum of an arithmetic sequence; the common difference is -1. The number of terms to be added is $n = 11$, with the first term $a = 20$ and the last term $a_{11} = 10$. The sum S is

$$S = \frac{n}{2}(a + a_{11}) = \frac{11}{2}(20 + 10) = 165$$

In all, 165 tiles will be required. ◀

11.2 Assess Your Understanding

Concepts and Vocabulary

1. In a(n) _____ sequence, the difference between successive terms is a constant.

2. *True or False:* In an arithmetic sequence the sum of the first and last terms equals twice the sum of all the terms.

Skill Building

In Problems 3–12, show that each sequence is arithmetic. Find the common difference and write out the first four terms.

3. $\{n + 4\}$ **4.** $\{n - 5\}$ **5.** $\{2n - 5\}$ **6.** $\{3n + 1\}$ **7.** $\{6 - 2n\}$

8. $\{4 - 2n\}$ **9.** $\left\{\frac{1}{2} - \frac{1}{3}n\right\}$ **10.** $\left\{\frac{2}{3} + \frac{n}{4}\right\}$ **11.** $\{\ln 3^n\}$ **12.** $\{e^{\ln n}\}$

In Problems 13–20, find the nth term of the arithmetic sequence whose initial term a and common difference d are given. What is the fifth term?

13. $a = 2;\quad d = 3$ **14.** $a = -2;\quad d = 4$ **15.** $a = 5;\quad d = -3$ **16.** $a = 6;\quad d = -2$

17. $a = 0;\quad d = \frac{1}{2}$ **18.** $a = 1;\quad d = -\frac{1}{3}$ **19.** $a = \sqrt{2};\quad d = \sqrt{2}$ **20.** $a = 0;\quad d = \pi$

In Problems 21–26, find the indicated term in each arithmetic sequence.

21. 12th term of $2, 4, 6, \ldots$

22. 8th term of $-1, 1, 3, \ldots$

23. 10th term of $1, -2, -5, \ldots$

24. 9th term of $5, 0, -5, \ldots$

25. 8th term of $a, a + b, a + 2b, \ldots$

26. 7th term of $2\sqrt{5}, 4\sqrt{5}, 6\sqrt{5}, \ldots$

In Problems 27–34, find the first term and the common difference of the arithmetic sequence described. Give a recursive formula for the sequence. Find a formula for the nth term.

27. 8th term is 8; 20th term is 44

28. 4th term is 3; 20th term is 35

29. 9th term is -5; 15th term is 31

30. 8th term is 4; 18th term is -96

31. 15th term is 0; 40th term is -50

32. 5th term is -2; 13th term is 30

33. 14th term is -1; 18th term is -9

34. 12th term is 4; 18th term is 28

In Problems 35–42, find each sum.

35. $1 + 3 + 5 + \cdots + (2n - 1)$

36. $2 + 4 + 6 + \cdots + 2n$

37. $7 + 12 + 17 + \cdots + (2 + 5n)$

38. $-1 + 3 + 7 + \cdots + (4n - 5)$

39. $2 + 4 + 6 + \cdots + 70$

40. $1 + 3 + 5 + \cdots + 59$

41. $5 + 9 + 13 + \cdots + 49$

42. $2 + 5 + 8 + \cdots + 41$

For Problems 43–48, use a graphing utility to find the sum of each sequence.

43. $\{3.45n + 4.12\},\quad n = 20$

44. $\{2.67n - 1.23\},\quad n = 25$

45. $2.8 + 5.2 + 7.6 + \cdots + 36.4$

46. $5.4 + 7.3 + 9.2 + \cdots + 32$

47. $4.9 + 7.48 + 10.06 + \cdots + 66.82$

48. $3.71 + 6.9 + 10.09 + \cdots + 80.27$

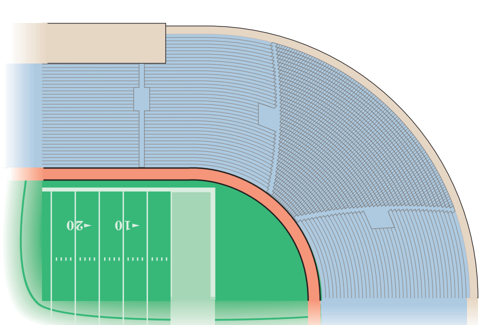
Applications and Extensions

49. Find x so that $x + 3, 2x + 1$, and $5x + 2$ are consecutive terms of an arithmetic sequence.

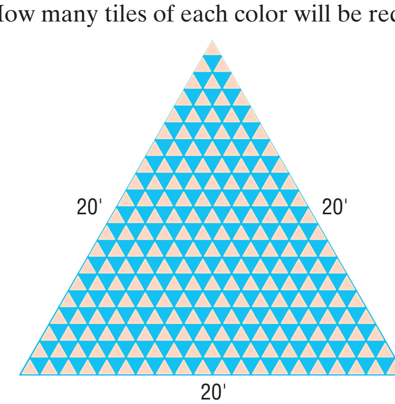
50. Find x so that $2x, 3x + 2$, and $5x + 3$ are consecutive terms of an arithmetic sequence.

51. Drury Lane Theater The Drury Lane Theater has 25 seats in the first row and 30 rows in all. Each successive row contains one additional seat. How many seats are in the theater?

52. Football Stadium The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?



53. Creating a Mosaic A mosaic is designed in the shape of an equilateral triangle, 20 feet on each side. Each tile in the mosaic is in the shape of an equilateral triangle, 12 inches to a side. The tiles are to alternate in color as shown in the illustration. How many tiles of each color will be required?



54. Constructing a Brick Staircase A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two less bricks than the prior step.
(a) How many bricks are required for the top step?
(b) How many bricks are required to build the staircase?

55. Stadium Construction How many rows are in the corner section of a stadium containing 2040 seats if the first row has 10 seats and each successive row has 4 additional seats?

56. Salary Suppose that you just received a job offer with a starting salary of \$35,000 per year and a guaranteed raise of \$1400 per year. How many years will it take before your aggregate salary is \$280,000?

[Hint: Your aggregate salary after 2 years is \$35,000 + (\$35,000 + \$1400).]

Discussion and Writing

57. Make up an arithmetic sequence. Give it to a friend and ask for its twentieth term.

58. Describe the similarities and differences between arithmetic sequences and linear functions.

11.3 Geometric Sequences; Geometric Series

OBJECTIVES
1. Determine If a Sequence Is Geometric
2. Find a Formula for a Geometric Sequence
3. Find the Sum of a Geometric Sequence
4. Find the Sum of a Geometric Series

1 Determine If a Sequence Is Geometric

When the ratio of successive terms of a sequence is always the same nonzero number, the sequence is called **geometric**. A **geometric sequence**[*] may be defined recursively as $a_1 = a$, $\dfrac{a_n}{a_{n-1}} = r$, or as

$$a_1 = a, \qquad a_n = ra_{n-1} \tag{1}$$

where $a_1 = a$ and $r \neq 0$ are real numbers. The number a is the first term, and the nonzero number r is called the **common ratio**.

The terms of a geometric sequence with first term a and common ratio r follow the pattern

$$a, \quad ar, \quad ar^2, \quad ar^3, \ldots$$

EXAMPLE 1 Determining If a Sequence Is Geometric

The sequence

$$2,\ 6,\ 18,\ 54,\ 162, \ldots$$

is geometric since the ratio of successive terms is $3\left(\dfrac{6}{2} = \dfrac{18}{6} = \dfrac{54}{18} = \cdots = 3\right)$. The first term is 2, and the common ratio is 3. ◀

EXAMPLE 2 Determining If a Sequence Is Geometric

Show that the following sequence is geometric.

$$\{s_n\} = 2^{-n}$$

Find the first term and the common ratio.

[*]Sometimes called a **geometric progression**.

Solution The first term is $s_1 = 2^{-1} = \frac{1}{2}$. The nth and $(n-1)$st terms of the sequence $\{s_n\}$ are

$$s_n = 2^{-n} \quad \text{and} \quad s_{n-1} = 2^{-(n-1)}$$

Their ratio is

$$\frac{s_n}{s_{n-1}} = \frac{2^{-n}}{2^{-(n-1)}} = 2^{-n+(n-1)} = 2^{-1} = \frac{1}{2}$$

Because the ratio of successive terms is a nonzero constant, the sequence $\{s_n\}$ is geometric with common ratio $\frac{1}{2}$. ◀

EXAMPLE 3 Determining If a Sequence Is Geometric

Show that the following sequence is geometric.

$$\{t_n\} = \{4^n\}$$

Find the first term and the common ratio.

Solution The first term is $t_1 = 4^1 = 4$. The nth and $(n-1)$st terms are

$$t_n = 4^n \quad \text{and} \quad t_{n-1} = 4^{n-1}$$

Their ratio is

$$\frac{t_n}{t_{n-1}} = \frac{4^n}{4^{n-1}} = 4^{n-(n-1)} = 4$$

The sequence, $\{t_n\}$, is a geometric sequence with common ratio 4. ◀

NOW WORK PROBLEM 7.

2 Find a Formula for a Geometric Sequence

Suppose that a is the first term of a geometric sequence with common ratio $r \neq 0$. We seek a formula for the nth term a_n. To see the pattern, we write down the first few terms:

$$\begin{aligned}
a_1 &= a = a \cdot 1 = ar^0 \\
a_2 &= ra_1 = ar^1 \\
a_3 &= ra_2 = r(ar) = ar^2 \\
a_4 &= ra_3 = r(ar^2) = ar^3 \\
a_5 &= ra_4 = r(ar^3) = ar^4 \\
&\vdots \\
a_n &= ra_{n-1} = r(ar^{n-2}) = ar^{n-1}
\end{aligned}$$

We are led to the following result:

Theorem

nth Term of a Geometric Sequence

For a geometric sequence $\{a_n\}$ whose first term is a and whose common ratio is r, the nth term is determined by the formula

$$a_n = ar^{n-1}, \qquad r \neq 0 \tag{2}$$

EXAMPLE 4

Finding a Particular Term of a Geometric Sequence

(a) Find the ninth term of the geometric sequence: $10, 9, \frac{81}{10}, \frac{729}{100} \ldots$

(b) Find a recursive formula for this sequence.

Solution (a) The first term of this geometric sequence is $a = 10$ and the common ratio is $\frac{9}{10}$. (Use $\frac{9}{10}$, or $\frac{\frac{81}{10}}{9} = \frac{9}{10}$, or any two successive terms.) By formula (2), the nth term is

$$a_n = 10\left(\frac{9}{10}\right)^{n-1}$$

The ninth term is

$$a_9 = 10\left(\frac{9}{10}\right)^{9-1} = 10\left(\frac{9}{10}\right)^{8} \approx 4.3046721$$

(b) The first term in the sequence is 10 and the common ratio is $r = \frac{9}{10}$. Using formula (1), the recursive formula is $a_1 = 10$, $a_n = \frac{9}{10}a_{n-1}$. ◀

— Exploration —

Use a graphing utility to find the ninth term of the sequence given in Example 4. Use it to find the twentieth and fiftieth terms. Now use a graphing utility to graph the recursive formula found in Example 4(b). Conclude that the graph of the recursive formula behaves like the graph of an exponential function. How is r, the common ratio, related to a, the base of the exponential function $y = a^x$?

NOW WORK PROBLEMS 29 AND 37.

3 Find the Sum of a Geometric Sequence

The next result gives us a formula for finding the sum of the first n terms of a geometric sequence.

Theorem

Sum of n Terms of a Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with first term a and common ratio r, where $r \neq 0, r \neq 1$. The sum S_n of the first n terms of $\{a_n\}$ is

$$S_n = a \cdot \frac{1 - r^n}{1 - r}, \qquad r \neq 0, 1 \tag{3}$$

Proof The sum S_n of the first n terms of $\{a_n\} = \{ar^{n-1}\}$ is

$$S_n = a + ar + \cdots + ar^{n-1} \tag{4}$$

Multiply each side by r to obtain

$$rS_n = ar + ar^2 + \cdots + ar^n \tag{5}$$

Now, subtract (5) from (4). The result is

$$S_n - rS_n = a - ar^n$$
$$(1 - r)S_n = a(1 - r^n)$$

Since $r \neq 1$, we can solve for S_n.

$$S_n = a \cdot \frac{1 - r^n}{1 - r}$$

■

EXAMPLE 5 Finding the Sum of n Terms of a Geometric Sequence

Find the sum S_n of the first n terms of the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$; that is, find

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n$$

Solution The sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$ is a geometric sequence with $a = \frac{1}{2}$ and $r = \frac{1}{2}$. The sum S_n that we seek is the sum of the first n terms of the sequence, so we use formula (3) to get

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{2}\right)^k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2}\left[\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\right] \quad \text{Formula (3)}$$

$$= \frac{1}{2}\left[\frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}}\right]$$

$$= 1 - \left(\frac{1}{2}\right)^n$$

◀

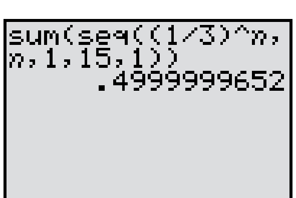
NOW WORK PROBLEM 43.

EXAMPLE 6 Using a Graphing Utility to Find the Sum of a Geometric Sequence

Use a graphing utility to find the sum of the first 15 terms of the sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$; that is, find

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \left(\frac{1}{3}\right)^{15}$$

Figure 14



Solution Figure 14 shows the result obtained using a TI-84 Plus graphing calculator. The sum of the first 15 terms of the sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$ is 0.4999999652. ◀

NOW WORK PROBLEM 49.

4 Find the Sum of a Geometric Series

An infinite sum of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

with first term a and common ratio r, is called an **infinite geometric series** and is denoted by

$$\sum_{k=1}^{\infty} ar^{k-1}$$

Based on formula (3), the sum S_n of the first n terms of a geometric series is

$$S_n = a \cdot \frac{1 - r^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r} \qquad \textbf{(6)}$$

If this finite sum S_n approaches a number L as $n \to \infty$, then we say the infinite geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ **converges**. We call L the **sum of the infinite geometric series**, and we write

$$L = \sum_{k=1}^{\infty} ar^{k-1}$$

If a series does not converge, it is called a **divergent series**.

Theorem

Sum of an Infinite Geometric Series

If $|r| < 1$, the sum of the infinite geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ is

$$\boxed{\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1 - r}} \qquad \textbf{(7)}$$

Intuitive Proof Since $|r| < 1$, it follows that $|r^n|$ approaches 0 as $n \to \infty$. Then, based on formula (6), the sum S_n approaches $\frac{a}{1 - r}$ as $n \to \infty$. ■

EXAMPLE 7 Finding the Sum of a Geometric Series

Find the sum of the geometric series: $2 + \frac{4}{3} + \frac{8}{9} + \cdots$

Solution The first term is $a = 2$, and the common ratio is

$$r = \frac{\frac{4}{3}}{2} = \frac{4}{6} = \frac{2}{3}$$

Since $|r| < 1$, we use formula (7) to find that

$$2 + \frac{4}{3} + \frac{8}{9} + \cdots = \frac{2}{1 - \frac{2}{3}} = 6$$

◄

Exploration

Use a graphing utility to graph $U_n = 2\left(\frac{2}{3}\right)^{n-1}$ in sequence mode. TRACE the graph for large values of n. What happens to the value of U_n as n increases without bound? What can you conclude about $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$?

NOW WORK PROBLEM 55.

EXAMPLE 8 Repeating Decimals

Show that the repeating decimal 0.999 . . . equals 1.

Solution $0.999\ldots = \dfrac{9}{10} + \dfrac{9}{100} + \dfrac{9}{1000} + \cdots$

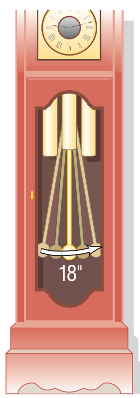
The decimal 0.999 . . . is a geometric series with first term $\frac{9}{10}$ and common ratio $\frac{1}{10}$. Using formula (7), we find

$$0.999\ldots = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

◀

EXAMPLE 9 Pendulum Swings

Figure 15



Initially, a pendulum swings through an arc of 18 inches. See Figure 15. On each successive swing, the length of the arc is 0.98 of the previous length.

(a) What is the length of the arc of the 10th swing?

(b) On which swing is the length of the arc first less than 12 inches?

(c) After 15 swings, what total distance will the pendulum have swung?

(d) When it stops, what total distance will the pendulum have swung?

Solution (a) The length of the first swing is 18 inches.

The length of the second swing is 0.98(18) inches.

The length of the third swing is $0.98(0.98)(18) = 0.98^2(18)$ inches.

The length of the arc of the 10th swing is

$$(0.98)^9(18) \approx 15.007 \text{ inches}$$

(b) The length of the arc of the nth swing is $(0.98)^{n-1}(18)$. For this to be exactly 12 inches requires that

$$
\begin{aligned}
(0.98)^{n-1}(18) &= 12 \\
(0.98)^{n-1} &= \frac{12}{18} = \frac{2}{3} && \text{Divide both sides by 18.} \\
n - 1 &= \log_{0.98}\left(\frac{2}{3}\right) && \text{Express as a logarithm.} \\
n &= 1 + \frac{\ln\left(\frac{2}{3}\right)}{\ln 0.98} \approx 1 + 20.07 = 21.07 && \text{Solve for } n\text{; use the Change of Base Formula.}
\end{aligned}
$$

The length of the arc of the pendulum exceeds 12 inches on the 21st swing and is first less than 12 inches on the 22nd swing.

(c) After 15 swings, the pendulum will have swung the following total distance L:

$$L = \underset{\text{1st}}{18} + \underset{\text{2nd}}{0.98(18)} + \underset{\text{3rd}}{(0.98)^2(18)} + \underset{\text{4th}}{(0.98)^3(18)} + \cdots + \underset{\text{15th.}}{(0.98)^{14}(18)}$$

This is the sum of a geometric sequence. The common ratio is 0.98; the first term is 18. The sum has 15 terms, so

$$L = 18 \cdot \frac{1 - 0.98^{15}}{1 - 0.98} \approx 18(13.07) \approx 235.3 \text{ inches}$$

The pendulum will have swung through 235.3 inches after 15 swings.

(d) When the pendulum stops, it will have swung the following total distance T:

$$T = 18 + 0.98(18) + (0.98)^2(18) + (0.98)^3(18) + \cdots$$

This is the sum of a geometric series. The common ratio is $r = 0.98$; the first term is $a = 18$. The sum is

$$T = \frac{a}{1 - r} = \frac{18}{1 - 0.98} = 900$$

The pendulum will have swung a total of 900 inches when it finally stops.

NOW WORK PROBLEM 69.

Historical Feature

Fibonacci

Sequences are among the oldest objects of mathematical investigation, having been studied for over 3500 years. After the initial steps, however, little progress was made until about 1600.

Arithmetic and geometric sequences appear in the Rhind papyrus, a mathematical text containing 85 problems copied around 1650 BC by the Egyptian scribe Ahmes from an earlier work (see Historical Problem 1). Fibonacci (AD 1220) wrote about problems similar to those found in the Rhind papyrus, leading one to suspect that Fibonacci may have had material available that is now lost. This material would have been in the non-Euclidean Greek tradition of Heron (about AD 75) and Diophantus (about AD 250). One problem, again modified slightly, is still with us in the familiar puzzle rhyme "As I was going to St. Ives . . . " (see Historical Problem 2).

The Rhind papyrus indicates that the Egyptians knew how to add up the terms of an arithmetic or geometric sequence, as did the Babylonians. The rule for summing up a geometric sequence is found in Euclid's *Elements* (Book IX, 35, 36), where, like all Euclid's algebra, it is presented in a geometric form.

Investigations of other kinds of sequences began in the 1500s, when algebra became sufficiently developed to handle the more complicated problems. The development of calculus in the 1600s added a powerful new tool, especially for finding the sum of infinite series, and the subject continues to flourish today.

Historical Problems

1. *Arithmetic sequence problem from the Rhind papyrus (statement modified slightly for clarity)* One hundred loaves of bread are to be divided among five people so that the amounts that they receive form an arithmetic sequence. The first two together receive one-seventh of what the last three receive. How many loaves does each receive?
[*Partial answer:* First person receives $1\frac{2}{3}$ loaves.]
2. The following old English children's rhyme resembles one of the Rhind papyrus problems.

   As I was going to St. Ives
   I met a man with seven wives
   Each wife had seven sacks
   Each sack had seven cats
   Each cat had seven kits [kittens]
   Kits, cats, sacks, wives
   How many were going to St. Ives?

   (a) Assuming that the speaker and the cat fanciers met by traveling in opposite directions, what is the answer?
   (b) How many kittens are being transported?
   (c) Kits, cats, sacks, wives; how many?
   [**Hint**: It is easier to include the man, find the sum with the formula, and then subtract 1 for the man.]

11.3 Assess Your Understanding

Concepts and Vocabulary

1. In a(n) _____ sequence, the ratio of successive terms is a constant.
2. If $|r| < 1$, the sum of the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ is _____.
3. *True or False:* A geometric sequence may be defined recursively.
4. *True or False:* In a geometric sequence, the common ratio is always a positive number.

Skill Building

In Problems 5–14, show that each sequence is geometric. Find the common ratio and write out the first four terms.

5. $\{3^n\}$ **6.** $\{(-5)^n\}$ **7.** $\left\{-3\left(\frac{1}{2}\right)^n\right\}$ **8.** $\left\{\left(\frac{5}{2}\right)^n\right\}$ **9.** $\left\{\frac{2^{n-1}}{4}\right\}$

10. $\left\{\frac{3^n}{9}\right\}$ **11.** $\{2^{n/3}\}$ **12.** $\{3^{2n}\}$ **13.** $\left\{\frac{3^{n-1}}{2^n}\right\}$ **14.** $\left\{\frac{2^n}{3^{n-1}}\right\}$

In Problems 15–28, determine whether the given sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference; if it is geometric, find the common ratio.

15. $\{n+2\}$ **16.** $\{2n-5\}$ **17.** $\{4n^2\}$ **18.** $\{5n^2+1\}$ **19.** $\left\{3-\frac{2}{3}n\right\}$

20. $\left\{8-\frac{3}{4}n\right\}$ **21.** $1, 3, 6, 10, \ldots$ **22.** $2, 4, 6, 8, \ldots$ **23.** $\left\{\left(\frac{2}{3}\right)^n\right\}$ **24.** $\left\{\left(\frac{5}{4}\right)^n\right\}$

25. $-1, -2, -4, -8, \ldots$ **26.** $1, 1, 2, 3, 5, 8, \ldots$ **27.** $\{3^{n/2}\}$ **28.** $\{(-1)^n\}$

In Problems 29–36, find the fifth term and the nth term of the geometric sequence whose initial term a and common ratio r are given.

29. $a = 2;\quad r = 3$ **30.** $a = -2;\quad r = 4$ **31.** $a = 5;\quad r = -1$ **32.** $a = 6;\quad r = -2$

33. $a = 0;\quad r = \frac{1}{2}$ **34.** $a = 1;\quad r = -\frac{1}{3}$ **35.** $a = \sqrt{2};\quad r = \sqrt{2}$ **36.** $a = 0;\quad r = \frac{1}{\pi}$

In Problems 37–42, find the indicated term of each geometric sequence.

37. 7th term of $1, \frac{1}{2}, \frac{1}{4}, \ldots$ **38.** 8th term of $1, 3, 9, \ldots$ **39.** 9th term of $1, -1, 1, \ldots$

40. 10th term of $-1, 2, -4, \ldots$ **41.** 8th term of $0.4, 0.04, 0.004, \ldots$ **42.** 7th term of $0.1, 1.0, 10.0, \ldots$

In Problems 43–48, find each sum.

43. $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\frac{2^3}{4}+\cdots+\frac{2^{n-1}}{4}$ **44.** $\frac{3}{9}+\frac{3^2}{9}+\frac{3^3}{9}+\cdots+\frac{3^n}{9}$ **45.** $\sum_{k=1}^{n}\left(\frac{2}{3}\right)^k$

46. $\sum_{k=1}^{n} 4\cdot 3^{k-1}$ **47.** $-1-2-4-8-\cdots-(2^{n-1})$ **48.** $2+\frac{6}{5}+\frac{18}{25}+\cdots+2\left(\frac{3}{5}\right)^{n-1}$

For Problems 49–54, use a graphing utility to find the sum of each geometric sequence.

49. $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\frac{2^3}{4}+\cdots+\frac{2^{14}}{4}$ **50.** $\frac{3}{9}+\frac{3^2}{9}+\frac{3^3}{9}+\cdots+\frac{3^{15}}{9}$ **51.** $\sum_{n=1}^{15}\left(\frac{2}{3}\right)^n$

52. $\sum_{n=1}^{15} 4\cdot 3^{n-1}$ **53.** $-1-2-4-8-\cdots-2^{14}$ **54.** $2+\frac{6}{5}+\frac{18}{25}+\cdots+2\left(\frac{3}{5}\right)^{15}$

In Problems 55–64, find the sum of each infinite geometric series.

55. $1+\frac{1}{3}+\frac{1}{9}+\cdots$ **56.** $2+\frac{4}{3}+\frac{8}{9}+\cdots$ **57.** $8+4+2+\cdots$ **58.** $6+2+\frac{2}{3}+\cdots$

59. $2-\frac{1}{2}+\frac{1}{8}-\frac{1}{32}+\cdots$ **60.** $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\cdots$ **61.** $\sum_{k=1}^{\infty} 5\left(\frac{1}{4}\right)^{k-1}$ **62.** $\sum_{k=1}^{\infty} 8\left(\frac{1}{3}\right)^{k-1}$

63. $\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1}$ **64.** $\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1}$

Applications and Extensions

65. Find x so that x, $x+2$, and $x+3$ are consecutive terms of a geometric sequence.

66. Find x so that $x-1$, x, and $x+2$ are consecutive terms of a geometric sequence.

67. Salary Increases Suppose that you have just been hired at an annual salary of \$18,000 and expect to receive annual increases of 5%. What will your salary be when you begin your fifth year?

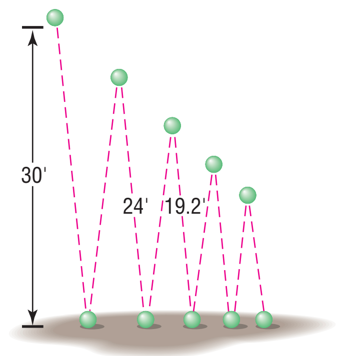
68. Equipment Depreciation A new piece of equipment cost a company \$15,000. Each year, for tax purposes, the company depreciates the value by 15%. What value should the company give the equipment after 5 years?

69. Pendulum Swings Initially, a pendulum swings through an arc of 2 feet. On each successive swing, the length of the arc is 0.9 of the previous length.

(a) What is the length of the arc of the 10th swing?

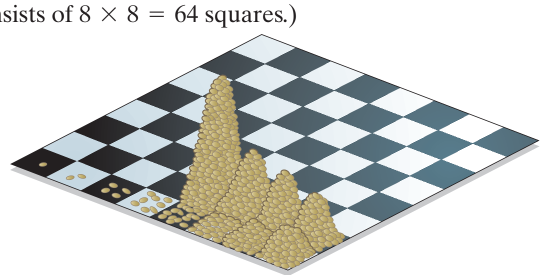
(b) On which swing is the length of the arc first less than 1 foot?
(c) After 15 swings, what total length will the pendulum have swung?
(d) When it stops, what total length will the pendulum have swung?

70. Bouncing Balls A ball is dropped from a height of 30 feet. Each time it strikes the ground, it bounces up to 0.8 of the previous height.

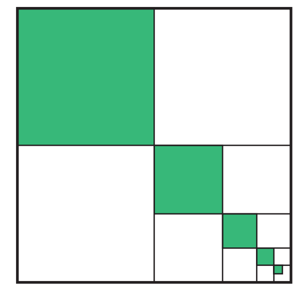


(a) What height will the ball bounce up to after it strikes the ground for the third time?
(b) How high will it bounce after it strikes the ground for the nth time?
(c) How many times does the ball need to strike the ground before its bounce is less than 6 inches?
(d) What total distance does the ball travel before it stops bouncing?

71. Grains of Wheat on a Chess Board In an old fable, a commoner who had saved the king's life was told he could ask the king for any just reward. Being a shrewd man, the commoner said, "A simple wish, sire. Place one grain of wheat on the first square of a chessboard, two grains on the second square, four grains on the third square, continuing until you have filled the board. This is all I seek." Compute the total number of grains needed to do this to see why the request, seemingly simple, could not be granted. (A chessboard consists of $8 \times 8 = 64$ squares.)



72. Look at the figure below. What fraction of the square is eventually shaded if the indicated shading process continues indefinitely?



73. Multiplier Suppose that, throughout the U.S. economy, individuals spend 90% of every additional dollar that they earn. Economists would say that an individual's **marginal propensity to consume** is 0.90. For example, if Jane earns an additional dollar, she will spend $0.9(1) = \$0.90$ of it. The individual that earns \$0.90 (from Jane) will spend 90% of it or \$0.81. This process of spending continues and results in an infinite geometric series as follows:

$$1, 0.90, 0.90^2, 0.90^3, 0.90^4, \ldots.$$

The sum of this infinite geometric series is called the **multiplier**. What is the multiplier if individuals spend 90% of every additional dollar that they earn?

74. Multiplier Refer to Problem 73. Suppose that the marginal propensity to consume throughout the U.S. economy is 0.95. What is the multiplier for the U.S. economy?

75. Stock Price One method of pricing a stock is to discount the stream of future dividends of the stock. Suppose that a stock pays \$$P$ per year in dividends and, historically, the dividend has been increased $i\%$ per year. If you desire an annual rate of return of $r\%$, this method of pricing a stock states that the price that you should pay is the present value of an infinite stream of payments:

$$\text{Price} = P + P\frac{1+i}{1+r} + P\left(\frac{1+i}{1+r}\right)^2 + P\left(\frac{1+i}{1+r}\right)^3 + \cdots$$

The price of the stock is the sum of an infinite geometric series. Suppose that a stock pays an annual dividend of \$4.00 and, historically, the dividend has been increased 3% per year. You desire an annual rate of return of 9%. What is the most you should pay for the stock?

76. Stock Price Refer to Problem 75. Suppose that a stock pays an annual dividend of \$2.50 and, historically, the dividend has increased 4% per year. You desire an annual rate of return of 11%. What is the most that you should pay for the stock?

Discussion and Writing

77. A Rich Man's Promise A rich man promises to give you \$1000 on September 1, 2001. Each day thereafter he will give you $\frac{9}{10}$ of what he gave you the previous day. What is the first date on which the amount you receive is less than 1¢? How much have you received when this happens?

78. Critical Thinking You are interviewing for a job and receive two offers:

A: \$20,000 to start, with guaranteed annual increases of 6% for the first 5 years

B: \$22,000 to start, with guaranteed annual increases of 3% for the first 5 years

Which offer is best if your goal is to be making as much as possible after 5 years? Which is best if your goal is to make as much money as possible over the contract (5 years)?

79. Critical Thinking Which of the following choices, A or B, results in more money?

A: To receive \$1000 on day 1, \$999 on day 2, \$998 on day 3, with the process to end after 1000 days

B: To receive \$1 on day 1, \$2 on day 2, \$4 on day 3, for 19 days

80. Critical Thinking You have just signed a 7-year professional football league contract with a beginning salary of \$2,000,000 per year. Management gives you the following options with regard to your salary over the 7 years.

1. A bonus of \$100,000 each year
2. An annual increase of 4.5% per year beginning after 1 year
3. An annual increase of \$95,000 per year beginning after 1 year

Which option provides the most money over the 7-year period? Which the least? Which would you choose? Why?

81. Can a sequence be both arithmetic and geometric? Give reasons for your answer.

82. Make up a geometric sequence. Give it to a friend and ask for its 20th term.

83. Make up two infinite geometric series, one that has a sum and one that does not. Give them to a friend and ask for the sum of each series.

84. Describe the similarities and differences between geometric sequences and exponential functions.

11.4 Mathematical Induction

OBJECTIVE 1 Prove Statements Using Mathematical Induction

1 Prove Statements Using Mathematical Induction

Mathematical induction is a method for proving that statements involving natural numbers are true for all natural numbers.*

For example, the statement "$2n$ is always an even integer" can be proved true for all natural numbers n by using mathematical induction. Also, the statement "the sum of the first n positive odd integers equals n^2," that is,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \tag{1}$$

can be proved for all natural numbers n by using mathematical induction.

Before stating the method of mathematical induction, let's try to gain a sense of the power of the method. We shall use the statement in equation (1) for this purpose by restating it for various values of $n = 1, 2, 3, \ldots$.

$n = 1$	The sum of the first positive odd integer is 1^2; $1 = 1^2$.
$n = 2$	The sum of the first 2 positive odd integers is 2^2; $1 + 3 = 4 = 2^2$.
$n = 3$	The sum of the first 3 positive odd integers is 3^2; $1 + 3 + 5 = 9 = 3^2$.
$n = 4$	The sum of the first 4 positive odd integers is 4^2; $1 + 3 + 5 + 7 = 16 = 4^2$.

Although from this pattern we might conjecture that statement (1) is true for any choice of n, can we really be sure that it does not fail for some choice of n? The method of proof by mathematical induction will, in fact, prove that the statement is true for all n.

*Recall that the natural numbers are the numbers $1, 2, 3, 4, \ldots$. In other words, the terms *natural numbers* and *positive integers* are synonymous.

Theorem

The Principle of Mathematical Induction

Suppose that the following two conditions are satisfied with regard to a statement about natural numbers:

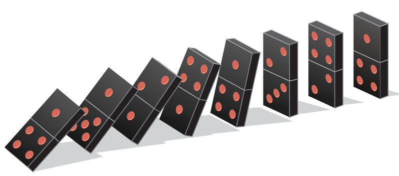
CONDITION I: The statement is true for the natural number 1.

CONDITION II: If the statement is true for some natural number k, it is also true for the next natural number $k + 1$.

Then the statement is true for all natural numbers.

We shall not prove this principle. However, we can provide a physical interpretation that will help us to see why the principle works. Think of a collection of natural numbers obeying a statement as a collection of infinitely many dominoes. See Figure 16.

Figure 16



Now, suppose that we are told two facts:

1. The first domino is pushed over.
2. If one domino falls over, say the kth domino, then so will the next one, the $(k + 1)$st domino.

Is it safe to conclude that *all* the dominoes fall over? The answer is yes, because if the first one falls (Condition I), then the second one does also (by Condition II); and if the second one falls, then so does the third (by Condition II); and so on.

Now let's prove some statements about natural numbers using mathematical induction.

EXAMPLE 1 Using Mathematical Induction

Show that the following statement is true for all natural numbers n.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \qquad \textbf{(2)}$$

Solution We need to show first that statement (2) holds for $n = 1$. Because $1 = 1^2$, statement (2) is true for $n = 1$. Condition I holds.

Next, we need to show that Condition II holds. Suppose that we know for some k that

$$1 + 3 + \cdots + (2k - 1) = k^2 \qquad \textbf{(3)}$$

We wish to show that, based on equation (3), statement (2) holds for $k + 1$. We look at the sum of the first $k + 1$ positive odd integers to determine whether this sum equals $(k + 1)^2$.

$$\begin{aligned} 1 + 3 + \cdots + (2k - 1) + [2(k + 1) - 1] &= \underbrace{[1 + 3 + \cdots + (2k - 1)]}_{= k^2 \text{ by equation (3)}} + (2k + 1) \\ &= k^2 + (2k + 1) \\ &= k^2 + 2k + 1 = (k + 1)^2 \end{aligned}$$

Conditions I and II are satisfied; by the Principle of Mathematical Induction, statement (2) is true for all natural numbers n. ◀

EXAMPLE 2 Using Mathematical Induction

Show that the following statement is true for all natural numbers n.

$$2^n > n$$

Solution First, we show that the statement $2^n > n$ holds when $n = 1$. Because $2^1 = 2 > 1$, the inequality is true for $n = 1$. Condition I holds.

Next, we assume, for some natural number k, that $2^k > k$. We wish to show that the formula holds for $k + 1$; that is, we wish to show that $2^{k+1} > k + 1$. Now

$$2^{k+1} = 2 \cdot 2^k \underset{\substack{\uparrow \\ \text{We know that} \\ 2^k > k.}}{>} 2 \cdot k = k + k \underset{\substack{\uparrow \\ k \geq 1}}{\geq} k + 1$$

If $2^k > k$, then $2^{k+1} > k + 1$, so Condition II of the Principle of Mathematical Induction is satisfied. The statement $2^n > n$ is true for all natural numbers n. ◀

EXAMPLE 3 Using Mathematical Induction

Show that the following formula is true for all natural numbers n.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{4}$$

Solution First, we show that formula (4) is true when $n = 1$. Because

$$\frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$$

Condition I of the Principle of Mathematical Induction holds.

Next, we assume that formula (4) holds for some k, and we determine whether the formula then holds for $k + 1$. We assume that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \quad \text{for some } k \tag{5}$$

Now we need to show that

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$$

We do this as follows:

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \underbrace{[1 + 2 + 3 + \cdots + k]}_{= \frac{k(k+1)}{2} \text{ by equation (5)}} + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2} \end{aligned}$$

Condition II also holds. As a result, formula (4) is true for all natural numbers n. ◀

NOW WORK PROBLEM 1.

EXAMPLE 4 Using Mathematical Induction

Show that $3^n - 1$ is divisible by 2 for all natural numbers n.

Solution First, we show that the statement is true when $n = 1$. Because $3^1 - 1 = 3 - 1 = 2$ is divisible by 2, the statement is true when $n = 1$. Condition I is satisfied.

Next, we assume that the statement holds for some k, and we determine whether the statement then holds for $k+1$. We assume that $3^k - 1$ is divisible by 2 for some k. We need to show that $3^{k+1} - 1$ is divisible by 2. Now

$$3^{k+1} - 1 = 3^{k+1} - 3^k + 3^k - 1 \quad \text{Subtract and add } 3^k.$$
$$= 3^k(3-1) + (3^k - 1) = 3^k \cdot 2 + (3^k - 1)$$

Because $3^k \cdot 2$ is divisible by 2 and $3^k - 1$ is divisible by 2, it follows that $3^k \cdot 2 + (3^k - 1) = 3^{k+1} - 1$ is divisible by 2. Condition II is also satisfied. As a result, the statement "$3^n - 1$ is divisible by 2" is true for all natural numbers n. ◀

WARNING The conclusion that a statement involving natural numbers is true for all natural numbers is made only after both Conditions I and II of the Principle of Mathematical Induction have been satisfied. Problem 27 demonstrates a statement for which only Condition I holds, but the statement is not true for all natural numbers. Problem 28 demonstrates a statement for which only Condition II holds, but the statement is not true for any natural number. ■

11.4 Assess Your Understanding

Skill Building

In Problems 1–26, use the Principle of Mathematical Induction to show that the given statement is true for all natural numbers n.

1. $2 + 4 + 6 + \cdots + 2n = n(n+1)$
2. $1 + 5 + 9 + \cdots + (4n-3) = n(2n-1)$
3. $3 + 4 + 5 + \cdots + (n+2) = \frac{1}{2}n(n+5)$
4. $3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$
5. $2 + 5 + 8 + \cdots + (3n-1) = \frac{1}{2}n(3n+1)$
6. $1 + 4 + 7 + \cdots + (3n-2) = \frac{1}{2}n(3n-1)$
7. $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$
8. $1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{1}{2}(3^n - 1)$
9. $1 + 4 + 4^2 + \cdots + 4^{n-1} = \frac{1}{3}(4^n - 1)$
10. $1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$
11. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
12. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$
13. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$
14. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$
15. $4 + 3 + 2 + \cdots + (5-n) = \frac{1}{2}n(9-n)$
16. $-2 - 3 - 4 - \cdots - (n+1) = -\frac{1}{2}n(n+3)$
17. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$
18. $1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \cdots + (2n-1)(2n) = \frac{1}{3}n(n+1)(4n-1)$
19. $n^2 + n$ is divisible by 2.
20. $n^3 + 2n$ is divisible by 3.
21. $n^2 - n + 2$ is divisible by 2.
22. $n(n+1)(n+2)$ is divisible by 6.
23. If $x > 1$, then $x^n > 1$.
24. If $0 < x < 1$, then $0 < x^n < 1$.
25. $a - b$ is a factor of $a^n - b^n$.
    [Hint: $a^{k+1} - b^{k+1} = a(a^k - b^k) + b^k(a - b)$]
26. $a + b$ is a factor of $a^{2n+1} + b^{2n+1}$.

Applications and Extensions

27. Show that the statement "$n^2 - n + 41$ is a prime number" is true for $n = 1$, but is not true for $n = 41$.

28. Show that the formula

$$2 + 4 + 6 + \cdots + 2n = n^2 + n + 2$$

obeys Condition II of the Principle of Mathematical Induction. That is, show that if the formula is true for some k it is also true for $k + 1$. Then show that the formula is false for $n = 1$ (or for any other choice of n).

29. Use mathematical induction to prove that if $r \neq 1$ then

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1 - r^n}{1 - r}$$

30. Use mathematical induction to prove that

$$a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = na + d\frac{n(n - 1)}{2}$$

31. Extended Principle of Mathematical Induction The Extended Principle of Mathematical Induction states that if Conditions I and II hold, that is,

(I) A statement is true for a natural number j.

(II) If the statement is true for some natural number $k \geq j$, then it is also true for the next natural number $k + 1$.

then the statement is true for all natural numbers $\geq j$.

Use the Extended Principle of Mathematical Induction to show that the number of diagonals in a convex polygon of n sides is $\frac{1}{2}n(n - 3)$.

[**Hint:** Begin by showing that the result is true when $n = 4$ (Condition I).]

32. Geometry Use the Extended Principle of Mathematical Induction to show that the sum of the interior angles of a convex polygon of n sides equals $(n - 2) \cdot 180°$.

Discussion and Writing

33. How would you explain the Principle of Mathematical Induction to a friend?

11.5 The Binomial Theorem

OBJECTIVES 1 Evaluate $\binom{n}{j}$

2 Use the Binomial Theorem

Formulas have been given for expanding $(x + a)^n$ for $n = 2$ and $n = 3$. The *Binomial Theorem** is a formula for the expansion of $(x + a)^n$ for any positive integer n. If $n = 1, 2, 3$, and 4, the expansion of $(x + a)^n$ is straightforward.

$$(x + a)^1 = x + a$$
Two terms, beginning with x^1 and ending with a^1

$$(x + a)^2 = x^2 + 2ax + a^2$$
Three terms, beginning with x^2 and ending with a^2

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$
Four terms, beginning with x^3 and ending with a^3

$$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$$
Five terms, beginning with x^4 and ending with a^4

Notice that each expansion of $(x + a)^n$ begins with x^n and ends with a^n. As you read from left to right, the powers of x are decreasing by one, while the powers of a are increasing by one. Also, the number of terms that appears equals $n + 1$. Notice, too, that the degree of each monomial in the expansion equals n. For example, in the

*The name *binomial* is derived from the fact that $x + a$ is a binomial; that is, it contains two terms.

expansion of $(x + a)^3$, each monomial $(x^3, 3ax^2, 3a^2x, a^3)$ is of degree 3. As a result, we might conjecture that the expansion of $(x + a)^n$ would look like this:

$$(x + a)^n = x^n + \_\_\_\_ ax^{n-1} + \_\_\_\_ a^2x^{n-2} + \cdots + \_\_\_\_ a^{n-1}x + a^n$$

where the blanks are numbers to be found. This is, in fact, the case, as we shall see shortly.

Before we can fill in the blanks, we need to introduce the symbol $\binom{n}{j}$.

1 Evaluate $\binom{n}{j}$

We define the symbol $\binom{n}{j}$, read "n taken j at a time," as follows:

NOTE On a graphing calculator, the symbol $\binom{n}{j}$ may be denoted by the key $\boxed{nCr}$.

If j and n are integers with $0 \le j \le n$, the symbol $\binom{n}{j}$ is defined as

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} \qquad \textbf{(1)}$$

EXAMPLE 1

Evaluating $\binom{n}{j}$

Find:

(a) $\binom{3}{1}$ (b) $\binom{4}{2}$ (c) $\binom{8}{7}$ (d) $\binom{65}{15}$

Solution

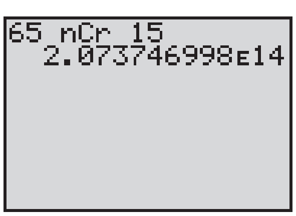
(a) $\binom{3}{1} = \dfrac{3!}{1!(3-1)!} = \dfrac{3!}{1!\,2!} = \dfrac{3 \cdot 2 \cdot 1}{1(2 \cdot 1)} = \dfrac{6}{2} = 3$

(b) $\binom{4}{2} = \dfrac{4!}{2!(4-2)!} = \dfrac{4!}{2!\,2!} = \dfrac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(2 \cdot 1)} = \dfrac{24}{4} = 6$

(c) $\binom{8}{7} = \dfrac{8!}{7!(8-7)!} = \dfrac{8!}{7!\,1!} = \dfrac{8 \cdot \cancel{7!}}{\cancel{7!} \cdot 1!} = \dfrac{8}{1} = 8$

$8! = 8 \cdot 7!$

Figure 17



(d) Figure 17 shows the solution using a TI-84 Plus graphing calculator. So,

$$\binom{65}{15} \approx 2.073746998 \times 10^{14}$$

NOW WORK PROBLEM 5.

Four useful formulas involving the symbol $\binom{n}{j}$ are

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{n-1} = n \qquad \binom{n}{n} = 1$$

Proof $$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!\,n!} = \frac{1}{1} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$$

You are asked to show the remaining two formulas in Problem 45. ■

Suppose that we arrange the various values of the symbol $\binom{n}{j}$ in a triangular display, as shown next and in Figure 18.

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

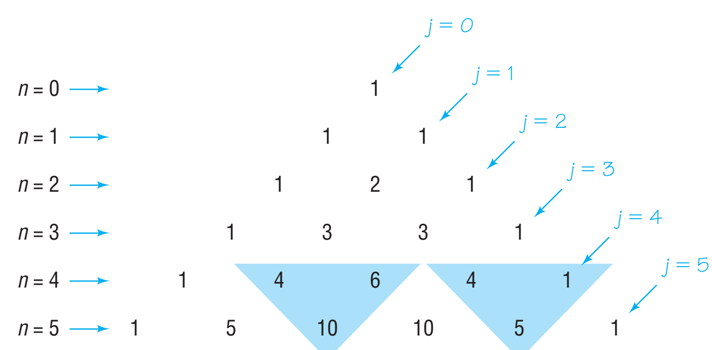
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Figure 18
Pascal triangle



This display is called the **Pascal triangle**, named after Blaise Pascal (1623–1662), a French mathematician.

The Pascal triangle has 1's down the sides. To get any other entry, add the two nearest entries in the row above it. The shaded triangles in Figure 18 illustrate this feature of the Pascal triangle. Based on this feature, the row corresponding to $n = 6$ is found as follows:

$n = 5 \rightarrow$ 1 5 10 10 5 1

$n = 6 \rightarrow$ 1 6 15 20 15 6 1

Later we shall prove that this addition always works (see the theorem on page 868).

Although the Pascal triangle provides an interesting and organized display of the symbol $\binom{n}{j}$, in practice it is not all that helpful. For example, if you wanted to know the value of $\binom{12}{5}$, you would need to produce 13 rows of the triangle before seeing the answer. It is much faster to use the definition (1).

2 Use the Binomial Theorem

Now we are ready to state the **Binomial Theorem**.

Theorem

Binomial Theorem

Let x and a be real numbers. For any positive integer n, we have

$$(x+a)^n = \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n$$
$$= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}a^j \qquad \textbf{(2)}$$

Now you know why we needed to introduce the symbol $\binom{n}{j}$; these symbols are the numerical coefficients that appear in the expansion of $(x+a)^n$. Because of this, the symbol $\binom{n}{j}$ is called a **binomial coefficient**.

EXAMPLE 2

Expanding a Binomial

Use the Binomial Theorem to expand $(x+2)^5$.

Solution In the Binomial Theorem, let $a = 2$ and $n = 5$. Then

$$(x+2)^5 = \binom{5}{0}x^5 + \binom{5}{1}2x^4 + \binom{5}{2}2^2x^3 + \binom{5}{3}2^3x^2 + \binom{5}{4}2^4x + \binom{5}{5}2^5$$

↑ Use equation (2)

$$= 1\cdot x^5 + 5\cdot 2x^4 + 10\cdot 4x^3 + 10\cdot 8x^2 + 5\cdot 16x + 1\cdot 32$$

↑ Use row $n = 5$ of the Pascal triangle or formula (1) for $\binom{n}{j}$.

$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

EXAMPLE 3

Expanding a Binomial

Expand $(2y-3)^4$ using the Binomial Theorem.

Solution First, we rewrite the expression $(2y-3)^4$ as $[2y+(-3)]^4$. Now we use the Binomial Theorem with $n = 4$, $x = 2y$, and $a = -3$.

$$[2y+(-3)]^4 = \binom{4}{0}(2y)^4 + \binom{4}{1}(-3)(2y)^3 + \binom{4}{2}(-3)^2(2y)^2 + \binom{4}{3}(-3)^3(2y) + \binom{4}{4}(-3)^4$$
$$= 1\cdot 16y^4 + 4(-3)8y^3 + 6\cdot 9\cdot 4y^2 + 4(-27)2y + 1\cdot 81$$

↑ Use row $n = 4$ of the Pascal triangle or formula (1) for $\binom{n}{j}$.

$$= 16y^4 - 96y^3 + 216y^2 - 216y + 81$$

In this expansion, note that the signs alternate due to the fact that $a = -3 < 0$.

NOW WORK PROBLEM 21.

EXAMPLE 4 Finding a Particular Coefficient in a Binomial Expansion

Find the coefficient of y^8 in the expansion of $(2y + 3)^{10}$.

Solution We write out the expansion using the Binomial Theorem.

$$(2y+3)^{10} = \binom{10}{0}(2y)^{10} + \binom{10}{1}(2y)^9(3)^1 + \binom{10}{2}(2y)^8(3)^2 + \binom{10}{3}(2y)^7(3)^3 + \binom{10}{4}(2y)^6(3)^4 + \cdots + \binom{10}{9}(2y)(3)^9 + \binom{10}{10}(3)^{10}$$

From the third term in the expansion, the coefficient of y^8 is

$$\binom{10}{2}(2)^8(3)^2 = \frac{10!}{2!\,8!}\cdot 2^8\cdot 9 = \frac{10\cdot 9\cdot \cancel{8!}}{2\cdot \cancel{8!}}\cdot 2^8\cdot 9 = 103{,}680$$

As this solution demonstrates, we can use the Binomial Theorem to find a particular term in an expansion without writing the entire expansion.

Based on the expansion of $(x + a)^n$, the term containing x^j is

$$\boxed{\binom{n}{n-j}a^{n-j}x^j \qquad (3)}$$

For example, we can solve Example 4 by using formula (3) with $n = 10$, $a = 3$, $x = 2y$, and $j = 8$. Then the term containing y^8 is

$$\binom{10}{10-8}3^{10-8}(2y)^8 = \binom{10}{2}\cdot 3^2\cdot 2^8\cdot y^8 = \frac{10!}{2!\,8!}\cdot 9\cdot 2^8y^8$$
$$= \frac{10\cdot 9\cdot \cancel{8!}}{2\cdot \cancel{8!}}\cdot 9\cdot 2^8y^8 = 103{,}680y^8$$

EXAMPLE 5 Finding a Particular Term in a Binomial Expansion

Find the sixth term in the expansion of $(x + 2)^9$.

Solution A We expand using the Binomial Theorem until the sixth term is reached.

$$(x+2)^9 = \binom{9}{0}x^9 + \binom{9}{1}x^8\cdot 2 + \binom{9}{2}x^7\cdot 2^2 + \binom{9}{3}x^6\cdot 2^3 + \binom{9}{4}x^5\cdot 2^4 + \binom{9}{5}x^4\cdot 2^5 + \cdots$$

The sixth term is

$$\binom{9}{5}x^4\cdot 2^5 = \frac{9!}{5!\,4!}\cdot x^4\cdot 32 = 4032x^4$$

Solution B The sixth term in the expansion of $(x + 2)^9$, which has 10 terms total, contains x^4. (Do you see why?) By formula (3), the sixth term is

$$\binom{9}{9-4}2^{9-4}x^4 = \binom{9}{5}2^5x^4 = \frac{9!}{5!\,4!}\cdot 32x^4 = 4032x^4$$

NOW WORK PROBLEMS 29 AND 35.

Next we show that the *triangular addition* feature of the Pascal triangle illustrated in Figure 18 always works.

Theorem

If n and j are integers with $1 \leq j \leq n$, then

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j} \tag{4}$$

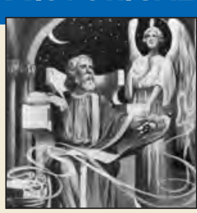
Proof

$$\begin{aligned}
\binom{n}{j-1} + \binom{n}{j} &= \frac{n!}{(j-1)![n-(j-1)]!} + \frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)!} + \frac{n!}{j!(n-j)!} \\
&= \frac{jn!}{j(j-1)!(n-j+1)!} + \frac{(n-j+1)n!}{j!(n-j+1)(n-j)!} \\
&= \frac{jn!}{j!(n-j+1)!} + \frac{(n-j+1)n!}{j!(n-j+1)!} \\
&= \frac{jn! + (n-j+1)n!}{j!(n-j+1)!} \\
&= \frac{n!(j+n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!(n+1)}{j!(n-j+1)!} = \frac{(n+1)!}{j![(n+1)-j]!} = \binom{n+1}{j}
\end{aligned}$$

Multiply the first term by $\frac{j}{j}$ and the second term by $\frac{n-j+1}{n-j+1}$. Now the denominators are equal.

■

Historical Feature



Omar Khayyám (1050–1123)

The case $n = 2$ of the Binomial Theorem, $(a + b)^2$, was known to Euclid in 300 BC, but the general law seems to have been discovered by the Persian mathematician and astronomer Omar Khayyám (1050–1123), who is also well known as the author of the *Rubáiyát*, a collection of four-line poems making observations on the human condition. Omar Khayyám did not state the Binomial Theorem explicitly, but he claimed to have a method for extracting third, fourth, fifth roots, and so on. A little study shows that one must know the Binomial Theorem to create such a method.

The heart of the Binomial Theorem is the formula for the numerical coefficients, and, as we saw, they can be written out in a symmetric triangular form. The Pascal triangle appears first in the books of Yang Hui (about 1270) and Chu Shih-chieh (1303). Pascal's name is attached to the triangle because of the many applications he made of it, especially to counting and probability. In establishing these results, he was one of the earliest users of mathematical induction.

Many people worked on the proof of the Binomial Theorem, which was finally completed for all n (including complex numbers) by Niels Abel (1802–1829).

11.5 Assess Your Understanding

Concepts and Vocabulary

1. The _____ _____ is a triangular display of the binomial coefficients.

2. $\binom{6}{2} =$ _____

3. *True of False:* $\binom{n}{j} = \frac{j!}{(n-j)!\,n!}$

4. The _____ _____ can be used to expand expressions like $(2x + 3)^6$.

Skill Building

In Problems 5–16, evaluate each expression.

5. $\binom{5}{3}$ **6.** $\binom{7}{3}$ **7.** $\binom{7}{5}$ **8.** $\binom{9}{7}$

9. $\binom{50}{49}$ **10.** $\binom{100}{98}$ **11.** $\binom{1000}{1000}$ **12.** $\binom{1000}{0}$

13. $\binom{55}{23}$ **14.** $\binom{60}{20}$ **15.** $\binom{47}{25}$ **16.** $\binom{37}{19}$

In Problems 17–28, expand each expression using the Binomial Theorem.

17. $(x+1)^5$ **18.** $(x-1)^5$ **19.** $(x-2)^6$ **20.** $(x+3)^5$

21. $(3x+1)^4$ **22.** $(2x+3)^5$ **23.** $(x^2+y^2)^5$ **24.** $(x^2-y^2)^6$

25. $(\sqrt{x}+\sqrt{2})^6$ **26.** $(\sqrt{x}-\sqrt{3})^4$ **27.** $(ax+by)^5$ **28.** $(ax-by)^4$

In Problems 29–42, use the Binomial Theorem to find the indicated coefficient or term.

29. The coefficient of x^6 in the expansion of $(x+3)^{10}$

30. The coefficient of x^3 in the expansion of $(x-3)^{10}$

31. The coefficient of x^7 in the expansion of $(2x-1)^{12}$

32. The coefficient of x^3 in the expansion of $(2x+1)^{12}$

33. The coefficient of x^7 in the expansion of $(2x+3)^9$

34. The coefficient of x^2 in the expansion of $(2x-3)^9$

35. The fifth term in the expansion of $(x+3)^7$

36. The third term in the expansion of $(x-3)^7$

37. The third term in the expansion of $(3x-2)^9$

38. The sixth term in the expansion of $(3x+2)^8$

39. The coefficient of x^0 in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$

40. The coefficient of x^0 in the expansion of $\left(x-\frac{1}{x^2}\right)^{9}$

41. The coefficient of x^4 in the expansion of $\left(x-\frac{2}{\sqrt{x}}\right)^{10}$

42. The coefficient of x^2 in the expansion of $\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)^{8}$

Applications and Extensions

43. Use the Binomial Theorem to find the numerical value of $(1.001)^5$ correct to five decimal places.
[**Hint:** $(1.001)^5 = (1+10^{-3})^5$]

44. Use the Binomial Theorem to find the numerical value of $(0.998)^6$ correct to five decimal places.

45. Show that $\binom{n}{n-1} = n$ and $\binom{n}{n} = 1$.

46. Show that if n and j are integers with $0 \le j \le n$ then

$$\binom{n}{j} = \binom{n}{n-j}$$

Conclude that the Pascal triangle is symmetric with respect to a vertical line drawn from the topmost entry.

47. If n is a positive integer, show that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

[**Hint:** $2^n = (1+1)^n$; now use the Binomial Theorem.]

48. If n is a positive integer, show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$$

49. $$\binom{5}{0}\left(\frac{1}{4}\right)^5 + \binom{5}{1}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) + \binom{5}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 + \binom{5}{3}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 + \binom{5}{4}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4 + \binom{5}{5}\left(\frac{3}{4}\right)^5 = ?$$

50. Stirling's Formula An approximation for $n!$, when n is large, is given by

$$n! \approx \sqrt{2n\pi}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n-1}\right)$$

Calculate 12!, 20!, and 25! on your calculator. Then use Stirling's formula to approximate 12!, 20!, and 25!.

Chapter Review

Things to Know

Sequence (p. 832)	A function whose domain is the set of positive integers.
Factorials (p. 834)	$0! = 1, 1! = 1, n! = n(n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ if $n \geq 2$
Arithmetic sequence (pp. 844 and 846)	$a_1 = a,\ a_n = a_{n-1} + d$, where $a =$ first term, $d =$ common difference $a_n = a + (n-1)d$
Sum of the first n terms of an arithmetic sequence (p. 847)	$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + a_n)$
Geometric sequence (pp. 850 and 852)	$a_1 = a,\ \ a_n = ra_{n-1}$, where $a =$ first term, $r =$ common ratio $a_n = ar^{n-1},\ \ r \neq 0$
Sum of the first n terms of a geometric sequence (p. 852)	$S_n = a\frac{1-r^n}{1-r},\ \ r \neq 0, 1$
Infinite geometric series (p. 854)	$a + ar + \cdots + ar^{n-1} + \cdots = \sum_{k=1}^{\infty} ar^{k-1}$
Sum of an infinite geometric series (p. 854)	$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r},\ \ \lvert r\rvert < 1$
Principle of Mathematical Induction (p. 860)	Suppose the following two conditions are satisfied. Condition I: The statement is true for the natural number 1. Condition II: If the statement is true for some natural number k, it is also true for $k + 1$. Then the statement is true for all natural numbers n.
Binomial coefficient (p. 864)	$\binom{n}{j} = \frac{n!}{j!(n-j)!}$
Pascal triangle (p. 865)	See Figure 18.
Binomial Theorem (p. 866)	$(x+a)^n = \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n$

Objectives

Section		You should be able to . . .	Review Exercises
11.1	1	Write the first several terms of a sequence (p. 832)	1–4
	2	Write the terms of a sequence defined by a recursive formula (p. 835)	5–8
	3	Use summation notation (p. 836)	9–12
	4	Find the sum of a sequence algebraically and using a graphing utility (p. 837)	25–30
	5	Solve annuity and amortization problems (p. 838)	66, 67
11.2	1	Determine if a sequence is arithmetic (p. 844)	13–24
	2	Find a formula for an arithmetic sequence (p. 845)	31, 32, 35, 37–40, 63, 64
	3	Find the sum of an arithmetic sequence (p. 847)	13, 14, 19, 20, 63, 64
11.3	1	Determine if a sequence is geometric (p. 850)	13–24
	2	Find a formula for a geometric sequence (p. 851)	17, 18, 21, 22, 33–36
	3	Find the sum of a geometric sequence (p. 852)	17, 18, 21, 22, 65(a)-(c)
	4	Find the sum of a geometric series (p. 854)	41–46, 65(d)
11.4	1	Prove statements using mathematical induction (p. 859)	47–52
11.5	1	Evaluate $\binom{n}{j}$ (p. 864)	53–54
	2	Use the Binomial Theorem (p. 866)	55–62

Review Exercises

In Problems 1–8, write down the first five terms of each sequence.

1. $\left\{(-1)^n\left(\dfrac{n+3}{n+2}\right)\right\}$ **2.** $\{(-1)^{n+1}(2n+3)\}$ **3.** $\left\{\dfrac{2^n}{n^2}\right\}$ **4.** $\left\{\dfrac{e^n}{n}\right\}$

5. $a_1 = 3;\quad a_n = \dfrac{2}{3}a_{n-1}$ **6.** $a_1 = 4;\quad a_n = -\dfrac{1}{4}a_{n-1}$ **7.** $a_1 = 2;\quad a_n = 2 - a_{n-1}$ **8.** $a_1 = -3;\quad a_n = 4 + a_{n-1}$

In Problems 9 and 10, write out each sum.

9. $\displaystyle\sum_{k=1}^{4}(4k+2)$ **10.** $\displaystyle\sum_{k=1}^{3}(3-k^2)$

In Problems 11 and 12, express each sum using summation notation.

11. $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots + \dfrac{1}{13}$ **12.** $2 + \dfrac{2^2}{3} + \dfrac{2^3}{3^2} + \cdots + \dfrac{2^{n+1}}{3^n}$

In Problems 13–24, determine whether the given sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference and the sum of the first n terms. If the sequence is geometric, find the common ratio and the sum of the first n terms.

13. $\{n+5\}$ **14.** $\{4n+3\}$ **15.** $\{2n^3\}$ **16.** $\{2n^2-1\}$

17. $\{2^{3n}\}$ **18.** $\{3^{2n}\}$ **19.** $0, 4, 8, 12, \ldots$ **20.** $1, -3, -7, -11, \ldots$

21. $3, \dfrac{3}{2}, \dfrac{3}{4}, \dfrac{3}{8}, \dfrac{3}{16}, \ldots$ **22.** $5, -\dfrac{5}{3}, \dfrac{5}{9}, -\dfrac{5}{27}, \dfrac{5}{81}, \ldots$ **23.** $\dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}, \ldots$ **24.** $\dfrac{3}{2}, \dfrac{5}{4}, \dfrac{7}{6}, \dfrac{9}{8}, \dfrac{11}{10}, \ldots$

In Problems 25–30, evaluate each sum (a) algebraically and (b) using a graphing utility.

25. $\displaystyle\sum_{k=1}^{5}(k^2+12)$ **26.** $\displaystyle\sum_{k=1}^{3}(k+2)^2$ **27.** $\displaystyle\sum_{k=1}^{10}(3k-9)$

28. $\displaystyle\sum_{k=1}^{9}(-2k+8)$ **29.** $\displaystyle\sum_{k=1}^{7}\left(\frac{1}{3}\right)^k$ **30.** $\displaystyle\sum_{k=1}^{10}(-2)^k$

In Problems 31–36, find the indicated term in each sequence.
[Hint: Find the general term first.]

31. 9th term of $3, 7, 11, 15, \ldots$ **32.** 8th term of $1, -1, -3, -5, \ldots$ **33.** 11th term of $1, \dfrac{1}{10}, \dfrac{1}{100}, \ldots$

34. 11th term of $1, 2, 4, 8, \ldots$ **35.** 9th term of $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \ldots$ **36.** 9th term of $\sqrt{2}, 2, 2^{3/2}, \ldots$

In Problems 37–40, find a general formula for each arithmetic sequence.

37. 7th term is 31; 20th term is 96 **38.** 8th term is -20; 17th term is -47

39. 10th term is 0; 18th term is 8 **40.** 12th term is 30; 22nd term is 50

In Problems 41–46, find the sum of each infinite geometric series.

41. $3 + 1 + \dfrac{1}{3} + \dfrac{1}{9} + \cdots$ **42.** $2 + 1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots$ **43.** $2 - 1 + \dfrac{1}{2} - \dfrac{1}{4} + \cdots$

44. $6 - 4 + \dfrac{8}{3} - \dfrac{16}{9} + \cdots$ **45.** $\displaystyle\sum_{k=1}^{\infty}4\left(\frac{1}{2}\right)^{k-1}$ **46.** $\displaystyle\sum_{k=1}^{\infty}3\left(-\frac{3}{4}\right)^{k-1}$

In Problems 47–52, use the Principle of Mathematical Induction to show that the given statement is true for all natural numbers.

47. $3 + 6 + 9 + \cdots + 3n = \dfrac{3n}{2}(n+1)$ **48.** $2 + 6 + 10 + \cdots + (4n-2) = 2n^2$

49. $2 + 6 + 18 + \cdots + 2\cdot 3^{n-1} = 3^n - 1$ **50.** $3 + 6 + 12 + \cdots + 3\cdot 2^{n-1} = 3(2^n - 1)$

51. $1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \dfrac{1}{2}n(6n^2 - 3n - 1)$ **52.** $1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \cdots + n(n+2) = \dfrac{n}{6}(n+1)(2n+7)$

In Problems 53 and 54, evaluate each binomial coefficient.

53. $\binom{5}{2}$

54. $\binom{8}{6}$

In Problems 55–58, expand each expression using the Binomial Theorem.

55. $(x + 2)^5$

56. $(x - 3)^4$

57. $(2x + 3)^5$

58. $(3x - 4)^4$

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59. Find the coefficient of x^7 in the expansion of $(x + 2)^9$.

60. Find the coefficient of x^3 in the expansion of $(x - 3)^8$.

61. Find the coefficient of x^2 in the expansion of $(2x + 1)^7$.

62. Find the coefficient of x^6 in the expansion of $(2x + 1)^8$.

63. Constructing a Brick Staircase A brick staircase has a total of 25 steps. The bottom step requires 80 bricks. Each successive step requires three less bricks than the prior step.
(a) How many bricks are required for the top step?
(b) How many bricks are required to build the staircase?

64. Creating a Floor Design A mosaic tile floor is designed in the shape of a trapezoid 30 feet wide at the base and 15 feet wide at the top. The tiles, 12 inches by 12 inches, are to be placed so that each successive row contains one less tile than the row below. How many tiles will be required?

[**Hint:** Refer to Figure 13, page 848.]

65. Bouncing Balls A ball is dropped from a height of 20 feet. Each time it strikes the ground, it bounces up to three-quarters of the previous height.
(a) What height will the ball bounce up to after it strikes the ground for the third time?
(b) How high will it bounce after it strikes the ground for the nth time?
(c) How many times does the ball need to strike the ground before its bounce is less than 6 inches?
(d) What total distance does the ball travel before it stops bouncing?

66. Home Loan Mike and Yola borrowed \$190,000 at 6.75% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be \$1232.34.
(a) Find a recursive formula for the balance after each monthly payment has been made.
(b) Determine the balance after the first payment.
(c) Using a graphing utility, create a table showing the balance after each monthly payment.
(d) Using a graphing utility, determine when the balance will be below \$100,000.
(e) Using a graphing utility, determine when Mike and Yola will pay off the balance.
(f) Determine their interest expense when the loan is paid.
(g) Suppose that Mike and Yola decide to pay an additional \$100 each month on the loan. Answer parts (a) to (f) under this scenario.

67. Credit Card Debt Beth just charged \$5000 on a VISA card that charges 1.5% interest per month on any unpaid balance. She can afford to pay \$100 toward the balance each month. Her balance at the beginning of each month after making a \$100 payment is given by the recursively defined sequence

$$b = \$5000, \quad b_n = 1.015b_{n-1} - 100$$

(a) Determine Beth's balance after making the first payment; that is, determine b_2, the balance at the beginning of the second month.
(b) Using a graphing utility, graph the recursively defined sequence.
(c) Using a graphing utility, determine when Beth's balance will be below \$4000. How many payments of \$100 have been made?
(d) Using a graphing utility, determine the number of payments it will take until Beth pays off the balance. What is the total of all the payments?
(e) What was Beth's interest expense?

Chapter Test

In Problems 1 and 2, write down the first five terms of each sequence.

1. $\left\{\dfrac{n^2 - 1}{n + 8}\right\}$

2. $a_1 = 4, a_n = 3a_{n-1} + 2$

In Problems 3 and 4, write out each sum. Evaluate each sum.

3. $\sum_{k=1}^{3} (-1)^{k+1}\left(\dfrac{k + 1}{k^2}\right)$

4. $\sum_{k=1}^{4} \left[\left(\dfrac{2}{3}\right)^k - k\right]$

5. Write the following sum using summation notation.

$$-\frac{2}{5} + \frac{3}{6} - \frac{4}{7} + \cdots + \frac{11}{14}$$

6. Find the sum using a graphing utility.

$$\sum_{k=1}^{100} (-1)^k(k^2)$$

In Problems 7–12, determine whether the given sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference and the sum of the first n terms. If the sequence is geometric, find the common ratio and the sum of the first n terms.

7. $6, 12, 36, 144, \ldots$

8. $\left\{-\frac{1}{2} \cdot 4^n\right\}$

9. $-2, -10, -18, -26, \ldots$

10. $\left\{-\frac{n}{2} + 7\right\}$

11. $25, 10, 4, \frac{8}{5}, \ldots$

12. $\left\{\frac{2n-3}{2n+1}\right\}$

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13. Find the sum of the infinite geometric series $256 - 64 + 16 - 4 + \cdots$

14. Expand $(3m + 2)^5$ using the Binomial Theorem.

15. Use the Principle of Mathematical Induction to show that the given statement is true for all natural numbers.
$\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{n}\right) = n + 1$

16. A 2004 Dodge Durango sold for \$31,000. If the vehicle loses 15% of its value each year, how much will it be worth after 10 years?

17. A weightlifter begins his routine by benching 100 pounds and increases the weight by 30 pounds for each set. If he does 10 repetitions in each set, what is the total weight lifted after 5 sets?

Chapter Projects

1. Population Growth The size of the population of the United States essentially depends on its current population, the birth and death rates of the population, and immigration. Suppose that b represents the birth rate of the U.S. population and d represents its death rate. Then $r = b - d$ represents the growth rate of the population, where r varies from year to year. The U.S. population after n years can be modeled using the recursive function

$$p_n = (1 + r)p_{n-1} + I$$

where I represents net immigration into the United States.

(a) Using data from the National Center for Health Statistics *www.fedstats.gov*, determine the birth and death rates for all races for the most recent year that data are available. Birth rates are given as the number of live births per 1000 population, while death rates are given as the number of deaths per 100,000 population. Each must be computed as the number of births (deaths) per individual. For example, in 1990, the birth rate was 16.7 per 1000 and the death rate was 863.8 per 100,000, so $b = \frac{16.7}{1000} = 0.0167$, while $d = \frac{863.8}{100{,}000} = 0.008638$.

Next, using data from the Immigration and Naturalization Service *www.fedstats.gov*, determine the net immigration to the United States for the same year used to obtain b and d in part (a).

(b) Determine the value of r, the growth rate of the population.

(c) Find a recursive formula for the population of the United States.

(d) Use the recursive formula to predict the population of the United States in the following year. In other words, if data are available up to the year 2003, predict the U.S. population in 2004.

(e) Compare your prediction to actual data.

(f) Do you think the recursive formula found in part (c) will be useful in predicting future populations? Why or why not?

The following projects are available at the Instructor's Resource Center (IRC):

2. Project at Motorola ***Digital Wireless Communication***

3. Economics

4. Standardized Tests

Cumulative Review

1. Find all the solutions, real and complex, of the equation

$$|x^2| = 9$$

2. (a) Graph the circle $x^2 + y^2 = 100$ and the parabola $y = 3x^2$

(b) Solve the system of equations: $\begin{cases} x^2 + y^2 = 100 \\ y = 3x^2 \end{cases}$

(c) Where do the circle and the parabola intersect?

3. Solve the equation $2e^x = 5$.

4. Find an equation of the line with slope 5 and x-intercept 2.

5. Find the general equation of the circle whose center is the point $(-1, 2)$ if $(3, 5)$ is a point on the circle.

6. $f(x) = \dfrac{3x}{x-2}$, $g(x) = 2x + 1$

Find:

(a) $(f \circ g)(2)$ (b) $(g \circ f)(4)$ (c) $(f \circ g)(x)$

(d) The domain of $(f \circ g)(x)$

(e) $(g \circ f)(x)$

(f) The domain of $(g \circ f)(x)$

(g) The function g^{-1} and its domain

(h) The function f^{-1} and its domain

7. Find the equation of an ellipse with center at the origin, a focus at $(0, 3)$, and a vertex at $(0, 4)$.

8. Find the equation of a parabola with vertex at $(-1, 2)$ and focus at $(-1, 3)$.

9. Find the polar equation of a circle with center at $(0, 4)$ that passes through the pole. What is the rectangular equation?

10. Solve the equation

$$2\sin^2 x - \sin x - 3 = 0, \quad 0 \le x < 2\pi.$$

11. Find the exact value of $\cos^{-1}(-0.5)$.

12. If $\sin\theta = \dfrac{1}{4}$ and θ is in the second quadrant, find:

(a) $\cos\theta$ (b) $\tan\theta$

(c) $\sin(2\theta)$ (d) $\cos(2\theta)$

(e) $\sin\left(\dfrac{1}{2}\theta\right)$