4.1.1

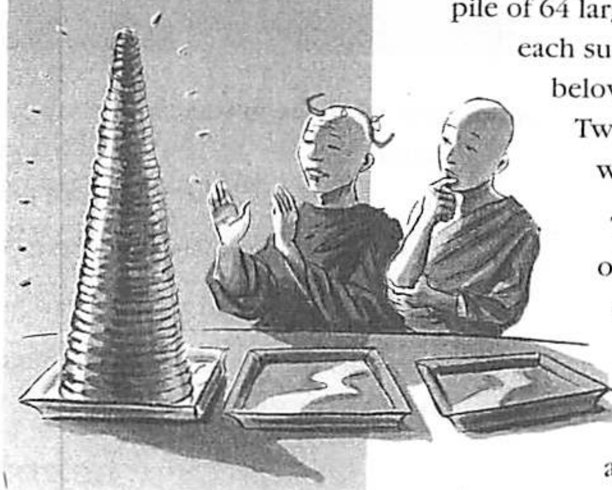
POW 1

The Tower of Hanoi

The Legend of the Golden Discs

Buddhism, one of the world's major religions, has roots in India and is practiced by over three hundred million people throughout the world. An ancient legend describes an important task once given to a group of Buddhist monks.

According to the legend, a Buddhist temple contained a pile of 64 large golden discs, one on top of another, with each successive disc slightly smaller than the one below it. This pile of discs sat upon a golden tray. Two empty golden trays lay next to the one with the pile of discs.



The monks' task was to move the pile of 64 discs from its original tray to one of the other trays. But according to the rules of the task, the monks could move only one disc at a time, taking it off the top of a pile. They could then place this disc either on an empty tray or on the top of an existing pile on one of the trays. Moreover, a disc placed on the top of an existing pile could not be larger than the disc below it.

The legend concludes with the promise that when the monks finish this task, the world will be filled with peace and harmony.

The Puzzle

As you will see, the monks could not possibly have finished the task. (How unfortunate for the state of the world!) But there is a mathematical puzzle, known as "The Tower of Hanoi," that is based on this legend. (Hanoi is the capital of Vietnam, which is located in southeast Asia. Many people in Vietnam are Buddhists.)

Continued on next page

4.3.1

POW 3

A Sticky Gum roblem" Revisited

you remember Ms. Hernandez and her twins? They re the main characters in a Year 1 POW called "A Sticky m Problem" (in the unit *The Game of Pig*). This POW olves a variation on that POW.

ıe Original POW

refresh your memory, here's the initial scenario from : Year 1 POW.

> Every time Ms. Hernandez passed a gum ball machine, her twins each wanted to get a gum ball, and they insisted on getting gum balls of the same color. Gum balls cost a penny each, and Ms. Hernandez had no control over which color she would get.

rst, Ms. Hernandez and the twins passed a gum ball achine with only two colors. Then they came across a achine with three colors. In each case, you needed to nd out the maximum amount Ms. Hernandez might have) spend in order to satisfy her twins.

hen Mr. Hodges came by the three-color gum ball ıachine. He had triplets, and so he needed to get three um balls that were the same color. You needed to find the ıaximum amount Mr. Hodges might have to spend in ırder to satisfy his triplets.

'inally, you were asked to generalize the problem. Your ;oal was to find a formula that would work for any ıumber of colors and any number of children. Your 'ormula needed to tell you the maximum amount that the parent might have to spend to provide each of the children with a gum ball of the same color.

Continued on next page

4.3.2

1. Before starting on the new problem, re-create the generalization for the old one.

   a. Find a formula for the maximum amount a parent might have to spend in terms of the number of colors and the number of children.

   b. Provide a proof of your generalization. That is, give a convincing argument to show that your formula is correct.

Some New Gum Ball Problems

Ms. Hernandez' twins have grown up a bit in the last three years, and they have changed in some ways. Now they each insist on getting a gum ball of a *different* color from the other twin. (Gum balls are still a penny each.)

One day, Ms. Hernandez and the twins passed a gum ball machine that contained exactly 20 gum balls: 8 yellow, 7 red, and 5 black. As before, Ms. Hernandez could not control which gum ball would come out of the machine.

2. If each twin wanted one gum ball, what's the maximum amount that Ms. Hernandez might have to spend so that the twins could each get a different color? Prove your answer.

3. Because the twins have grown, they now have bigger mouths, so sometimes they each want two gum balls. The two gum balls each twin gets must be the same color, so the flavors match, but one twin's pair of gum balls have to be a different color from the pair the other twin gets.

What's the maximum amount that Ms. Hernandez might have to spend? Prove your answer.

Continued on next page

4.3.3

New Generalizations

Now, make at least three generalizations about these new problems. You get to decide exactly what you want to generalize. Here are some options.

- Generalize the number of children.
- Generalize the number of gum balls each child wants.
- Generalize the number of gum balls of each color.
- Generalize the number of colors in the machine.

Your formula or procedure should tell you how to find the maximum amount that the parent might have to spend to provide all the children with the particular number of gum balls with each child getting gum balls of a different color.

Write a proof for each generalization you create. As a grand finale, try to generalize all of these variables.

Write-up

Your write-up for this problem should begin with the formula and proof for Question 1 and the answers (with proofs) for Questions 2 and 3.

Then present each generalization you found for the new type of problem, with a proof for each generalization. Also explain how you arrived at each generalization and how you discovered your proof.

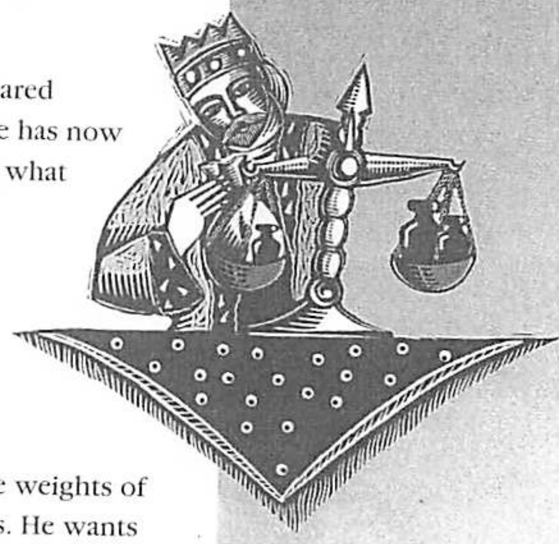
Adapted from "A Sticky Gum Problem" in *aha! Insight* by Martin Gardner, W. H. Freeman and Company, New York City/San Francisco, 1978.

4.6.1

POW 6

Which Weights Weigh What?

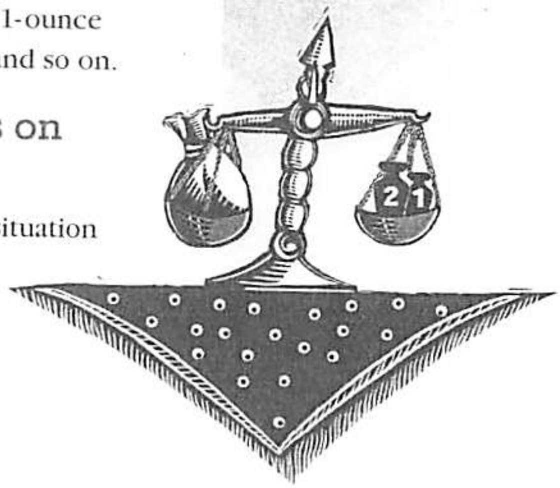
Do you remember the economical king with the different kinds of scales? (You first met him in Year 1 in *POW 14: Eight Bags of Gold* and *POW 15: Twelve Bags of Gold,* and he reappeared in Year 3 in *POW 14: And a Fortune, Too!*) He has now decided that an old-fashioned balance scale is what he wants. But he wants to be able to use it as efficiently as possible (of course).



In the king's country, it is customary to pay for food by weight, but the king doesn't trust the merchants' scales. He insists on using his balance scale to verify their claims.

He has decided that he only needs to verify the weights of packages that weigh whole numbers of ounces. He wants to be able to verify weights starting from 1 ounce and including all possibilities up to some still-undecided value. (He's the king, so he can make rules like that.) But he's concerned that in order to weigh different amounts, he might need a large set of standard weights—a 1-ounce weight, a 2-ounce weight, a 3-ounce weight, and so on.

Method 1: Combining Weights on One Side



One of the king's advisors points out that the situation is not so bad. For example, the king doesn't need a 3-ounce weight, because he can verify the weight of a 3-ounce package by balancing it against a combination of a 1-ounce and a 2-ounce weight, as shown here.

Continued on next page

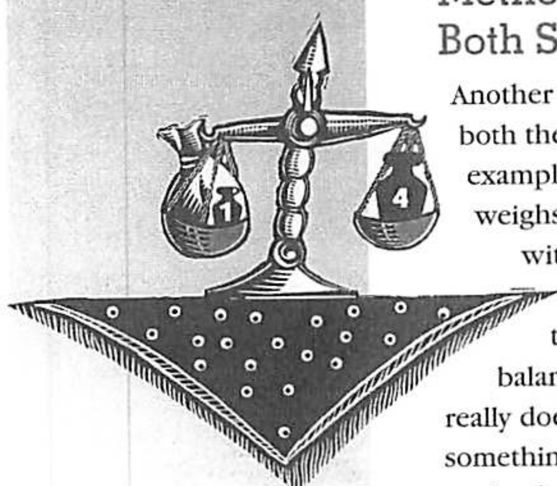
4.6.2

The king is relieved to hear this, but he still needs to decide what weights to buy. No specific set of weights will cover all possibilities, so the choice seems to depend on *how many* weights he is willing to buy.

Of course, if he buys only one weight, it will need to be a 1-ounce weight. Unfortunately, that will only allow him to verify the weight of a 1-ounce package.

Suppose the king is willing to buy two weights. Exactly which weights should he buy in order to verify the weight of as heavy a package as possible (and still include all whole-number possibilities of lesser weight)? And what if he buys three weights? Or four? The king isn't sure what to do.

Method 2: Using Weights on Both Sides



Another advisor suggests that perhaps the king can omit both the 2-ounce weight and the 3-ounce weight. For example, to verify a merchant's claim that a package weighs 3 ounces, the king can put the package together with a 1-ounce weight on one side of the balance scale and put a 4-ounce weight on the other side, as shown here. If the two sides balance, then the king will know that the package really does weigh 3 ounces. (But the King will need something other than the 1-ounce and 4-ounce weights to check a 2-ounce package.)

This method seems to provide more options, and so the king may be able to weigh more possibilities for a given number of weights. As with the first method, if the king buys only one weight, it will need to be a 1-ounce weight. But that will only allow him to verify the weight of a 1-ounce package.

Continued on next page

4.6.3

Suppose the king is willing to buy two weights. Which weights should he buy? And what if he buys three weights? Or four? Again, he isn't sure what to do.

Your Task: Consider Each Method

First consider Method 1, in which the weights must all go on one side of the balance scale and the package on the other side. Start with the specific cases of two weights, three weights, and four weights, and tell the king which weights he should choose in each case and what package weights his choice will allow him to verify. Then look for a general procedure that tells him what weights to choose and what package weights he will be able to verify.

Next, consider Method 2, which allows the king to put weights on both sides of the balance scale. (In Method 2, he can still put the weights all on one side if he chooses.) Again, start with specific cases of two weights, three weights, and four weights, and tell the king which weights he should choose in each case and what package weights his choice will allow him to verify. Then look for a general procedure that tells him what weights to choose and what package weights he will be able to verify.

Write-up

Your write-up should contain these components:

1. *Problem Statement*
2. *Process*
3. *Solution:* For each method, include a clear explanation of why the weights you choose are the best. State how high your choice of weights will allow the king to go, and explain this result. Include a general solution if you can.
4. *Evaluation*
5. *Self-assessment*

4.10.1

POW 10

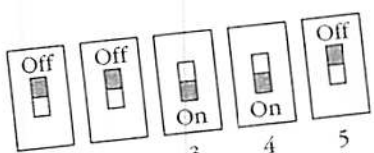
The King's Switches

Our favorite king has decided that he is tired of having his gold stolen. He is about to leave on a vacation, so he has installed an alarm system to prevent intruders from getting into his royal vault while he is gone.

The alarm system is activated and deactivated by the use of five switches. The only way to deactivate the system is to turn off all five switches.

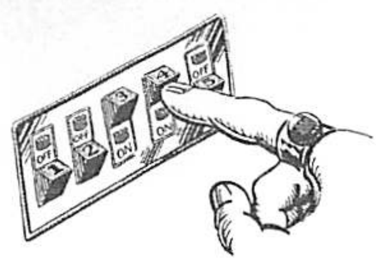
You might think that a thief could simply turn off the switches, but that isn't so easy to do. The system is set up so that the switches can only be changed according to these rules.

- The switch on the left may be changed (turned on or off) no matter how the other switches are set.
- Any other switch may be changed only if the switch to its immediate left is on and all other switches to its left are off.



For example, if the switches are currently set as shown in the diagram, then you can change either switch 1 or switch 4, but you can't change switch 2, switch 3, or switch 5.

1. When the king leaves for his vacation, all five switches are turned on. What is the minimum number of moves required to deactivate the king's alarm system?



Of course, this POW isn't only about the case of five switches, and it isn't only about getting a numerical answer. So here's more of the POW.

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4.10.2

Suppose you have n switches, all turned on, and you want to turn them all off. Assume that you can only change a switch according to the rules in the king's alarm system.

2. Find the minimum number of moves required for the cases $n = 1, 2, 3$, and 4, and prove your results. Also prove the result you found in Question 1 for the case $n = 5$.

3. Describe a pattern you see in your results or a procedure for finding new results based on results for smaller values of n. You might also make a guess about a general formula. You can get more data by considering cases where n is more than 5 and then use this information either to make guesses or to test ideas.

4. Based on your pattern, procedure, or formula, find the number of moves that would be required for $n = 10$.

5. Try to *prove* that your pattern, procedure, or general formula works in all cases. (That's hard!)

Write-up

In answering Questions 1 through 5, address these issues.

- The minimum number of moves required to deactivate the system for the specific cases you studied, including all cases up to $n = 5$
- Any ideas you have about a general pattern, procedure, or formula for describing the number of moves required, and an explanation of how to get the number of moves for $n = 10$
- Proofs, for each case up to $n = 5$, that your answer represents the minimum number of moves by which the system can be deactivated