Polynomial and Rational Functions

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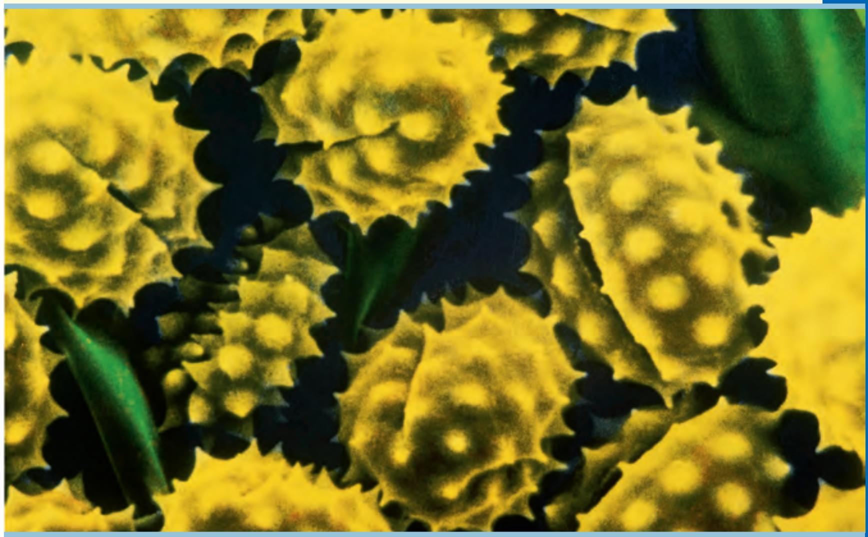


Ragweed plus bad air will equal fall misery

A hot, dry, smoggy summer is adding up to an insufferable fall for allergy sufferers.

Experts say people with allergies to ragweed, who normally sneeze and wheeze through September, will be dabbing their eyes and nose well into October, even though pollen counts are actually no worse than usual.

The difference this year is that a summer of heavy smog has made the pollen in the air feel much thicker, making it more irritating.

"Smog plays an extremely important role in allergy suffering," says Frances Coates, owner of Ottawa-based Aerobiology Research, which measures pollen levels. "It holds particles in the air a lot longer."

Low rainfall delayed the ragweed season until the end of August. It usually hits early in the month.

SOURCE: Adapted by Louise Surette from *Toronto Star*, September 8, 2001.

See Chapter Project 1.

A LOOK BACK In Chapter 2, we began our discussion of functions. We defined domain and range and independent and dependent variables; we found the value of a function and graphed functions. We continued our study of functions by listing the properties that a function might have, like being even or odd, and we created a library of functions, naming key functions and listing their properties, including their graphs.

A LOOK AHEAD In this chapter, we look at two general classes of functions: polynomial functions and rational functions, and examine their properties. Polynomial functions are arguably the simplest expressions in algebra. For this reason, they are often used to approximate other, more complicated, functions. Rational functions are simply ratios of polynomial functions.

We also introduce methods that can be used to determine the zeros of polynomial functions that are not factored completely. We end the chapter with a discussion of complex zeros.

OUTLINE

3.1 Quadratic Functions and Models

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intercepts (Section 1.2, pp. 15–17)
- Quadratic Equations (Appendix, Section A.5, pp. 988–995)
- Completing the Square (Appendix, Section A.5, pp. 991–992)
- Graphing Techniques: Transformations (Section 2.6, pp. 118–126)

Now work the 'Are You Prepared?' problems on page 163.

OBJECTIVES
1. Graph a Quadratic Function Using Transformations
2. Identify the Vertex and Axis of Symmetry of a Quadratic Function
3. Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts
4. Use the Maximum or Minimum Value of a Quadratic Function to Solve Applied Problems
5. Use a Graphing Utility to Find the Quadratic Function of Best Fit to Data

A *quadratic function* is a function that is defined by a second-degree polynomial in one variable.

A **quadratic function** is a function of the form

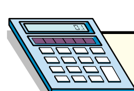
$$f(x) = ax^2 + bx + c \qquad \textbf{(1)}$$

where a, b, and c are real numbers and $a \neq 0$. The domain of a quadratic function is the set of all real numbers.

Many applications require a knowledge of quadratic functions. For example, suppose that Texas Instruments collects the data shown in Table 1, which relate the number of calculators sold at the price p (in dollars) per calculator. Since the price of a product determines the quantity that will be purchased, we treat price as the independent variable. The relationship between the number x of calculators sold and the price p per calculator may be approximated by the linear equation

$$x = 21{,}000 - 150p$$

Table 1

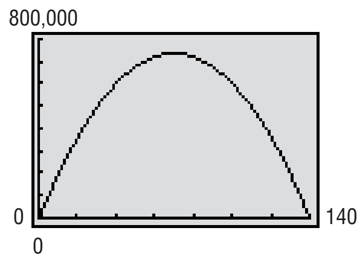


Price per Calculator, p (Dollars)	Number of Calculators, x
60	11,100
65	10,115
70	9,652
75	8,731
80	8,087
85	7,205
90	6,439

Then the revenue R derived from selling x calculators at the price p per calculator is equal to the unit selling price p of the product times the number x of units actually sold. That is,

$$
\begin{aligned}
R &= xp \\
R(p) &= (21{,}000 - 150p)p \\
&= -150p^2 + 21{,}000p
\end{aligned}
$$

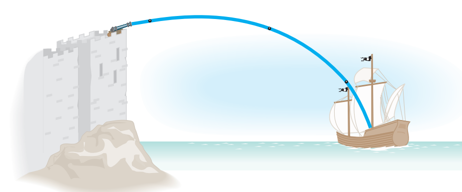
Figure 1



So, the revenue R is a quadratic function of the price p. Figure 1 illustrates the graph of this revenue function, whose domain is $0 \le p \le 140$, since both x and p must be non-negative. Later in this section we shall determine the price p that maximizes revenue.

A second situation in which a quadratic function appears involves the motion of a projectile. Based on Newton's second law of motion (force equals mass times acceleration, $F = ma$), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function. See Figure 2 for an illustration. Later in this section we shall analyze the path of a projectile.

Figure 2
Path of a cannonball



1 Graph a Quadratic Function Using Transformations

We know how to graph the quadratic function $f(x) = x^2$. Figure 3 shows the graph of three functions of the form $f(x) = ax^2$, $a > 0$, for $a = 1$, $a = \frac{1}{2}$, and $a = 3$. Notice that the larger the value of a, the "narrower" the graph is, and the smaller the value of a, the "wider" the graph is.

Figure 3

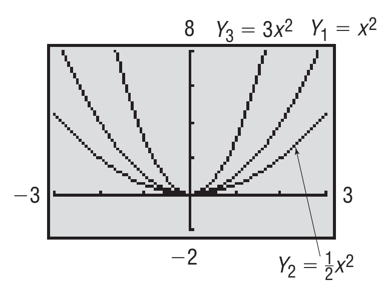


Figure 4

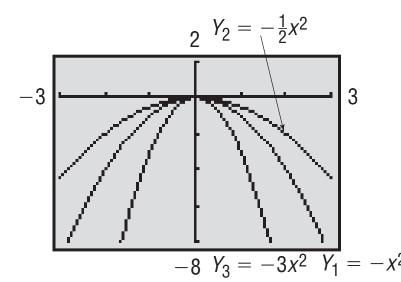
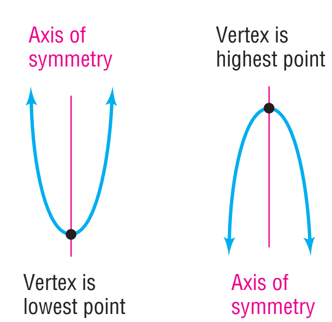


Figure 4 shows the graphs of $f(x) = ax^2$ for $a < 0$. Notice that these graphs are reflections about the x-axis of the graphs in Figure 3. Based on the results of these two figures, we can draw some general conclusions about the graph of $f(x) = ax^2$. First, as $|a|$ increases, the graph becomes "taller" (a vertical stretch), and as $|a|$ gets closer to zero, the graph gets "shorter" (a vertical compression). Second, if a is positive, then the graph opens "up," and if a is negative, then the graph opens "down."

The graphs in Figures 3 and 4 are typical of the graphs of all quadratic functions, which we call **parabolas**.* Refer to Figure 5, where two parabolas are pictured. The one on the left **opens up** and has a lowest point; the one on the right **opens down** and has a highest point. The lowest or highest point of a parabola is called the **vertex**.

Figure 5
Graphs of a quadratic function, $f(x) = ax^2 + bx + c$, $a \neq 0$



(a) Opens up $a > 0$ **(b)** Opens down $a < 0$

*We shall study parabolas using a geometric definition later in this book.

The vertical line passing through the vertex in each parabola in Figure 5 is called the **axis of symmetry** (usually abbreviated to **axis**) of the parabola. Because the parabola is symmetric about its axis, the axis of symmetry of a parabola can be used to find additional points on the parabola when graphing by hand.

The parabolas shown in Figure 5 are the graphs of a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$. Notice that the coordinate axes are not included in the figure. Depending on the values of a, b, and c, the axes could be placed anywhere. The important fact is that the shape of the graph of a quadratic function will look like one of the parabolas in Figure 5.

In the following example, we use techniques from Section 2.6 to graph a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$. In so doing, we shall complete the square and write the function f in the form $f(x) = a(x - h)^2 + k$.

EXAMPLE 1 Graphing a Quadratic Function Using Transformations

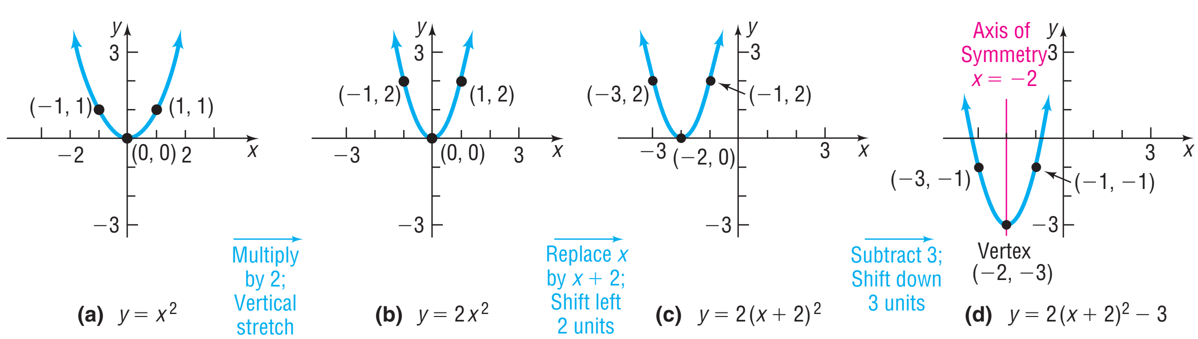
Graph the function $f(x) = 2x^2 + 8x + 5$. Find the vertex and axis of symmetry.

Solution We begin by completing the square on the right side.

$$\begin{aligned} f(x) &= 2x^2 + 8x + 5 \\ &= 2(x^2 + 4x) + 5 && \text{Factor out the 2 from } 2x^2 + 8x. \\ &= 2(x^2 + 4x + 4) + 5 - 8 && \text{Complete the square of } 2(x^2 + 4x)\text{. Notice that the factor of 2 requires that 8 be added and subtracted.} \\ &= 2(x + 2)^2 - 3 && \textbf{(2)} \end{aligned}$$

The graph of f can be obtained in three stages, as shown in Figure 6. Now compare this graph to the graph in Figure 5(a). The graph of $f(x) = 2x^2 + 8x + 5$ is a parabola that opens up and has its vertex (lowest point) at $(-2, -3)$. Its axis of symmetry is the line $x = -2$.

Figure 6



✔ **CHECK:** Use a graphing utility to graph $f(x) = 2x^2 + 8x + 5$ and use the MINIMUM command to locate its vertex. ◀

NOW WORK PROBLEM 27.

The method used in Example 1 can be used to graph any quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, as follows:

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c && \text{Factor out } a \text{ from } ax^2 + bx. \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) && \text{Complete the square by adding and subtracting } a\left(\frac{b^2}{4a^2}\right)\text{. Look closely at this step!} \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} && \text{Factor} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && c - \frac{b^2}{4a} = c\cdot\frac{4a}{4a} - \frac{b^2}{4a} = \frac{4ac - b^2}{4a}
\end{aligned}
$$

Based on these results, we conclude the following:

If $h = -\dfrac{b}{2a}$ and $k = \dfrac{4ac - b^2}{4a}$, then

$$\boxed{f(x) = ax^2 + bx + c = a(x - h)^2 + k} \qquad (3)$$

The graph of $f(x) = a(x - h)^2 + k$ is the parabola $y = ax^2$ shifted horizontally h units (replace x by $x - h$) and vertically k units (add k). As a result, the vertex is at (h, k), and the graph opens up if $a > 0$ and down if $a < 0$. The axis of symmetry is the vertical line $x = h$.

For example, compare equation (3) with equation (2) of Example 1.

$$
\begin{aligned}
f(x) &= 2(x + 2)^2 - 3 \\
&= 2(x - (-2))^2 - 3 \\
&= a(x - h)^2 + k
\end{aligned}
$$

We conclude that $a = 2$, so the graph opens up. Also, we find that $h = -2$ and $k = -3$, so its vertex is at $(-2, -3)$.

2 Identify the Vertex and Axis of Symmetry of a Quadratic Function

We do not need to complete the square to obtain the vertex. In almost every case, it is easier to obtain the vertex of a quadratic function f by remembering that its x-coordinate is $h = -\dfrac{b}{2a}$. The y-coordinate can then be found by evaluating f at $-\dfrac{b}{2a}$ to find $k = f\left(-\dfrac{b}{2a}\right)$.

We summarize these remarks as follows:

Properties of the Graph of a Quadratic Function

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

$$\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \qquad \text{Axis of symmetry: the line } x = -\frac{b}{2a} \qquad (4)$$

Parabola opens up if $a > 0$; the vertex is a minimum point.

Parabola opens down if $a < 0$; the vertex is a maximum point.

EXAMPLE 2 **Locating the Vertex without Graphing**

Without graphing, locate the vertex and axis of symmetry of the parabola defined by $f(x) = -3x^2 + 6x + 1$. Does it open up or down?

Solution For this quadratic function, $a = -3$, $b = 6$, and $c = 1$. The x-coordinate of the vertex is

$$h = -\frac{b}{2a} = -\frac{6}{-6} = 1$$

The y-coordinate of the vertex is

$$k = f\left(-\frac{b}{2a}\right) = f(1) = -3 + 6 + 1 = 4$$

The vertex is located at the point $(1, 4)$. The axis of symmetry is the line $x = 1$. Because $a = -3 < 0$, the parabola opens down. ◀

3 Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts

The information we gathered in Example 2, together with the location of the intercepts, usually provides enough information to graph $f(x) = ax^2 + bx + c$, $a \neq 0$, by hand.

The y-intercept is the value of f at $x = 0$; that is, $f(0) = c$.

The x-intercepts, if there are any, are found by solving the quadratic equation

$$ax^2 + bx + c = 0$$

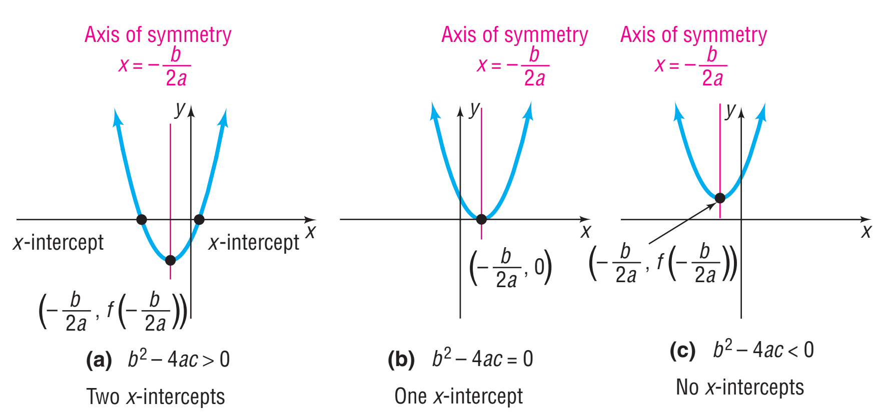
This equation has two, one, or no real solutions, depending on whether the discriminant $b^2 - 4ac$ is positive, 0, or negative. The graph of f has x-intercepts, as follows:

The x-intercepts of a Quadratic Function

1. If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct x-intercepts and so will cross the x-axis in two places.
2. If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one x-intercept and touches the x-axis at its vertex.
3. If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no x-intercept and so will not cross or touch the x-axis.

Figure 7 illustrates these possibilities for parabolas that open up.

Figure 7 $f(x) = ax^2 + bx + c$, $a > 0$



EXAMPLE 3 Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts

Use the information from Example 2 and the locations of the intercepts to graph $f(x) = -3x^2 + 6x + 1$. Determine the domain and the range of f. Determine where f is increasing and where it is decreasing.

Solution In Example 2, we found the vertex to be at $(1, 4)$ and the axis of symmetry to be $x = 1$. The y-intercept is found by letting $x = 0$. The y-intercept is $f(0) = 1$. The x-intercepts are found by solving the equation $f(x) = 0$. This results in the equation

$$-3x^2 + 6x + 1 = 0 \qquad a = -3, b = 6, c = 1$$

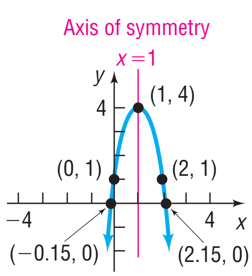
The discriminant $b^2 - 4ac = (6)^2 - 4(-3)(1) = 36 + 12 = 48 > 0$, so the equation has two real solutions and the graph has two x-intercepts. Using the quadratic formula, we find that

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-6 + \sqrt{48}}{-6} = \frac{-6 + 4\sqrt{3}}{-6} \approx -0.15$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-6 - \sqrt{48}}{-6} = \frac{-6 - 4\sqrt{3}}{-6} \approx 2.15$$

Figure 8



The x-intercepts are approximately -0.15 and 2.15.

The graph is illustrated in Figure 8. Notice how we used the y-intercept and the axis of symmetry, $x = 1$, to obtain the additional point $(2, 1)$ on the graph.

The domain of f is the set of all real numbers. Based on the graph, the range of f is the interval $(-\infty, 4]$. The function f is increasing on the interval $(-\infty, 1)$ and decreasing on the interval $(1, \infty)$.

✔ **CHECK:** Graph $f(x) = -3x^2 + 6x + 1$ using a graphing utility. Use ZERO (or ROOT) to locate the two x-intercepts. Use MAXIMUM to locate the vertex. ◀

NOW WORK PROBLEM 35.

If the graph of a quadratic function has only one x-intercept or no x-intercepts, it is usually necessary to plot an additional point to obtain the graph by hand.

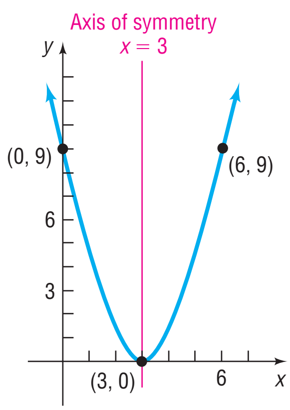
EXAMPLE 4 Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts

Graph $f(x) = x^2 - 6x + 9$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. Determine the domain and the range of f. Determine where f is increasing and where it is decreasing.

Solution For $f(x) = x^2 - 6x + 9$, we have $a = 1$, $b = -6$, and $c = 9$. Since $a = 1 > 0$, the parabola opens up. The x-coordinate of the vertex is

$$h = -\frac{b}{2a} = -\frac{-6}{2(1)} = 3$$

Figure 9



The y-coordinate of the vertex is

$$k = f(3) = (3)^2 - 6(3) + 9 = 0$$

So, the vertex is at $(3, 0)$. The axis of symmetry is the line $x = 3$. The y-intercept is $f(0) = 9$. Since the vertex $(3, 0)$ lies on the x-axis, the graph touches the x-axis at the x-intercept. By using the axis of symmetry and the y-intercept at $(0, 9)$, we can locate the additional point $(6, 9)$ on the graph. See Figure 9.

The domain of f is the set of all real numbers. Based on the graph, the range of f is the interval $[0, \infty)$. The function f is decreasing on the interval $(-\infty, 3)$ and increasing on the interval $(3, \infty)$.

Graph the function in Example 4 by completing the square and using transformations. Which method do you prefer?

NOW WORK PROBLEM 43.

EXAMPLE 5 Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts

Graph $f(x) = 2x^2 + x + 1$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. Determine the domain and the range of f. Determine where f is increasing and where it is decreasing.

Solution For $f(x) = 2x^2 + x + 1$, we have $a = 2$, $b = 1$, and $c = 1$. Since $a = 2 > 0$, the parabola opens up. The x-coordinate of the vertex is

$$h = -\frac{b}{2a} = -\frac{1}{4}$$

The y-coordinate of the vertex is

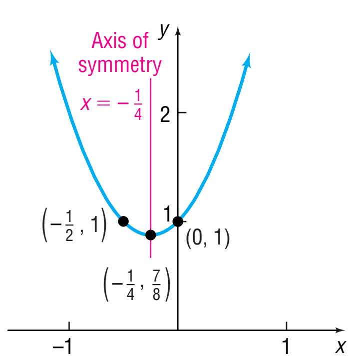
$$k = f\left(-\frac{1}{4}\right) = 2\left(\frac{1}{16}\right) + \left(-\frac{1}{4}\right) + 1 = \frac{7}{8}$$

So, the vertex is at $\left(-\frac{1}{4}, \frac{7}{8}\right)$. The axis of symmetry is the line $x = -\frac{1}{4}$. The y-intercept is $f(0) = 1$. The x-intercept(s), if any, obey the equation $2x^2 + x + 1 = 0$. Since the discriminant $b^2 - 4ac = (1)^2 - 4(2)(1) = -7 < 0$, this equation has no real solutions, and therefore the graph has no x-intercepts. We use the point $(0, 1)$ and the axis of symmetry $x = -\frac{1}{4}$ to locate the additional point $\left(-\frac{1}{2}, 1\right)$ on the graph. See Figure 10.

NOTE
In Example 5, since the vertex is above the x-axis and the parabola opens up, we can conclude the graph of the quadratic function will have no x-intercepts.

The domain of f is the set of all real numbers. Based on the graph, the range of f is the interval $\left[\frac{7}{8}, \infty\right)$. The function f is decreasing on the interval $\left(-\infty, -\frac{1}{4}\right)$ and increasing on the interval $\left(-\frac{1}{4}, \infty\right)$.

Figure 10



NOW WORK PROBLEM 47.

Given the vertex (h, k) and one additional point on the graph of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, we can use

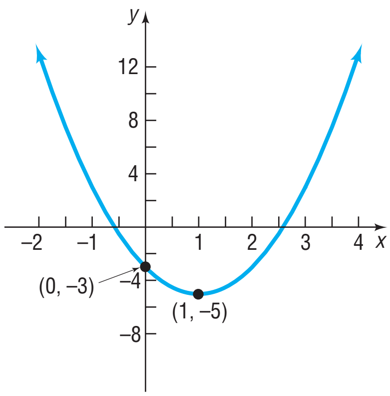
$$\boxed{f(x) = a(x - h)^2 + k \qquad (5)}$$

to obtain the quadratic function.

EXAMPLE 6 Finding the Quadratic Function Given Its Vertex and One Other Point

Determine the quadratic function whose vertex is $(1, -5)$ and whose y-intercept is -3. The graph of the parabola is shown in Figure 11.

Figure 11



Solution The vertex is $(1, -5)$, so $h = 1$ and $k = -5$. Substitute these values into equation (5).

$$f(x) = a(x - h)^2 + k \qquad \text{Equation (5)}$$
$$f(x) = a(x - 1)^2 - 5 \qquad h = 1, k = -5$$

To determine the value of a, we use the fact that $f(0) = -3$ (the y-intercept).

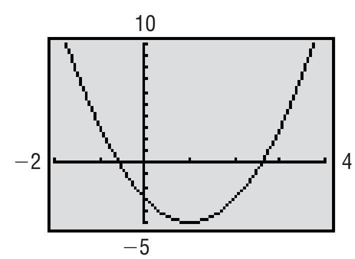
$$\begin{aligned} f(x) &= a(x - 1)^2 - 5 \\ -3 &= a(0 - 1)^2 - 5 && x = 0, y = f(0) = -3 \\ -3 &= a - 5 \\ a &= 2 \end{aligned}$$

The quadratic function whose graph is shown in Figure 11 is

$$f(x) = a(x - h)^2 + k = 2(x - 1)^2 - 5 = 2x^2 - 4x - 3$$

✔ **CHECK:** Figure 12 shows the graph $f(x) = 2x^2 - 4x - 3$ using a graphing utility.

Figure 12



◀

NOW WORK PROBLEM 53.

Summary

Steps for Graphing a Quadratic Function $f(x) = ax^2 + bx + c, a \neq 0$, by Hand.

Option 1

STEP 1: Complete the square in x to write the quadratic function in the form $f(x) = a(x - h)^2 + k$.

STEP 2: Graph the function in stages using transformations.

Option 2

STEP 1: Determine the vertex $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

STEP 2: Determine the axis of symmetry, $x = -\dfrac{b}{2a}$.

STEP 3: Determine the y-intercept, $f(0)$.

STEP 4: (a) If $b^2 - 4ac > 0$, then the graph of the quadratic function has two x-intercepts, which are found by solving the equation $ax^2 + bx + c = 0$.
(b) If $b^2 - 4ac = 0$, the vertex is the x-intercept.
(c) If $b^2 - 4ac < 0$, there are no x-intercepts.

STEP 5: Determine an additional point by using the y-intercept and the axis of symmetry.

STEP 6: Plot the points and draw the graph.

4 Use the Maximum or Minimum Value of a Quadratic Function to Solve Applied Problems

When a mathematical model leads to a quadratic function, the properties of this quadratic function can provide important information about the model. For example, for a quadratic revenue function, we can find the maximum revenue; for a quadratic cost function, we can find the minimum cost.

To see why, recall that the graph of a quadratic function

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

is a parabola with vertex at $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$. This vertex is the highest point on the graph if $a < 0$ and the lowest point on the graph if $a > 0$. If the vertex is the highest point ($a < 0$), then $f\left(-\dfrac{b}{2a}\right)$ is the **maximum value** of f. If the vertex is the lowest point ($a > 0$), then $f\left(-\dfrac{b}{2a}\right)$ is the **minimum value** of f.

This property of the graph of a quadratic function enables us to answer questions involving optimization (finding maximum or minimum values) in models involving quadratic functions.

EXAMPLE 7 Finding the Maximum or Minimum Value of a Quadratic Function

Determine whether the quadratic function

$$f(x) = x^2 - 4x - 5$$

has a maximum or minimum value. Then find the maximum or minimum value.

Solution We compare $f(x) = x^2 - 4x - 5$ to $f(x) = ax^2 + bx + c$. We conclude that $a = 1, b = -4$, and $c = -5$. Since $a > 0$, the graph of f opens up, so the vertex is a minimum point. The minimum value occurs at

$$x = -\frac{b}{2a} \underset{\uparrow}{=} -\frac{-4}{2(1)} = \frac{4}{2} = 2$$

$a = 1, b = -4$

The minimum value is

$$f\left(-\frac{b}{2a}\right) = f(2) = 2^2 - 4(2) - 5 = 4 - 8 - 5 = -9$$

Table 2

X	Y1	
-1	0	
0	-5	
1	-8	
2	-9	
3	-8	
4	-5	
5	0	

$Y_1 = X^2 - 4X - 5$

✔ **CHECK:** We can support the algebraic solution using the TABLE feature on a graphing utility. Create Table 2 by letting $Y_1 = x^2 - 4x - 5$. From the table we see that the smallest value of y occurs when $x = 2$, leading us to investigate the number 2 further. The symmetry about $x = 2$ confirms that 2 is the x-coordinate of the vertex of the parabola. We conclude that the minimum value is -9 and occurs at $x = 2$. ◀

NOW WORK PROBLEM 61.

EXAMPLE 8 Maximizing Revenue

The marketing department at Texas Instruments has found that, when certain calculators are sold at a price of p dollars per unit, the revenue R (in dollars) as a function of the price p is

$$R(p) = -150p^2 + 21{,}000p$$

What unit price should be established to maximize revenue? If this price is charged, what is the maximum revenue?

Solution The revenue R is

$$R(p) = -150p^2 + 21{,}000p \qquad R(p) = ap^2 + bp + c$$

The function R is a quadratic function with $a = -150$, $b = 21{,}000$, and $c = 0$. Because $a < 0$, the vertex is the highest point on the parabola. The revenue R is therefore a maximum when the price p is

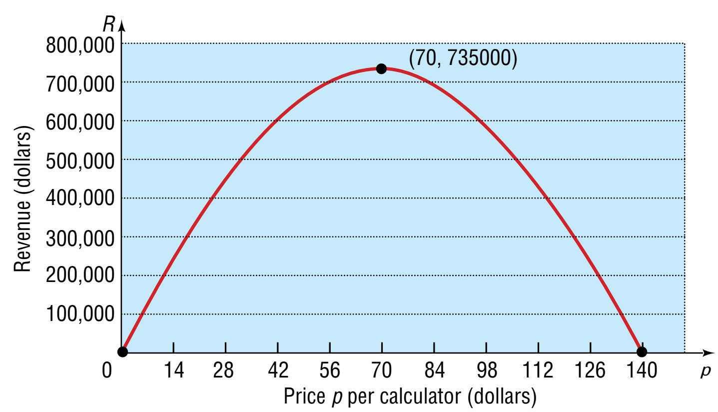
$$p = -\frac{b}{2a} \underset{a=-150,\ b=21{,}000}{=} -\frac{21{,}000}{2(-150)} = -\frac{21{,}000}{-300} = \$70.00$$

The maximum revenue R is

$$R(70) = -150(70)^2 + 21{,}000(70) = \$735{,}000$$

See Figure 13 for an illustration.

Figure 13



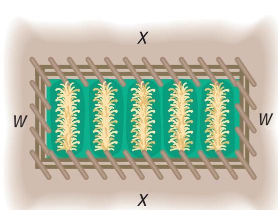
NOW WORK PROBLEM 71.

EXAMPLE 9 Maximizing the Area Enclosed by a Fence

A farmer has 2000 yards of fence to enclose a rectangular field. What are the dimensions of the rectangle that encloses the most area?

Solution Figure 14 illustrates the situation. The available fence represents the perimeter of the rectangle. If x is the length and w is the width, then

Figure 14



$$2x + 2w = 2000 \tag{6}$$

The area A of the rectangle is

$$A = xw$$

To express A in terms of a single variable, we solve equation (6) for w and substitute the result in $A = xw$. Then A involves only the variable x. [You could also solve equation (6) for x and express A in terms of w alone. Try it!]

$$2x + 2w = 2000 \qquad \text{Equation (6)}$$
$$2w = 2000 - 2x \qquad \text{Solve for } w.$$
$$w = \frac{2000 - 2x}{2} = 1000 - x$$

Then the area A is

$$A = xw = x(1000 - x) = -x^2 + 1000x$$

Now, A is a quadratic function of x.

$$A(x) = -x^2 + 1000x \qquad a = -1,\ b = 1000,\ c = 0$$

Figure 15

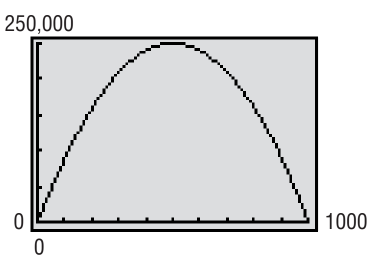


Figure 15 shows a graph of $A(x) = -x^2 + 1000x$ using a graphing utility. Since $a < 0$, the vertex is a maximum point on the graph of A. The maximum value occurs at

$$x = -\frac{b}{2a} = -\frac{1000}{2(-1)} = 500$$

The maximum value of A is

$$A\left(-\frac{b}{2a}\right) = A(500) = -500^2 + 1000(500) = -250{,}000 + 500{,}000 = 250{,}000$$

The largest rectangle that can be enclosed by 2000 yards of fence has an area of 250,000 square yards. Its dimensions are 500 yards by 500 yards. ◀

NOW WORK PROBLEM 77.

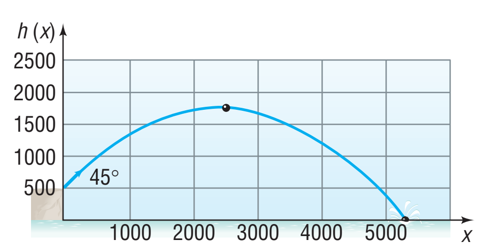
EXAMPLE 10 Analyzing the Motion of a Projectile

A projectile is fired from a cliff 500 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 400 feet per second. In physics, it is established that the height h of the projectile above the water is given by

$$h(x) = \frac{-32x^2}{(400)^2} + x + 500$$

where x is the horizontal distance of the projectile from the base of the cliff. See Figure 16.

Figure 16



(a) Find the maximum height of the projectile.

(b) How far from the base of the cliff will the projectile strike the water?

Solution (a) The height of the projectile is given by a quadratic function.

$$h(x) = \frac{-32x^2}{(400)^2} + x + 500 = \frac{-1}{5000}x^2 + x + 500$$

We are looking for the maximum value of h. Since $a < 0$, the maximum value is obtained at the vertex. We compute

$$x = -\frac{b}{2a} = -\frac{1}{2\left(-\frac{1}{5000}\right)} = \frac{5000}{2} = 2500$$

The maximum height of the projectile is

$$h(2500) = \frac{-1}{5000}(2500)^2 + 2500 + 500 = -1250 + 2500 + 500 = 1750 \text{ ft}$$

(b) The projectile will strike the water when the height is zero. To find the distance x traveled, we need to solve the equation

$$h(x) = \frac{-1}{5000}x^2 + x + 500 = 0$$

We find the discriminant first.

$$b^2 - 4ac = 1^2 - 4\left(\frac{-1}{5000}\right)(500) = 1.4$$

Then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1.4}}{2\left(-\frac{1}{5000}\right)} \approx \begin{cases} -458 \\ 5458 \end{cases}$$

We discard the negative solution and find that the projectile will strike the water at a distance of about 5458 feet from the base of the cliff. ◀

— Seeing the Concept —

Graph

$$h(x) = \frac{-1}{5000}x^2 + x + 500, \quad 0 \le x \le 5500$$

Use MAXIMUM to find the maximum height of the projectile, and use ROOT or ZERO to find the distance from the base of the cliff to where it strikes the water. Compare your results with those obtained in Example 10.

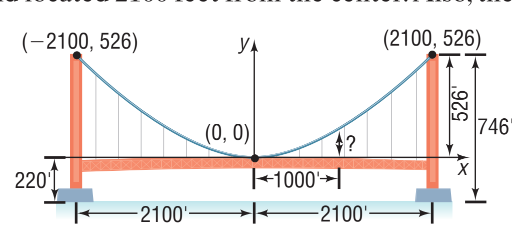
NOW WORK PROBLEM 81.

EXAMPLE 11 The Golden Gate Bridge

The Golden Gate Bridge, a suspension bridge, spans the entrance to San Francisco Bay. Its 746-foot-tall towers are 4200 feet apart. The bridge is suspended from two huge cables more than 3 feet in diameter; the 90-foot-wide roadway is 220 feet above the water. The cables are parabolic in shape* and touch the road surface at the center of the bridge. Find the height of the cable at a distance of 1000 feet from the center.

Solution See Figure 17. We begin by choosing the placement of the coordinate axes so that the x-axis coincides with the road surface and the origin coincides with the center of the bridge. As a result, the twin towers will be vertical (height $746 - 220 = 526$ feet above the road) and located 2100 feet from the center. Also, the cable, which has the

Figure 17



*A cable suspended from two towers is in the shape of a **catenary**, but when a horizontal roadway is suspended from the cable, the cable takes the shape of a parabola.

shape of a parabola, will extend from the towers, open up, and have its vertex at $(0, 0)$. The choice of placement of the axes enables us to identify the equation of the parabola as $y = ax^2$, $a > 0$. We can also see that the points $(-2100, 526)$ and $(2100, 526)$ are on the graph.

Based on these facts, we can find the value of a in $y = ax^2$.

$$y = ax^2$$
$$526 = a(2100)^2 \qquad x = 2100, y = 526$$
$$a = \frac{526}{(2100)^2}$$

The equation of the parabola is therefore

$$y = \frac{526}{(2100)^2}x^2$$

The height of the cable when $x = 1000$ is

$$y = \frac{526}{(2100)^2}(1000)^2 \approx 119.3 \text{ feet}$$

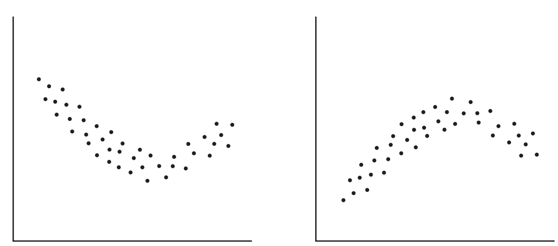
The cable is 119.3 feet high at a distance of 1000 feet from the center of the bridge. ◄

NOW WORK PROBLEM 83.

5 Use a Graphing Utility to Find the Quadratic Function of Best Fit to Data

In Section 2.4, we found the line of best fit for data that appeared to be linearly related. It was noted that data may also follow a nonlinear relation. Figures 18(a) and (b) show scatter diagrams of data that follow a quadratic relation.

Figure 18



$y = ax^2 + bx + c, a > 0$
(a)

$y = ax^2 + bx + c, a < 0$
(b)

EXAMPLE 12 Fitting a Quadratic Function to Data

A farmer collected the data given in Table 3, which shows crop yields Y for various amounts of fertilizer used, x.

(a) With a graphing utility, draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.

(b) Use a graphing utility to find the quadratic function of best fit to these data.

(c) Use the function found in part (b) to determine the optimal amount of fertilizer to apply.

Table 3

Plot	Fertilizer, x (Pounds/100 ft^2)	Yield (Bushels)
1	0	4
2	0	6
3	5	10
4	5	7
5	10	12
6	10	10
7	15	15
8	15	17
9	20	18
10	20	21
11	25	20
12	25	21
13	30	21
14	30	22
15	35	21
16	35	20
17	40	19
18	40	19

(d) Use the function found in part (b) to predict crop yield when the optimal amount of fertilizer is applied.

(e) Draw the quadratic function of best fit on the scatter diagram.

Solution

(a) Figure 19 shows the scatter diagram, from which it appears that the data follow a quadratic relation, with $a < 0$.

(b) Upon executing the QUADratic REGression program, we obtain the results shown in Figure 20. The output that the utility provides shows us the equation $y = ax^2 + bx + c$. The quadratic function of best fit is

$$Y(x) = -0.0171x^2 + 1.0765x + 3.8939$$

where x represents the amount of fertilizer used and Y represents crop yield.

(c) Based on the quadratic function of best fit, the optimal amount of fertilizer to apply is

$$x = -\frac{b}{2a} = -\frac{1.0765}{2(-0.0171)} \approx 31.5 \text{ pounds of fertilizer per 100 square feet}$$

(d) We evaluate the function $Y(x)$ for $x = 31.5$.

$$Y(31.5) = -0.0171(31.5)^2 + 1.0765(31.5) + 3.8939 \approx 20.8 \text{ bushels}$$

If we apply 31.5 pounds of fertilizer per 100 square feet, the crop yield will be 20.8 bushels according to the quadratic function of best fit.

(e) Figure 21 shows the graph of the quadratic function found in part (b) drawn on the scatter diagram.

Figure 19

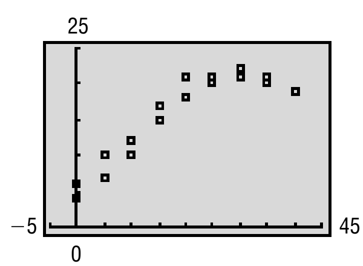


Figure 20

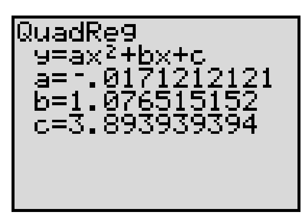
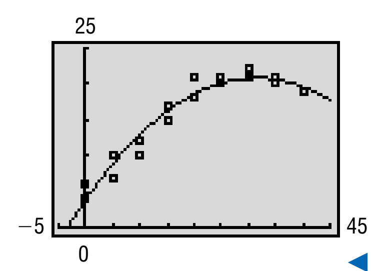


Figure 21



Look again at Figure 20. Notice that the output given by the graphing calculator does not include r, the correlation coefficient. Recall that the correlation coefficient is a measure of the strength of a *linear* relation that exists between two variables. The graphing calculator does not provide an indication of how well the function fits the data in terms of r since a quadratic function cannot be expressed as a linear function.

NOW WORK PROBLEM **101.**

3.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. List the intercepts of the equation $y = x^2 - 9$. (pp. 15–17)

2. Solve the equation $2x^2 + 7x - 4 = 0$. (pp. 988–995)

3. To complete the square of $x^2 - 5x$, you add the number _____. (pp. 991–992)

4. To graph $y = (x - 4)^2$, you shift the graph of $y = x^2$ to the _____ a distance of _____ units. (pp. 118–120)

Concepts and Vocabulary

5. The graph of a quadratic function is called a(n) _____.

6. The vertical line passing through the vertex of a parabola is called the _____.

7. The x-coordinate of the vertex of $f(x) = ax^2 + bx + c$, $a \neq 0$, is _____.

8. *True or False:* The graph of $f(x) = 2x^2 + 3x - 4$ opens up.

9. *True or False:* The y-coordinate of the vertex of $f(x) = -x^2 + 4x + 5$ is $f(2)$.

10. *True or False:* If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$, $a \neq 0$, will touch the x-axis at its vertex.

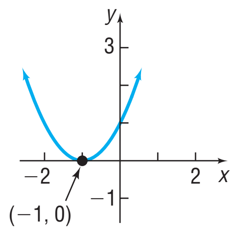
Skill Building

In Problems 11–18, match each graph to one the following functions without using a graphing utility.

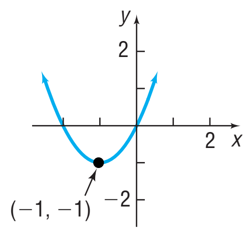
11. $f(x) = x^2 - 1$ **12.** $f(x) = -x^2 - 1$ **13.** $f(x) = x^2 - 2x + 1$ **14.** $f(x) = x^2 + 2x + 1$

15. $f(x) = x^2 - 2x + 2$ **16.** $f(x) = x^2 + 2x$ **17.** $f(x) = x^2 - 2x$ **18.** $f(x) = x^2 + 2x + 2$

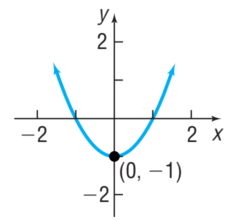
A.



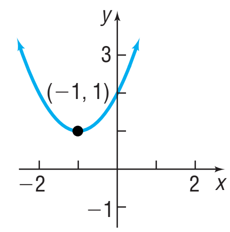
B.



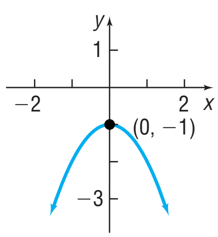
C.



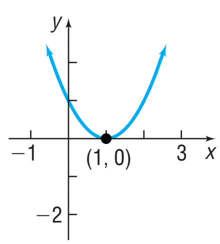
D.



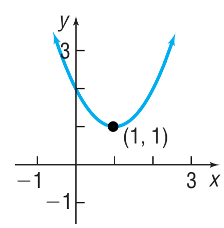
E.



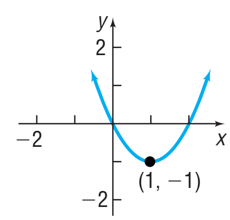
F.



G.



H.



In Problems 19–34, graph the function f by starting with the graph of $y = x^2$ and using transformations (shifting, compressing, stretching, and/or reflection). Verify your results using a graphing utility.
[**Hint:** *If necessary, write f in the form $f(x) = a(x - h)^2 + k$.*]

19. $f(x) = \frac{1}{4}x^2$ **20.** $f(x) = 2x^2$ **21.** $f(x) = \frac{1}{4}x^2 - 2$ **22.** $f(x) = 2x^2 - 3$

23. $f(x) = \frac{1}{4}x^2 + 2$ **24.** $f(x) = 2x^2 + 4$ **25.** $f(x) = \frac{1}{4}x^2 + 1$ **26.** $f(x) = -2x^2 - 2$

27. $f(x) = x^2 + 4x + 2$ **28.** $f(x) = x^2 - 6x - 1$ **29.** $f(x) = 2x^2 - 4x + 1$ **30.** $f(x) = 3x^2 + 6x$

31. $f(x) = -x^2 - 2x$ **32.** $f(x) = -2x^2 + 6x + 2$ **33.** $f(x) = \frac{1}{2}x^2 + x - 1$ **34.** $f(x) = \frac{2}{3}x^2 + \frac{4}{3}x - 1$

In Problems 35–52, graph each quadratic function by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. Determine the domain and the range of the function. Determine where the function is increasing and where it is decreasing. Verify your results using a graphing utility.

35. $f(x) = x^2 + 2x$ **36.** $f(x) = x^2 - 4x$ **37.** $f(x) = -x^2 - 6x$

38. $f(x) = -x^2 + 4x$ **39.** $f(x) = 2x^2 - 8x$ **40.** $f(x) = 3x^2 + 18x$

41. $f(x) = x^2 + 2x - 8$ **42.** $f(x) = x^2 - 2x - 3$ **43.** $f(x) = x^2 + 2x + 1$

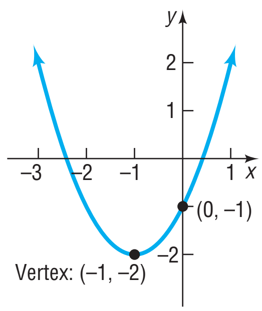
44. $f(x) = x^2 + 6x + 9$ **45.** $f(x) = 2x^2 - x + 2$ **46.** $f(x) = 4x^2 - 2x + 1$

47. $f(x) = -2x^2 + 2x - 3$ **48.** $f(x) = -3x^2 + 3x - 2$ **49.** $f(x) = 3x^2 + 6x + 2$

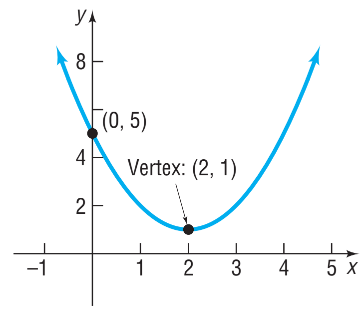
50. $f(x) = 2x^2 + 5x + 3$ **51.** $f(x) = -4x^2 - 6x + 2$ **52.** $f(x) = 3x^2 - 8x + 2$

In Problems 53–58, determine the quadratic function whose graph is given.

53.



54.



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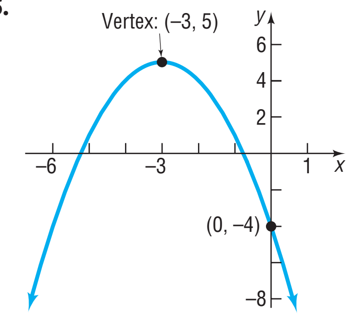

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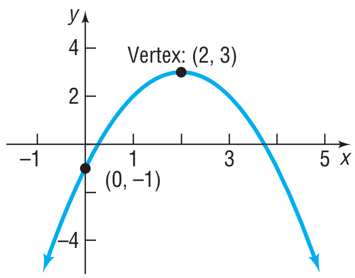
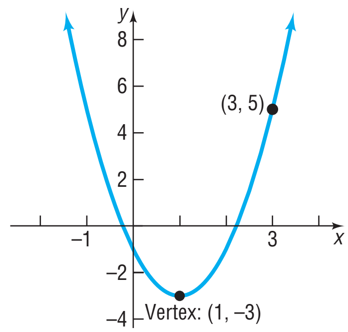
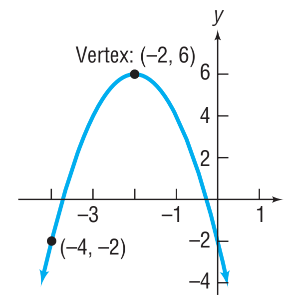


57.



58.



In Problems 59–66, determine, without graphing, whether the given quadratic function has a maximum value or a minimum value and then find the value. Use the TABLE feature on a graphing utility to support your answer.

59. $f(x) = 2x^2 + 12x$ **60.** $f(x) = -2x^2 + 12x$ **61.** $f(x) = 2x^2 + 12x - 3$ **62.** $f(x) = 4x^2 - 8x + 3$

63. $f(x) = -x^2 + 10x - 4$ **64.** $f(x) = -2x^2 + 8x + 3$ **65.** $f(x) = -3x^2 + 12x + 1$ **66.** $f(x) = 4x^2 - 4x$

Applications and Extensions

67. The graph of the function $f(x) = ax^2 + bx + c$ has vertex at $(0, 2)$ and passes through the point $(1, 8)$. Find a, b, and c.

68. The graph of the function $f(x) = ax^2 + bx + c$ has vertex at $(1, 4)$ and passes through the point $(-1, -8)$. Find a, b, and c.

Answer Problems 69 and 70 using the following: A quadratic function of the form $f(x) = ax^2 + bx + c$ with $b^2 - 4ac > 0$ may also be written in the form $f(x) = a(x - r_1)(x - r_2)$, where r_1 and r_2 are the x-intercepts of the graph of the quadratic function.

69. (a) Find a quadratic function whose x-intercepts are -3 and 1 with $a = 1$; $a = 2$; $a = -2$; $a = 5$.
(b) How does the value of a affect the intercepts?
(c) How does the value of a affect the axis of symmetry?
(d) How does the value of a affect the vertex?
(e) Compare the x-coordinate of the vertex with the midpoint of the x-intercepts. What might you conclude?

70. (a) Find a quadratic function whose x-intercepts are -5 and 3 with $a = 1$; $a = 2$; $a = -2$; $a = 5$.
(b) How does the value of a affect the intercepts?
(c) How does the value of a affect the axis of symmetry?
(d) How does the value of a affect the vertex?
(e) Compare the x-coordinate of the vertex with the midpoint of the x-intercepts. What might you conclude?

71. Maximizing Revenue Suppose that the manufacturer of a gas clothes dryer has found that, when the unit price is p dollars, the revenue R (in dollars) is

$$R(p) = -4p^2 + 4000p$$

What unit price for the dryer should be established to maximize revenue? What is the maximum revenue?

72. Maximizing Revenue The John Deere company has found that the revenue from sales of heavy-duty tractors is a function of the unit price p (in dollars) that it charges. If the revenue R is

$$R(p) = -\frac{1}{2}p^2 + 1900p$$

what unit price p (in dollars) should be charged to maximize revenue? What is the maximum revenue?

73. Demand Equation The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{6}x + 100, \qquad 0 \le x \le 600$$

(a) Express the revenue R as a function of x. (Remember, $R = xp$.)
(b) What is the revenue if 200 units are sold?
(c) What quantity x maximizes revenue? What is the maximum revenue?
(d) What price should the company charge to maximize revenue?

74. Demand Equation The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{3}x + 100, \qquad 0 \le x \le 300$$

(a) Express the revenue R as a function of x.
(b) What is the revenue if 100 units are sold?
(c) What quantity x maximizes revenue? What is the maximum revenue?
(d) What price should the company charge to maximize revenue?

75. Demand Equation The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$x = -5p + 100, \qquad 0 \le p \le 20$$

(a) Express the revenue R as a function of x.
(b) What is the revenue if 15 units are sold?
(c) What quantity x maximizes revenue? What is the maximum revenue?
(d) What price should the company charge to maximize revenue?

76. Demand Equation The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

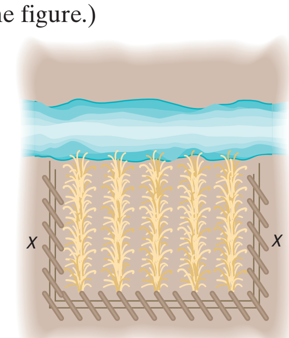
$$x = -20p + 500, \qquad 0 \le p \le 25$$

(a) Express the revenue R as a function of x.
(b) What is the revenue if 20 units are sold?
(c) What quantity x maximizes revenue? What is the maximum revenue?
(d) What price should the company charge to maximize revenue?

77. Enclosing a Rectangular Field David has available 400 yards of fencing and wishes to enclose a rectangular area.
(a) Express the area A of the rectangle as a function of the width w of the rectangle.
(b) For what value of w is the area largest?
(c) What is the maximum area?

78. Enclosing a Rectangular Field Beth has 3000 feet of fencing available to enclose a rectangular field.
(a) Express the area A of the rectangle as a function of x where x is the length of the rectangle.
(b) For what value of x is the area largest?
(c) What is the maximum area?

79. Enclosing the Most Area with a Fence A farmer with 4000 meters of fencing wants to enclose a rectangular plot that borders on a river. If the farmer does not fence the side along the river, what is the largest area that can be enclosed? (See the figure.)



80. Enclosing the Most Area with a Fence A farmer with 2000 meters of fencing wants to enclose a rectangular plot that borders on a straight highway. If the farmer does not fence the side along the highway, what is the largest area that can be enclosed?

81. Analyzing the Motion of a Projectile A projectile is fired from a cliff 200 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 50 feet per second. The height h of the projectile above the water is given by

$$h(x) = \frac{-32x^2}{(50)^2} + x + 200$$

where x is the horizontal distance of the projectile from the base of the cliff.
(a) How far from the base of the cliff is the height of the projectile a maximum?
(b) Find the maximum height of the projectile.
(c) How far from the base of the cliff will the projectile strike the water?
(d) Using a graphing utility, graph the function h, $0 \le x \le 200$.
(e) When the height of the projectile is 100 feet above the water, how far is it from the cliff?

82. Analyzing the Motion of a Projectile A projectile is fired at an inclination of 45° to the horizontal, with a muzzle velocity of 100 feet per second. The height h of the projectile is given by

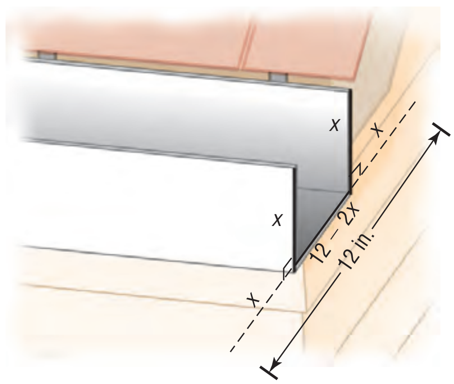
$$h(x) = \frac{-32x^2}{(100)^2} + x$$

where x is the horizontal distance of the projectile from the firing point.
(a) How far from the firing point is the height of the projectile a maximum?
(b) Find the maximum height of the projectile.
(c) How far from the firing point will the projectile strike the ground?
(d) Using a graphing utility, graph the function h, $0 \le x \le 350$.
(e) When the height of the projectile is 50 feet above the ground, how far has it traveled horizontally?

83. Suspension Bridge A suspension bridge with weight uniformly distributed along its length has twin towers that extend 75 meters above the road surface and are 400 meters apart. The cables are parabolic in shape and are suspended from the tops of the towers. The cables touch the road surface at the center of the bridge. Find the height of the cables at a point 100 meters from the center. (Assume that the road is level.)

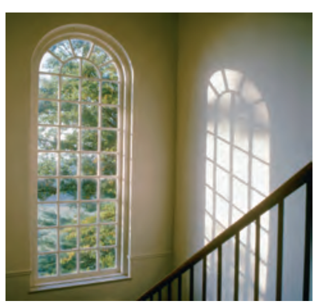
84. Architecture A parabolic arch has a span of 120 feet and a maximum height of 25 feet. Choose suitable rectangular coordinate axes and find the equation of the parabola. Then calculate the height of the arch at points 10 feet, 20 feet, and 40 feet from the center.

85. Constructing Rain Gutters A rain gutter is to be made of aluminum sheets that are 12 inches wide by turning up the edges 90°. What depth will provide maximum cross-sectional area and hence allow the most water to flow?

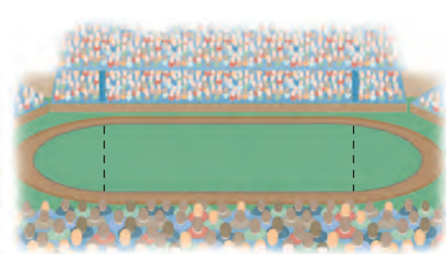


86. Norman Windows A Norman window has the shape of a rectangle surmounted by a semicircle of diameter equal to the width of the rectangle (see the figure). If the perimeter of the window is 20 feet, what dimensions will admit the most light (maximize the area)?

[Hint: Circumference of a circle $= 2\pi r$; area of a circle $= \pi r^2$, where r is the radius of the circle.]

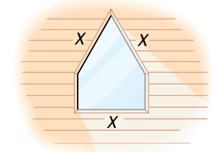


87. Constructing a Stadium A track and field playing area is in the shape of a rectangle with semicircles at each end (see the figure). The inside perimeter of the track is to be 1500 meters. What should the dimensions of the rectangle be so that the area of the rectangle is a maximum?



88. Architecture A special window has the shape of a rectangle surmounted by an equilateral triangle (see the figure). If the perimeter of the window is 16 feet, what dimensions will admit the most light?

[Hint: Area of an equilateral triangle $= \left(\frac{\sqrt{3}}{4}\right)x^2$, where x is the length of a side of the triangle.]



89. Chemical Reactions A self-catalytic chemical reaction results in the formation of a compound that causes the formation ratio to increase. If the reaction rate V is given by

$$V(x) = kx(a - x), \qquad 0 \le x \le a$$

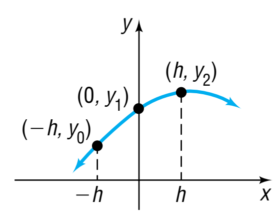
where k is a positive constant, a is the initial amount of the compound, and x is the variable amount of the compound, for what value of x is the reaction rate a maximum?

90. Calculus: Simpson's Rule The figure shows the graph of $y = ax^2 + bx + c$. Suppose that the points $(-h, y_0)$, $(0, y_1)$, and (h, y_2) are on the graph. It can be shown that the area enclosed by the parabola, the x-axis, and the lines $x = -h$ and $x = h$ is

$$\text{Area} = \frac{h}{3}(2ah^2 + 6c)$$

Show that this area may also be given by

$$\text{Area} = \frac{h}{3}(y_0 + 4y_1 + y_2)$$



91. Use the result obtained in Problem 90 to find the area enclosed by $f(x) = -5x^2 + 8$, the x-axis, and the lines $x = -1$ and $x = 1$.

92. Use the result obtained in Problem 90 to find the area enclosed by $f(x) = 2x^2 + 8$, the x-axis, and the lines $x = -2$ and $x = 2$.

93. Use the result obtained in Problem 90 to find the area enclosed by $f(x) = x^2 + 3x + 5$, the x-axis, and the lines $x = -4$ and $x = 4$.

94. Use the result obtained in Problem 90 to find the area enclosed by $f(x) = -x^2 + x + 4$, the x-axis, and the lines $x = -1$ and $x = 1$.

95. A rectangle has one vertex on the line $y = 10 - x, x > 0$, another at the origin, one on the positive x-axis, and one on the positive y-axis. Find the largest area A that can be enclosed by the rectangle.

96. Let $f(x) = ax^2 + bx + c$, where a, b, and c are odd integers. If x is an integer, show that $f(x)$ must be an odd integer.

[Hint: x is either an even integer or an odd integer.]

97. Hunting The function $H(x) = -1.01x^2 + 114.3x + 451.0$ models the number of individuals who engage in hunting activities whose annual income is x thousand dollars.

SOURCE: Based on data obtained from the National Sporting Goods Association.

(a) What is the income level for which there are the most hunters? Approximately how many hunters earn this amount?
(b) Using a graphing utility, graph $H = H(x)$. Are the number of hunters increasing or decreasing for individuals earning between \$20,000 and \$40,000?

98. Advanced Degrees The function $P(x) = -0.008x^2 + 0.868x - 11.884$ models the percentage of the U.S. population in March 2000 whose age is given by x that has earned an advanced degree (more than a bachelor's degree).

SOURCE: Based on data obtained from the U.S. Census Bureau.

(a) What is the age for which the highest percentage of Americans have earned an advanced degree? What is the highest percentage?
(b) Using a graphing utility, graph $P = P(x)$. Is the percentage of Americans that have earned an advanced degree increasing or decreasing for individuals between the ages of 40 and 50?

99. Male Murder Victims The function $M(x) = 0.76x^2 - 107.00x + 3854.18$ models the number of male murder victims who are x years of age $(20 \le x < 90)$.

SOURCE: Based on data obtained from the Federal Bureau of Investigation.

(a) Use the model to approximate the number of male murder victims who are $x = 23$ years of age.
(b) At what age is the number of male murder victims 1456?
(c) Using a graphing utility, graph $M = M(x)$.
(d) Based on the graph drawn in part (c), describe what happens to the number of male murder victims as age increases.

100. Health Care Expenditures The function $H(x) = 0.004x^2 - 0.197x + 5.406$ models the percentage of total income that an individual who is x years of age spends on health care.

SOURCE: Based on data obtained from the Bureau of Labor Statistics.

(a) Use the model to approximate the percentage of total income that an individual who is $x = 45$ years of age spends on health care.
(b) At what age is the percentage of income spent on health care 10%?
(c) Using a graphing utility, graph $H = H(x)$.
(d) Based on the graph drawn in part (c), describe what happens to the percentage of income spent on health care as individuals age.

101. Life Cycle Hypothesis An individual's income varies with his or her age. The following table shows the median income I of individuals of different age groups within the United States for 2001. For each age group, let the class midpoint represent the independent variable, x. For the class "65 years and older," we will assume that the class midpoint is 69.5.

Age	Class Midpoint, x	Median Income, I
15–24 years	19.5	\$9,301
25–34 years	29.5	\$30,510
35–44 years	39.5	\$38,340
45–54 years	49.5	\$41,104
55–64 years	59.5	\$35,637
65 years and older	69.5	\$19,688

SOURCE: U.S. Census Bureau

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Use a graphing utility to find the quadratic function of best fit to these data.
(c) Use the function found in part (b) to determine the age at which an individual can expect to earn the most income.
(d) Use the function found in part (b) to predict the peak income earned.
(e) With a graphing utility, graph the quadratic function of best fit on the scatter diagram.

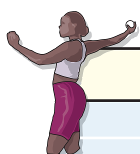
102. Life Cycle Hypothesis An individual's income varies with his or her age. The following table shows the median income I of individuals of different age groups within the United States for 2000. For each age group, the class midpoint represents the independent variable, x. For the age group "65 years and older," we will assume that the class midpoint is 69.5.

Age	Class Midpoint, x	Median Income, I
15–24 years	19.5	\$9,548
25–34 years	29.5	\$30,633
35–44 years	39.5	\$37,088
45–54 years	49.5	\$41,072
55–64 years	59.5	\$34,414
65 years and older	69.5	\$19,167

SOURCE: U.S. Census Bureau

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Use a graphing utility to find the quadratic function of best fit to these data.
(c) Use the function found in part (b) to determine the age at which an individual can expect to earn the most income.
(d) Use the function found in part (b) to predict the peak income earned.
(e) With a graphing utility, graph the quadratic function of best fit on the scatter diagram.
(f) Compare the results of parts (c) and (d) in 2000 to those of 2001 in Problem 101.

103. Height of a Ball A shot-putter throws a ball at an inclination of 45° to the horizontal. The following data represent the height of the ball h at the instant that it has traveled x feet horizontally.



Distance, x	Height, h
20	25
40	40
60	55
80	65
100	71
120	77
140	77
160	75
180	71
200	64

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Use a graphing utility to find the quadratic function of best fit to these data.
(c) Use the function found in part (b) to determine how far the ball will travel before it reaches its maximum height.
(d) Use the function found in part (b) to find the maximum height of the ball.
(e) With a graphing utility, graph the quadratic function of best fit on the scatter diagram.

104. Miles per Gallon An engineer collects the following data showing the speed s of a Ford Taurus and its average miles per gallon, M.

Speed, s	Miles per Gallon, M
30	18
35	20
40	23
40	25
45	25
50	28
55	30
60	29
65	26
65	25
70	25

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Use a graphing utility to find the quadratic function of best fit to these data.
(c) Use the function found in part (b) to determine the speed that maximizes miles per gallon.
(d) Use the function found in part (b) to predict miles per gallon for a speed of 63 miles per hour.
(e) With a graphing utility, graph the quadratic function of best fit on the scatter diagram.

Discussion and Writing

105. Make up a quadratic function that opens down and has only one x-intercept. Compare yours with others in the class. What are the similarities? What are the differences?

106. On one set of coordinate axes, graph the family of parabolas $f(x) = x^2 + 2x + c$ for $c = -3, c = 0$, and $c = 1$. Describe the characteristics of a member of this family.

107. On one set of coordinate axes, graph the family of parabolas $f(x) = x^2 + bx + 1$ for $b = -4, b = 0$, and $b = 4$. Describe the general characteristics of this family.

108. State the circumstances that cause the graph of a quadratic function $f(x) = ax^2 + bx + c$ to have no x-intercepts.

109. Why does the graph of a quadratic function open up if $a > 0$ and down if $a < 0$?

110. Refer to Example 8 on page 159. Notice that if the price charged for the calculators is \$0 or \$140 the revenue is \$0. It is easy to explain why revenue would be \$0 if the price charged is \$0, but how can revenue be \$0 if the price charged is \$140?

'Are You Prepared?' Answers

1. $(0, -9), (-3, 0), (3, 0)$ **2.** $\left\{-4, \frac{1}{2}\right\}$ **3.** $\frac{25}{4}$ **4.** right; 4

3.2 Polynomial Functions and Models

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Polynomials (Appendix, Section A.3, pp. 966–968)
- Using a Graphing Utility to Approximate Local Maxima and Local Minima (Section 2.3, pp. 84–85)
- Graphing Techniques: Transformations (Section 2.6, pp. 118–126)
- Intercepts (Section 1.2, pp. 15–17)

Now work the 'Are You Prepared?' problems on page 182.

OBJECTIVES
1. Identify Polynomial Functions and Their Degree
2. Graph Polynomial Functions Using Transformations
3. Identify the Zeros of a Polynomial Function and Their Multiplicity
4. Analyze the Graph of a Polynomial Function
5. Find the Cubic Function of Best Fit to Data

1 Identify Polynomial Functions and Their Degree

Polynomial functions are among the simplest expressions in algebra. They are easy to evaluate: only addition and repeated multiplication are required. Because of this, they are often used to approximate other, more complicated functions. In this section, we investigate properties of this important class of functions.

A **polynomial function** is a function of the form

$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and n is a nonnegative integer. The domain is the set of all real numbers.

A polynomial function is a function whose rule is given by a polynomial in one variable. The **degree** of a polynomial function is the degree of the polynomial in one variable, that is, the largest power of x that appears.

EXAMPLE 1 Identifying Polynomial Functions

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

(a) $f(x) = 2 - 3x^4$ (b) $g(x) = \sqrt{x}$ (c) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

(d) $F(x) = 0$ (e) $G(x) = 8$ (f) $H(x) = -2x^3(x-1)^2$

Solution

(a) f is a polynomial function of degree 4.

(b) g is not a polynomial function because $g(x) = \sqrt{x} = x^{\frac{1}{2}}$, so the variable x is raised to the $\frac{1}{2}$ power, which is not a nonnegative integer.

(c) h is not a polynomial function. It is the ratio of two polynomials, and the polynomial in the denominator is of positive degree.

(d) F is the zero polynomial function; it is not assigned a degree.

(e) G is a nonzero constant function. It is a polynomial function of degree 0 since $G(x) = 8 = 8x^0$.

(f) $H(x) = -2x^3(x-1)^2 = -2x^3(x^2 - 2x + 1) = -2x^5 + 4x^4 - 2x^3$. So H is a polynomial function of degree 5. Do you see how to find the degree of H without multiplying out? ◀

NOW WORK PROBLEMS 11 AND 15.

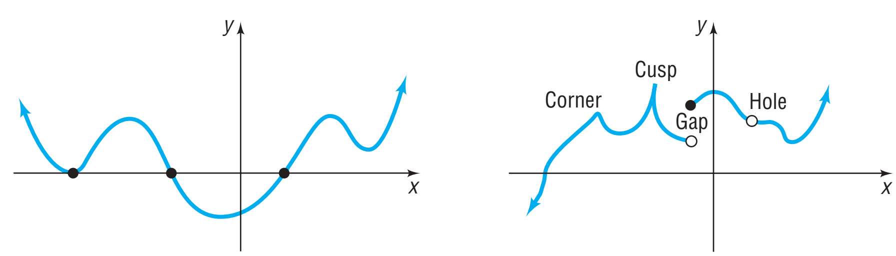
We have already discussed in detail polynomial functions of degrees 0, 1, and 2. See Table 4 for a summary of the properties of the graphs of these polynomial functions.

Table 4

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: Graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$

One objective of this section is to analyze the graph of a polynomial function. If you take a course in calculus, you will learn that the graph of every polynomial function is both smooth and continuous. By **smooth**, we mean that the graph contains no sharp corners or cusps; by **continuous**, we mean that the graph has no gaps or holes and can be drawn without lifting pencil from paper. See Figures 22(a) and (b).

Figure 22



(a) Graph of a polynomial function: smooth, continuous

(b) Cannot be the graph of a polynomial function

2 Graph Polynomial Functions Using Transformations

We begin the analysis of the graph of a polynomial function by discussing *power functions*, a special kind of polynomial function.

In Words
A power function is a function that is defined by a single monomial.

A **power function of degree** n is a function of the form

$$f(x) = ax^n \qquad (2)$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

Examples of power functions are

$f(x) = 3x$ (degree 1) $\quad f(x) = -5x^2$ (degree 2) $\quad f(x) = 8x^3$ (degree 3) $\quad f(x) = -5x^4$ (degree 4)

The graph of a power function of degree 1, $f(x) = ax$, is a straight line, with slope a, that passes through the origin. The graph of a power function of degree 2, $f(x) = ax^2$, is a parabola, with vertex at the origin, that opens up if $a > 0$ and down if $a < 0$.

If we know how to graph a power function of the form $f(x) = x^n$, then a compression or stretch and, perhaps, a reflection about the x-axis will enable us to obtain the graph of $g(x) = ax^n$. Consequently, we shall concentrate on graphing power functions of the form $f(x) = x^n$.

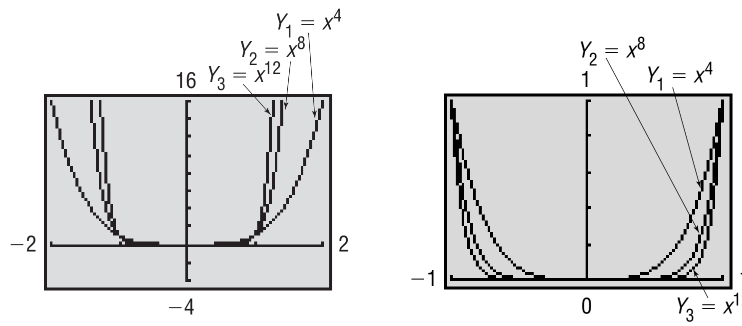
We begin with power functions of even degree of the form $f(x) = x^n$, $n \geq 2$ and n *even*.

Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2$, $-4 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^4$. On the same screen, graph $Y_2 = g(x) = x^8$. Now, also on the same screen, graph $Y_3 = h(x) = x^{12}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1$, $0 \leq y \leq 1$. What do you notice?

Result See Figures 23(a) and (b).

Figure 23



(a) (b)

Table 5

X	Y2	Y3
-1	1	1
1	1	1
.5	.00391	2.4E-4
.1	1E-8	1E-12
.01	1E-16	1E-24
.001	1E-24	1E-36
0	0	0

Y2=X^8

The domain of $f(x) = x^n$, $n \geq 2$ and n even, is the set of all real numbers, and the range is the set of nonnegative real numbers. Such a power function is an even function (do you see why?), so its graph is symmetric with respect to the y-axis. Its graph always contains the origin $(0, 0)$ and the points $(-1, 1)$ and $(1, 1)$.

For large n, it appears that the graph coincides with the x-axis near the origin, but it does not; the graph actually touches the x-axis only at the origin. See Table 5, where $Y_2 = x^8$ and $Y_3 = x^{12}$. For x close to 0, the values of y are positive and close to 0. Also, for large n, it may appear that for $x < -1$ or for $x > 1$ the graph is vertical, but it is not; it is only increasing very rapidly. If you TRACE along one of the graphs, these distinctions will be clear.

NOTE

Don't forget how graphing calculators express scientific notation. In Table 5, $-1E-8$ means -1×10^{-8}.

To summarize:

Properties of Power Functions, $f(x) = x^n$, n Is an Even Integer

1. The graph is symmetric with respect to the y-axis, so f is even.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

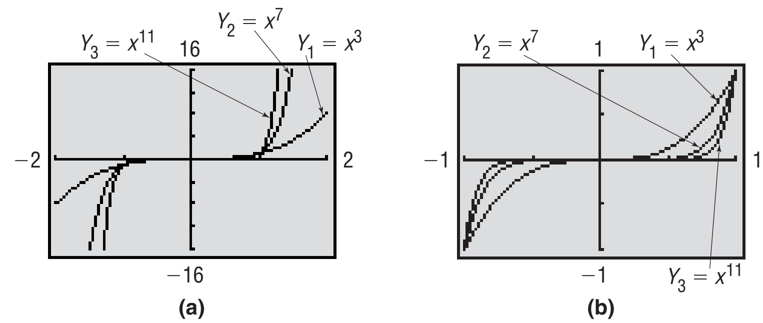
Now we consider power functions of odd degree of the form $f(x) = x^n$, n *odd*.

Exploration

Using your graphing utility and the viewing window $-2 \le x \le 2$, $-16 \le y \le 16$, graph the function $Y_1 = f(x) = x^3$. On the same screen, graph $Y_2 = g(x) = x^7$ and $Y_3 = h(x) = x^{11}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \le x \le 1$, $-1 \le y \le 1$. What do you notice?

Result The graphs on your screen should look like Figures 24(a) and (b).

Figure 24



The domain and the range of $f(x) = x^n$, $n \ge 3$ and n odd, are the set of real numbers. Such a power function is an odd function (do you see why?), so its graph is symmetric with respect to the origin. Its graph always contains the origin (0, 0) and the points $(-1, -1)$ and (1, 1).

It appears that the graph coincides with the x-axis near the origin, but it does not; the graph actually crosses the x-axis only at the origin. Also, it appears that as x increases the graph is vertical, but it is not; it is increasing very rapidly. TRACE along the graphs to verify these distinctions.

To summarize:

Properties of Power Functions, $f(x) = x^n$, n Is an Odd Integer

1. The graph is symmetric with respect to the origin, so f is odd.
2. The domain and the range are the set of all real numbers.
3. The graph always contains the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$, but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

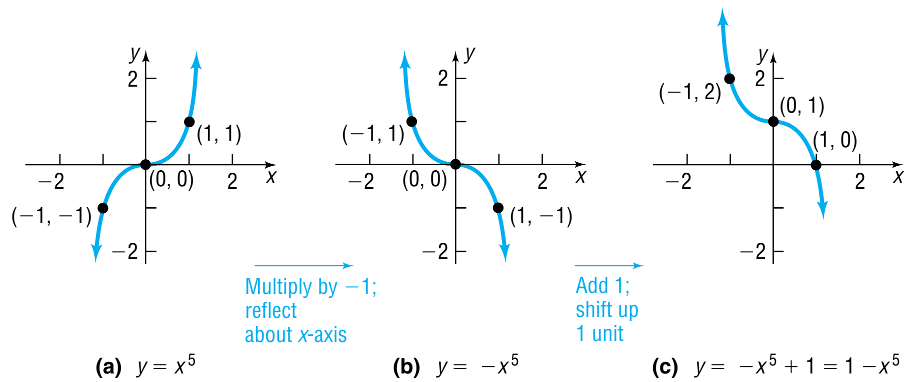
The methods of shifting, compression, stretching, and reflection studied in Section 2.6, when used with the facts just presented, will enable us to graph polynomial functions that are transformations of power functions.

EXAMPLE 2 Graphing Polynomial Functions Using Transformations

Graph: $f(x) = 1 - x^5$

Solution Figure 25 shows the required stages.

Figure 25



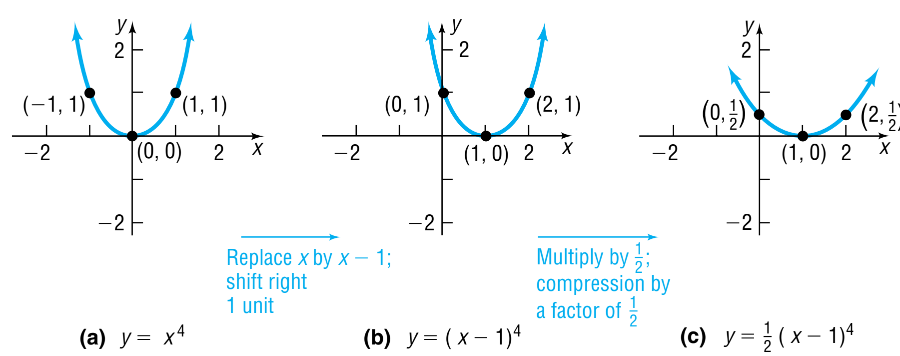
✔ **CHECK:** Verify the result of Example 2 by graphing $Y_1 = f(x) = 1 - x^5$. ◀

EXAMPLE 3 Graphing Polynomial Functions Using Transformations

Graph: $f(x) = \frac{1}{2}(x - 1)^4$

Solution Figure 26 shows the required stages.

Figure 26



✔ **CHECK:** Verify the result of Example 3 by graphing

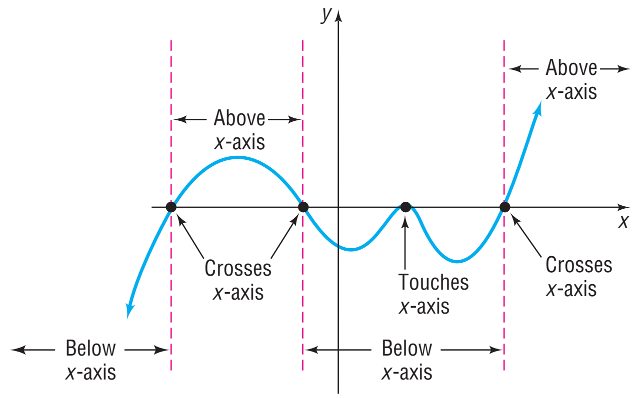
$$Y_1 = f(x) = \frac{1}{2}(x - 1)^4$$ ◀

NOW WORK PROBLEMS 23 AND 27.

3 Identify the Zeros of a Polynomial Function and Their Multiplicity

Figure 27 shows the graph of a polynomial function with four x-intercepts. Notice that at the x-intercepts the graph must either cross the x-axis or touch the x-axis. Consequently, between consecutive x-intercepts the graph is either above the x-axis or below the x-axis. We will make use of this property of the graph of a polynomial shortly.

Figure 27



If a polynomial function f is factored completely, it is easy to solve the equation $f(x) = 0$ and locate the x-intercepts of the graph. For example, if $f(x) = (x - 1)^2(x + 3)$, then the solutions of the equation

$$f(x) = (x - 1)^2(x + 3) = 0$$

are identified as 1 and -3. Based on this result, we make the following observations:

If f is a polynomial function and r is a real number for which $f(r) = 0$, then r is called a (real) **zero of f**, or **root of f**. If r is a (real) zero of f, then

(a) r is an x-intercept of the graph of f.

(b) $(x - r)$ is a factor of f.

So the real zeros of a function are the x-intercepts of its graph, and they are found by solving the equation $f(x) = 0$.

EXAMPLE 4 Finding a Polynomial from Its Zeros

(a) Find a polynomial of degree 3 whose zeros are -3, 2, and 5.

(b) Graph the polynomial found in part (a) to verify your result.

Solution

(a) If r is a zero of a polynomial f, then $x - r$ is a factor of f. This means that $x - (-3) = x + 3$, $x - 2$, and $x - 5$ are factors of f. As a result, any polynomial of the form

$$f(x) = a(x + 3)(x - 2)(x - 5)$$

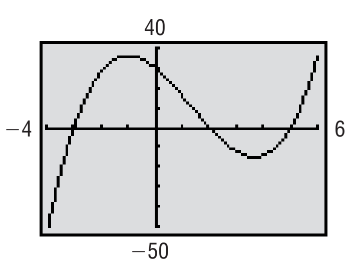
where a is any nonzero real number, qualifies.

(b) The value of a causes a stretch, compression, or reflection, but does not affect the x-intercepts. We choose to graph f with $a = 1$.

$$f(x) = (x + 3)(x - 2)(x - 5) = x^3 - 4x^2 - 11x + 30$$

Figure 28 shows the graph of f. Notice that the x-intercepts are -3, 2, and 5. ◀

Figure 28



— Seeing the Concept —

Graph the function found in Example 4 for $a = 2$ and $a = -1$. Does the value of a affect the zeros of f? How does the value of a affect the graph of f?

NOW WORK PROBLEM 37.

If the same factor $x - r$ occurs more than once, then r is called a **repeated**, or **multiple, zero of f**. More precisely, we have the following definition.

If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity m of f**.

EXAMPLE 5 Identifying Zeros and Their Multiplicities

For the polynomial

$$f(x) = 5(x - 2)(x + 3)^2\left(x - \frac{1}{2}\right)^4$$

2 is a zero of multiplicity 1 because the exponent on the factor $x - 2$ is 1.

-3 is a zero of multiplicity 2 because the exponent on the factor $x + 3$ is 2.

$\frac{1}{2}$ is a zero of multiplicity 4 because the exponent on the factor $x - \frac{1}{2}$ is 4. ◀

NOW WORK PROBLEM 45(a).

In Example 5 notice that, if you add the multiplicities $(1 + 2 + 4 = 7)$, you obtain the degree of the polynomial.

Suppose that it is possible to factor completely a polynomial function and, as a result, locate all the x-intercepts of its graph (the real zeros of the function). The following example illustrates the role that the multiplicity of an x-intercept plays.

EXAMPLE 6 Investigating the Role of Multiplicity

For the polynomial $f(x) = x^2(x - 2)$:

(a) Find the x- and y-intercepts of the graph of f.

(b) Using a graphing utility, graph the polynomial.

(c) For each x-intercept, determine whether it is of odd or even multiplicity.

Solution

(a) The y-intercept is $f(0) = 0^2(0 - 2) = 0$. The x-intercepts satisfy the equation

$$f(x) = x^2(x - 2) = 0$$

from which we find that

$$x^2 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 0 \quad \text{or} \quad x = 2$$

The x-intercepts are 0 and 2.

(b) See Figure 29 for the graph of f.

(c) We can see from the factored form of f that 0 is a zero or root of multiplicity 2, and 2 is a zero or root of multiplicity 1; so 0 is of even multiplicity and 2 is of odd multiplicity. ◀

Figure 29

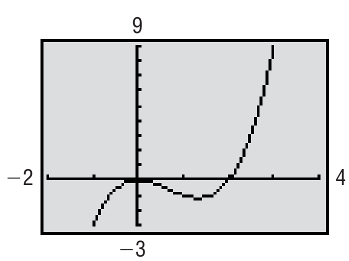


Table 6

X	Y1
-2	-16
-1	-3
0	0
1	-1
2	0
3	9
4	32

Y1≡X²(X−2)

We can use a TABLE to further analyze the graph. See Table 6. The sign of $f(x)$ is the same on each side of $x = 0$ and the graph of f just touches the x-axis at $x = 0$ (a zero of even multiplicity). The sign of $f(x)$ changes from one side of $x = 2$ to the other and the graph of f crosses the x-axis at $x = 2$ (a zero of odd multiplicity). These observations suggest the following result:

If r Is a Zero of Even Multiplicity

Sign of $f(x)$ does not change from one side of r to the other side of r.	Graph **touches** x-axis at r.

If r Is a Zero of Odd Multiplicity

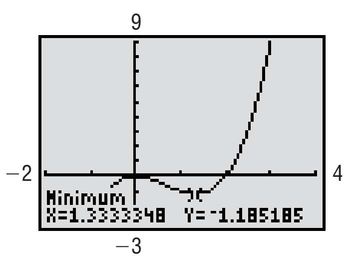
Sign of $f(x)$ changes from one side of r to the other side of r.	Graph **crosses** x-axis at r.

NOW WORK PROBLEM 45(b).

Points on the graph where the graph changes from an increasing function to a decreasing function, or vice versa, are called **turning points**.*

Look at Figure 30. The graph of $f(x) = x^2(x - 2) = x^3 - 2x^2$, has a turning point at $(0, 0)$. After utilizing MINIMUM, we find that the graph also has a turning point at $(1.33, -1.19)$, rounded to two decimal places.

Figure 30



Exploration

Graph $Y_1 = x^3$, $Y_2 = x^3 - x$, and $Y_3 = x^3 + 3x^2 + 4$. How many turning points do you see? How does the number of turning points relate to the degree? Graph $Y_1 = x^4$, $Y_2 = x^4 - \frac{4}{3}x^3$, and $Y_3 = x^4 - 2x^2$. How many turning points do you see? How does the number of turning points compare to the degree?

The following theorem from calculus supplies the answer.

Theorem

If f is a polynomial function of degree n, then f has at most $n - 1$ turning points.

One last remark about Figure 30. Notice that the graph of $f(x) = x^2(x - 2)$ looks somewhat like the graph of $y = x^3$. In fact, for very large values of x, either positive or negative, there is little difference.

Exploration

For each pair of functions Y_1 and Y_2 given in parts (a), (b), and (c), graph Y_1 and Y_2 on the same viewing window. Create a TABLE or TRACE for large positive and large negative values of x. What do you notice about the graphs of Y_1 and Y_2 as x becomes very large and positive or very large and negative?

(a) $Y_1 = x^2(x - 2)$; $Y_2 = x^3$

(b) $Y_1 = x^4 - 3x^3 + 7x - 3$; $Y_2 = x^4$

(c) $Y_1 = -2x^3 + 4x^2 - 8x + 10$; $Y_2 = -2x^3$

*Graphing utilities can be used to approximate turning points. Calculus is needed to find the exact location of turning points.

The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its **end behavior**.

Theorem

End Behavior

For large values of x, either positive or negative, that is, for large $|x|$, the graph of the polynomial

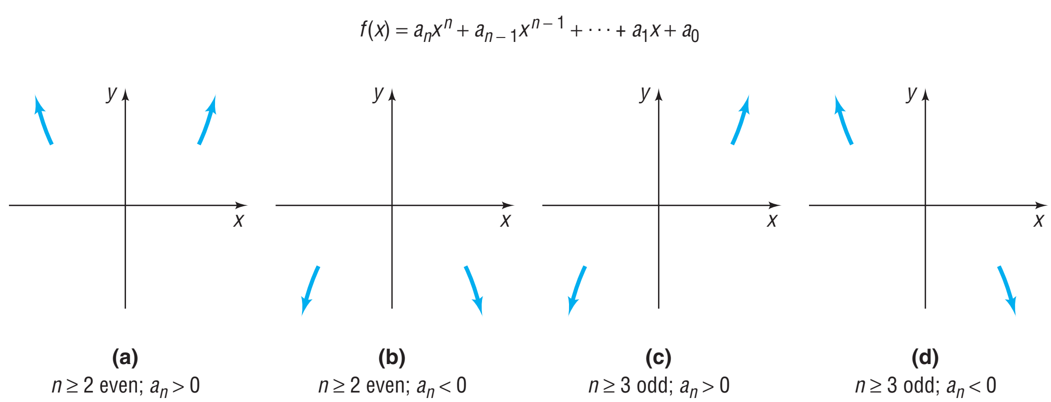
$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}$$

resembles the graph of the power function

$$\boxed{y = a_n x^n}$$

Look back at Figures 23 and 24. Based on this theorem and the previous discussion on power functions, the end behavior of a polynomial can be of only four types. See Figure 31.

Figure 31
End Behavior



For example, consider the polynomial function $f(x) = -2x^4 + x^3 + 4x^2 - 7x + 1$. The graph of f will resemble the graph of the power function $y = -2x^4$ for large $|x|$. The graph of f will look like Figure 31(b) for large $|x|$.

NOW WORK PROBLEM 45(c).

Summary

Graph of a Polynomial Function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $\quad a_n \neq 0$

Degree of the polynomial f: n

Maximum number of turning points: $n - 1$

At a zero of even multiplicity: The graph of f touches the x-axis.

At a zero of odd multiplicity: The graph of f crosses the x-axis.

Between zeros, the graph of f is either above or below the x-axis.

End behavior: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

4 Analyze the Graph of a Polynomial Function

EXAMPLE 7 Analyzing the Graph of a Polynomial Function

For the polynomial: $f(x) = x^4 - 2x^3 - 8x^2$

(a) Find the degree of the polynomial. Determine the end behavior; that is, find the power function that the graph of f resembles for large values of $|x|$.

(b) Find the x- and y-intercepts of the graph of f.

(c) Determine whether the graph crosses or touches the x-axis at each x-intercept.

(d) Use a graphing utility to graph f.

(e) Determine the number of turning points on the graph of f. Approximate the turning points, if any exist, rounded to two decimal places.

(f) Use the information obtained in parts (a) to (e) to draw a complete graph of f by hand.

(g) Find the domain of f. Use the graph to find the range of f.

(h) Use the graph to determine where f is increasing and where f is decreasing.

Solution

(a) The polynomial is of degree 4. End behavior: the graph of f resembles that of the power function $y = x^4$ for large values of $|x|$.

(b) The y-intercept is $f(0) = 0$. To find the x-intercepts, if any, we first factor f.

$$f(x) = x^4 - 2x^3 - 8x^2 = x^2(x^2 - 2x - 8) = x^2(x - 4)(x + 2)$$

We find the x-intercepts by solving the equation

$$f(x) = x^2(x - 4)(x + 2) = 0$$

So

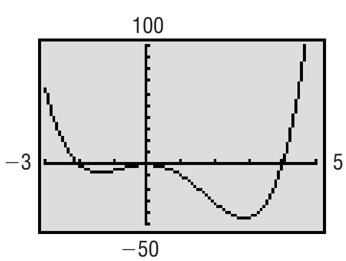
$$\begin{array}{ccccc} x^2 = 0 & \text{or} & x - 4 = 0 & \text{or} & x + 2 = 0 \\ x = 0 & \text{or} & x = 4 & \text{or} & x = -2 \end{array}$$

The x-intercepts are -2, 0, and 4.

(c) The x-intercept 0 is a zero of multiplicity 2, so the graph of f will touch the x-axis at 0; -2 and 4 are zeros of multiplicity 1, so the graph of f will cross the x-axis at -2 and 4.

(d) See Figure 32 for the graph of f.

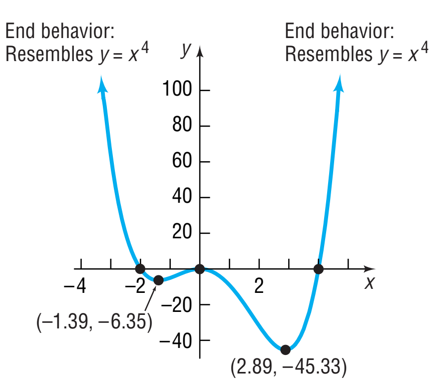
Figure 32



(e) From the graph of f shown in Figure 32, we can see that f has three turning points. Using MAXIMUM, one turning point is at $(0, 0)$. Using MINIMUM, the two remaining turning points are at $(-1.39, -6.35)$ and $(2.89, -45.33)$, rounded to two decimal places.

(f) Figure 33 shows a graph of f using the information obtained in parts (a) to (e).

Figure 33



(g) The domain of f is the set of all real numbers. The range of f is the interval $[-45.33, \infty)$.

(h) Based on the graph, f is decreasing on the intervals $(-\infty, -1.39)$ and $(0, 2.89)$ and is increasing on the intervals $(-1.39, 0)$ and $(2.89, \infty)$.

NOW WORK PROBLEM 69.

For polynomial functions that have noninteger coefficients and for polynomials that are not easily factored, we utilize the graphing utility early in the analysis of the graph. This is because the amount of information that can be obtained from algebraic analysis is limited.

EXAMPLE 8

Using a Graphing Utility to Analyze the Graph of a Polynomial Function

For the polynomial $f(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$:

(a) Find the degree of the polynomial. Determine the end behavior: that is, find the power function that the graph of f resembles for large values of $|x|$.

(b) Graph f using a graphing utility.

(c) Find the x- and y-intercepts of the graph.

(d) Use a TABLE to find points on the graph around each x-intercept.

(e) Determine the local maxima and local minima, if any exist, rounded to two decimal places. That is, locate any turning points.

(f) Use the information obtained in parts (a)–(e) to draw a complete graph of f by hand. Be sure to label the intercepts, turning points, and the points obtained in part (d).

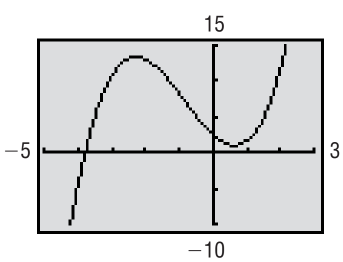
(g) Find the domain of f. Use the graph to find the range of f.

(h) Use the graph to determine where f is increasing and where f is decreasing.

Solution

(a) The degree of the polynomial is 3. End behavior: the graph of f resembles that of the power function $y = x^3$ for large values of $|x|$.

Figure 34



(b) See Figure 34 for the graph of f.

(c) The y-intercept is $f(0) = 2.484406$. In Example 7 we could easily factor $f(x)$ to find the x-intercepts. However, it is not readily apparent how $f(x)$ factors in this example. Therefore, we use a graphing utility's ZERO (or ROOT) feature and find the lone x-intercept to be -3.79, rounded to two decimal places.

Table 7

X	Y1
-4	-4.574
-2	13.035

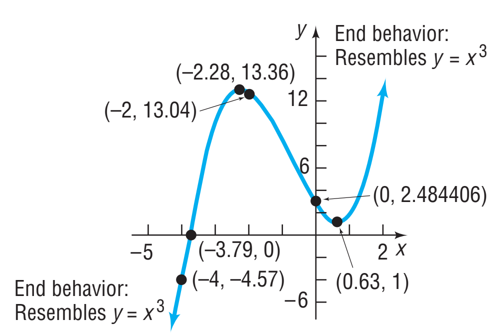
Y1=X^3+2.48X²-4...

(d) Table 7 shows values of x around the x-intercept. The points $(-4, -4.57)$ and $(-2, 13.04)$ are on the graph.

(e) From the graph we see that it has two turning points: one between -3 and -2 the other between 0 and 1. Rounded to two decimal places, the local maximum is 13.36 and occurs at $x = -2.28$; the local minimum is 1 and occurs at $x = 0.63$. The turning points are $(-2.28, 13.36)$ and $(0.63, 1)$.

(f) Figure 35 shows a graph of f drawn by hand using the information obtained in parts (a) to (e).

Figure 35



(g) The domain and the range of f are the set of all real numbers.

(h) Based on the graph, f is decreasing on the interval $(-2.28, 0.63)$ and is increasing on the intervals $(-\infty, -2.28)$ and $(0.63, \infty)$.

NOW WORK PROBLEM 81.

Summary

Steps for Graphing a Polynomial by Hand

To analyze the graph of a polynomial function $y = f(x)$, follow these steps:

STEP 1: End behavior: find the power function that the graph of f resembles for large values of x.

STEP 2: (a) Find the x-intercepts, if any, by solving the equation $f(x) = 0$.
(b) Find the y-intercept by letting $x = 0$ and finding the value of $f(0)$.

STEP 3: Determine whether the graph of f crosses or touches the x-axis at each x-intercept.

STEP 4: Use a graphing utility to graph f. Determine the number of turning points on the graph of f. Approximate any turning points rounded to two decimal places.

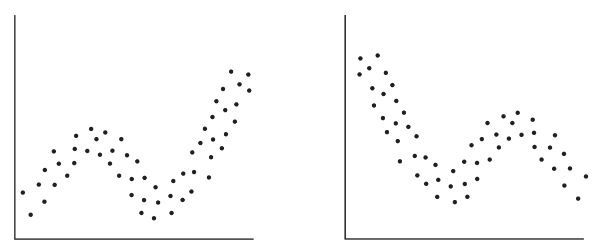
STEP 5: Use the information obtained in Steps 1 to 4 to draw a complete graph of f by hand.

5 Find the Cubic Function of Best Fit to Data

In Section 2.4 we found the line of best fit from data and in Section 3.1 we found the quadratic function of best fit. It is also possible to find polynomial functions of best fit. However, most statisticians do not recommend finding polynomials of best fit of degree higher than 3.

Data that follow a cubic relation should look like Figure 36(a) or (b).

Figure 36



$y = ax^3 + bx^2 + cx + d,\ a > 0$
(a)

$y = ax^3 + bx^2 + cx + d,\ a < 0$
(b)

EXAMPLE 9 A Cubic Function of Best Fit

The data in Table 8 represent the average miles per gallon of vans, pickups, and sport utility vehicles (SUVs) in the United States for 1991–2001, where 1 represents 1991, 2 represents 1992, and so on.

(a) Draw a scatter diagram of the data using the year x as the independent variable and average miles per gallon M as the dependent variable. Comment on the type of relation that may exist between the two variables x and M.

(b) Using a graphing utility, find the cubic function of best fit $M = M(x)$ to these data.

(c) Graph the cubic function of best fit on your scatter diagram.

(d) Use the function found in part (b) to predict the average miles per gallon in 2002 $(x = 12)$.

Table 8

Year, x	Average Miles per Gallon, M
1991, 1	17.0
1992, 2	17.3
1993, 3	17.4
1994, 4	17.3
1995, 5	17.3
1996, 6	17.2
1997, 7	17.2
1998, 8	17.2
1999, 9	17.0
2000, 10	17.4
2001, 11	17.6

SOURCE: United States Federal Highway Administration

Solution (a) Figure 37 shows the scatter diagram. A cubic relation may exist between the two variables.

(b) Upon executing the CUBIC REGression program, we obtain the results shown in Figure 38. The output the utility provides shows us the equation $y = ax^3 + bx^2 + cx + d$. The cubic function of best fit to the data is $M(x) = 0.0058x^3 - 0.1007x^2 + 0.4950x + 16.6258$.

(c) Figure 39 shows the graph of the cubic function of best fit on the scatter diagram. The function fits the data reasonably well.

Figure 37

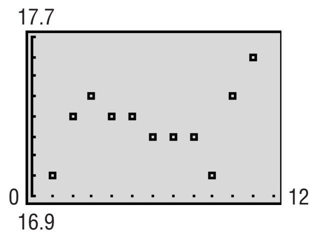


Figure 38

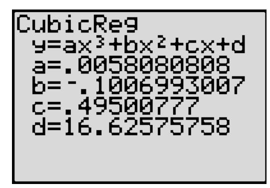
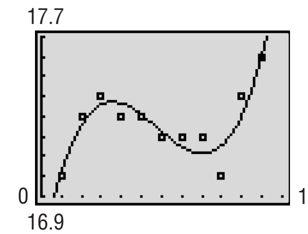


Figure 39



(d) We evaluate the function $M(x)$ at $x = 12$.

$$M(12) = 0.0058(12)^3 - 0.1007(12)^2 + 0.4950(12) + 16.6258 \approx 18.1$$

The model predicts that the average miles per gallon of vans, pickups, and SUVs will be 18.1 miles per gallon in 2002. ◀

3.2 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The intercepts of the equation $9x^2 + 4y = 36$ are _____. (pp. 15–17)
2. *True or False:* The expression $4x^3 - 3.6x^2 - \sqrt{2}$ is a polynomial. (pp. 966–968)
3. To graph $y = x^2 - 4$, you would shift the graph of $y = x^2$ _____ a distance of _____ units. (pp. 118–120)
4. Use a graphing utility to approximate (rounded to two decimal places) the local maxima and local minima of $f(x) = x^3 - 2x^2 - 4x + 5$, for $-10 < x < 10$ (pp. 84–85)

Concepts and Vocabulary

5. The graph of every polynomial function is both _____ and _____.
6. A number r for which $f(r) = 0$ is called a(n) _____ of the function f.
7. If r is a zero of even multiplicity for a function f, the graph of f _____ the x-axis at r.
8. *True or False:* The graph of $f(x) = x^2(x - 3)(x + 4)$ has exactly three x-intercepts.
9. *True or False:* The x-intercepts of the graph of a polynomial function are called turning points.
10. *True or False:* End behavior: the graph of the function $f(x) = 3x^4 - 6x^2 + 2x + 5$ resembles $y = x^4$ for large values of $|x|$.

Skill Building

In Problems 11–22, determine which functions are polynomial functions. For those that are, state the degree. For those that are not, tell why not.

11. $f(x) = 4x + x^3$
12. $f(x) = 5x^2 + 4x^4$
13. $g(x) = \dfrac{1 - x^2}{2}$
14. $h(x) = 3 - \dfrac{1}{2}x$
15. $f(x) = 1 - \dfrac{1}{x}$
16. $f(x) = x(x - 1)$

17. $g(x) = x^{3/2} - x^2 + 2$ **18.** $h(x) = \sqrt{x}(\sqrt{x} - 1)$ **19.** $F(x) = 5x^4 - \pi x^3 + \frac{1}{2}$

20. $F(x) = \frac{x^2 - 5}{x^3}$ **21.** $G(x) = 2(x - 1)^2(x^2 + 1)$ **22.** $G(x) = -3x^2(x + 2)^3$

In Problems 23–36, use transformations of the graph of $y = x^4$ or $y = x^5$ to graph each function.

23. $f(x) = (x + 1)^4$ **24.** $f(x) = (x - 2)^5$ **25.** $f(x) = x^5 - 3$ **26.** $f(x) = x^4 + 2$

27. $f(x) = \frac{1}{2}x^4$ **28.** $f(x) = 3x^5$ **29.** $f(x) = -x^5$ **30.** $f(x) = -x^4$

31. $f(x) = (x - 1)^5 + 2$ **32.** $f(x) = (x + 2)^4 - 3$ **33.** $f(x) = 2(x + 1)^4 + 1$ **34.** $f(x) = \frac{1}{2}(x - 1)^5 - 2$

35. $f(x) = 4 - (x - 2)^5$ **36.** $f(x) = 3 - (x + 2)^4$

In Problems 37–44, form a polynomial whose zeros and degree are given.

37. Zeros: $-1, 1, 3$; degree 3 **38.** Zeros: $-2, 2, 3$; degree 3 **39.** Zeros: $-3, 0, 4$; degree 3

40. Zeros: $-4, 0, 2$; degree 3 **41.** Zeros: $-4, -1, 2, 3$; degree 4 **42.** Zeros: $-3, -1, 2, 5$; degree 4

43. Zeros: -1, multiplicity 1; 3, multiplicity 2; degree 3 **44.** Zeros: -2, multiplicity 2; 4, multiplicity 1; degree 3

In Problems 45–56, for each polynomial function: (a) List each real zero and its multiplicity; (b) Determine whether the graph crosses or touches the x-axis at each x-intercept. (c) Find the power function that the graph of f resembles for large values of $|x|$.

45. $f(x) = 3(x - 7)(x + 3)^2$ **46.** $f(x) = 4(x + 4)(x + 3)^3$ **47.** $f(x) = 4(x^2 + 1)(x - 2)^3$

48. $f(x) = 2(x - 3)(x + 4)^3$ **49.** $f(x) = -2\left(x + \frac{1}{2}\right)^2(x^2 + 4)^2$ **50.** $f(x) = \left(x - \frac{1}{3}\right)^2(x - 1)^3$

51. $f(x) = (x - 5)^3(x + 4)^2$ **52.** $f(x) = \left(x + \sqrt{3}\right)^2(x - 2)^4$ **53.** $f(x) = 3(x^2 + 8)(x^2 + 9)^2$

54. $f(x) = -2(x^2 + 3)^3$ **55.** $f(x) = -2x^2(x^2 - 2)$ **56.** $f(x) = 4x(x^2 - 3)$

In Problems 57–80, for each polynomial function f:

(a) *Find the degree of the polynomial. Determine the end behavior: that is, find the power function that the graph of f resembles for large values of $|x|$.*
(b) *Find the x- and y-intercepts of the graph.*
(c) *Determine whether the graph crosses or touches the x-axis at each x-intercept.*
(d) *Graph f using a graphing utility.*
(e) *Determine the local maxima and local minima, if any exist, rounded to two decimal places. That is, locate any turning points.*
(f) *Use the information obtained in parts (a)–(e) to draw a complete graph of f by hand. Be sure to label the intercepts and turning points.*
(g) *Find the domain of f. Use the graph to find the range of f.*
(h) *Use the graph to determine where f is increasing and where f is decreasing.*

57. $f(x) = (x - 1)^2$ **58.** $f(x) = (x - 2)^3$ **59.** $f(x) = x^2(x - 3)$

60. $f(x) = x(x + 2)^2$ **61.** $f(x) = (x + 4)(x - 2)^2$ **62.** $f(x) = (x - 1)(x + 3)^2$

63. $f(x) = -2(x + 2)(x - 2)^3$ **64.** $f(x) = -\frac{1}{2}(x + 4)(x - 1)^3$ **65.** $f(x) = (x + 1)(x - 2)(x + 4)$

66. $f(x) = (x - 1)(x + 4)(x - 3)$ **67.** $f(x) = 4x - x^3$ **68.** $f(x) = x - x^3$

69. $f(x) = x^2(x - 2)(x + 2)$ **70.** $f(x) = x^2(x - 3)(x + 4)$ **71.** $f(x) = (x + 1)^2(x - 2)^2$

72. $f(x) = (x + 1)^3(x - 3)$ **73.** $f(x) = x^2(x - 3)(x + 1)$ **74.** $f(x) = x^2(x - 3)(x - 1)$

75. $f(x) = (x + 2)^2(x - 4)^2$ **76.** $f(x) = (x - 2)^2(x + 2)(x + 4)$ **77.** $f(x) = x^2(x - 2)(x^2 + 3)$

78. $f(x) = x^2(x^2 + 1)(x + 4)$ **79.** $f(x) = -x^2(x^2 - 1)(x + 1)$ **80.** $f(x) = -x^2(x^2 - 4)(x - 5)$

In Problems 81–90, for each polynomial function f:

(a) *Find the degree of the polynomial. Determine the end behavior: that is, find the power function that the graph of f resembles for large values of $|x|$.*
(b) *Graph f using a graphing utility.*
(c) *Find the x- and y-intercepts of the graph.*
(d) *Use a TABLE to find points on the graph around each x-intercept.*
(e) *Determine the local maxima and local minima, if any exist, rounded to two decimal places. That is, locate any turning points.*
(f) *Use the information obtained in parts (a)–(e) to draw a complete graph of f by hand. Be sure to label the intercepts, turning points, and the points obtained in part (d).*
(g) *Find the domain of f. Use the graph to find the range of f.*
(h) *Use the graph to determine where f is increasing and where f is decreasing.*

81. $f(x) = x^3 + 0.2x^2 - 1.5876x - 0.31752$

82. $f(x) = x^3 - 0.8x^2 - 4.6656x + 3.73248$

83. $f(x) = x^3 + 2.56x^2 - 3.31x + 0.89$

84. $f(x) = x^3 - 2.91x^2 - 7.668x - 3.8151$

85. $f(x) = x^4 - 2.5x^2 + 0.5625$

86. $f(x) = x^4 - 18.5x^2 + 50.2619$

87. $f(x) = 2x^4 - \pi x^3 + \sqrt{5}x - 4$

88. $f(x) = -1.2x^4 + 0.5x^2 - \sqrt{3}x + 2$

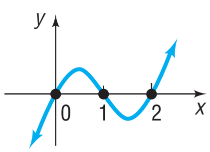
89. $f(x) = -2x^5 - \sqrt{2}x^2 - x - \sqrt{2}$

90. $f(x) = \pi x^5 + \pi x^4 + \sqrt{3}x + 1$

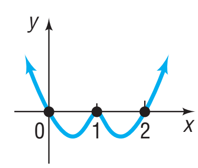
Applications and Extensions

In Problems 91–94, construct a polynomial function that might have this graph. (More than one answer may be possible.)

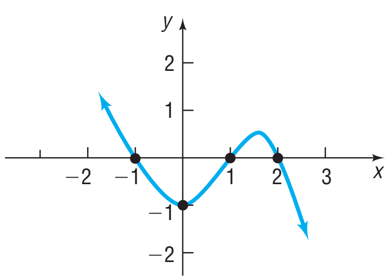
91.



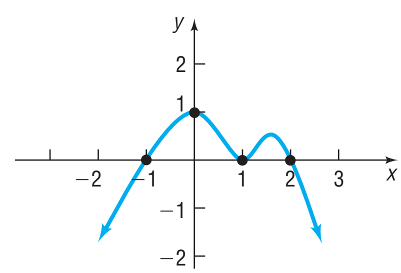
92.



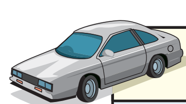
93.



94.



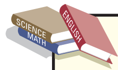
95. Cost of Manufacturing The following data represent the cost C (in thousands of dollars) of manufacturing Chevy Cavaliers and the number x of Cavaliers produced.



Number of Cavaliers Produced, x	Cost, C
0	10
1	23
2	31
3	38
4	43
5	50
6	59
7	70
8	85
9	105
10	135

(a) Draw a scatter diagram of the data using x as the independent variable and C as the dependent variable. Comment on the type of relation that may exist between the two variables C and x.
(b) Find the average rate of change of cost from four to five Cavaliers.
(c) What is the average rate of change in cost from eight to nine Cavaliers?
(d) Use a graphing utility to find the cubic function of best fit $C = C(x)$.
(e) Graph the cubic function of best fit on the scatter diagram.
(f) Use the function found in part (d) to predict the cost of manufacturing 11 Cavaliers.
(g) Interpret the y-intercept.

96. Cost of Printing The following data represent the weekly cost C (in thousands of dollars) of printing textbooks and the number x (in thousands of units) of texts printed.



Number of Text Books, x	Cost, C
0	100
5	128.1
10	144
13	153.5
17	161.2
18	162.6
20	166.3
23	178.9
25	190.2
27	221.8

(a) Draw a scatter diagram of the data using x as the independent variable and C as the dependent variable. Comment on the type of relation that may exist between the two variables C and x.
(b) Find the average rate of change in cost from 10,000 to 13,000 textbooks.
(c) What is the average rate of change in the cost of producing from 18,000 to 20,000 textbooks?
(d) Use a graphing utility to find the cubic function of best fit $C = C(x)$.
(e) Graph the cubic function of best fit on the scatter diagram.
(f) Use the function found in part (d) to predict the cost of printing 22,000 texts per week.
(g) Interpret the y-intercept.

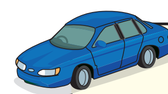
97. Motor Vehicle Thefts The following data represent the number T (in thousands) of motor vehicle thefts in the United States for the years 1989–2001, where $x = 1$ represents 1989, $x = 2$ represents 1990, and so on.

Year, x	Motor Vehicle Thefts, T
1989, 1	1565
1990, 2	1636
1991, 3	1662
1992, 4	1611
1993, 5	1563
1994, 6	1539
1995, 7	1472
1996, 8	1394
1997, 9	1354
1998, 10	1243
1999, 11	1152
2000, 12	1160
2001, 13	1226

SOURCE: U.S. Federal Bureau of Investigation

(a) Draw a scatter diagram of the data using x as the independent variable and T as the dependent variable. Comment on the type of relation that may exist between the two variables T and x.
(b) Use a graphing utility to find the cubic function of best fit $T = T(x)$.
(c) Graph the cubic function of best fit on the scatter diagram.
(d) Use the function found in part (b) to predict the number of motor vehicle thefts in 2005.
(e) Do you think the function found in part (b) will be useful in predicting motor vehicle thefts in 2010? Why?

98. Average Annual Miles The following data represent the average number of miles driven (in thousands) annually by vans, pickups, and sport utility vehicles (SUVs) for the years 1993–2001, where $x = 1$ represents 1993, $x = 2$ represents 1994, and so on.



Year, x	Average Miles Driven, M
1993, 1	12.4
1994, 2	12.2
1995, 3	12.0
1996, 4	11.8
1997, 5	12.1
1998, 6	12.2
1999, 7	12.0
2000, 8	11.7
2001, 9	11.1

SOURCE: United States Federal Highway Administration

(a) Draw a scatter diagram of the data using x as the independent variable and M as the dependent variable. Comment on the type of relation that may exist between the two variables M and x.
(b) Use a graphing utility to find the cubic function of best fit $M = M(x)$.
(c) Graph the cubic function of best fit on the scatter diagram.
(d) Use the function found in part (b) to predict the average miles driven by vans, pickups, and SUVs in 2005.
(e) Do you think the function found in part (b) will be useful in predicting the average miles driven by vans, pickups, and SUVs in 2010? Why?

Discussion and Writing

99. Can the graph of a polynomial function have no y-intercept? Can it have no x-intercepts? Explain.

100. Write a few paragraphs that provide a general strategy for graphing a polynomial function. Be sure to mention the following: degree, intercepts, end behavior, and turning points.

101. Make up a polynomial that has the following characteristics: crosses the x-axis at -1 and 4, touches the x-axis at 0 and 2, and is above the x-axis between 0 and 2. Give your polynomial to a fellow classmate and ask for a written critique of your polynomial.

102. Make up two polynomials, not of the same degree, with the following characteristics: crosses the x-axis at -2, touches the x-axis at 1, and is above the x-axis between -2 and 1. Give your polynomials to a fellow classmate and ask for a written critique of your polynomials.

103. The graph of a polynomial function is always smooth and continuous. Name a function studied earlier that is smooth and not continuous. Name one that is continuous, but not smooth.

104. Which of the following statements are true regarding the graph of the cubic polynomial $f(x) = x^3 + bx^2 + cx + d$? (Give reasons for your conclusions.)

(a) It intersects the y-axis in one and only one point.
(b) It intersects the x-axis in at most three points.
(c) It intersects the x-axis at least once.
(d) For $|x|$ very large, it behaves like the graph of $y = x^3$.
(e) It is symmetric with respect to the origin.
(f) It passes through the origin.

'Are You Prepared?' Answers

1. $(-2, 0), (2, 0), (0, 9)$ **2.** True **3.** down; 4 **4.** Local maximum of 6.48 at $x = -0.67$; local minimum of -3 at $x = 2$

3.3 Properties of Rational Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Rational Expressions (Appendix, Section A.3, pp. 971–975)
- Polynomial Division (Appendix, Section A.4, pp. 977–979)
- Graph of $f(x) = \frac{1}{x}$ (Section 1.2, Example 13, p. 21)
- Graphing Techniques: Transformations (Section 2.6, pp. 118–126)

Now work the 'Are You Prepared?' problems on page 195.

OBJECTIVES
1 Find the Domain of a Rational Function
2 Find the Vertical Asymptotes of a Rational Function
3 Find the Horizontal or Oblique Asymptotes of a Rational Function

Ratios of integers are called *rational numbers*. Similarly, ratios of polynomial functions are called *rational functions*.

A **rational function** is a function of the form

$$\boxed{R(x) = \frac{p(x)}{q(x)}}$$

where p and q are polynomial functions and q is not the zero polynomial. The domain is the set of all real numbers except those for which the denominator q is 0.

1 Find the Domain of a Rational Function

EXAMPLE 1 Finding the Domain of a Rational Function

(a) The domain of $R(x) = \dfrac{2x^2 - 4}{x + 5}$ is the set of all real numbers x except -5; that is, $\{x | x \neq -5\}$.

(b) The domain of $R(x) = \dfrac{1}{x^2 - 4}$ is the set of all real numbers x except -2 and 2, that is, $\{x | x \neq -2, x \neq 2\}$.

(c) The domain of $R(x) = \dfrac{x^3}{x^2 + 1}$ is the set of all real numbers.

(d) The domain of $R(x) = \dfrac{-x^2 + 2}{3}$ is the set of all real numbers.

(e) The domain of $R(x) = \dfrac{x^2 - 1}{x - 1}$ is the set of all real numbers x except 1, that is, $\{x | x \neq 1\}$. ◀

It is important to observe that the functions

$$R(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad f(x) = x + 1$$

are not equal, since the domain of R is $\{x | x \neq 1\}$ and the domain of f is the set of all real numbers.

NOW WORK PROBLEM 13.

If $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function and if p and q have no common factors, then the rational function R is said to be in **lowest terms**. For a rational function $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms, the zeros, if any, of the numerator are the x-intercepts of the graph of R and so will play a major role in the graph of R. The zeros of the denominator of R [that is, the numbers x, if any, for which $q(x) = 0$], although not in the domain of R, also play a major role in the graph of R. We will discuss this role shortly.

We have already discussed the properties of the rational function $f(x) = \dfrac{1}{x}$. (Refer to Example 13, page 21.) The next rational function that we take up is $H(x) = \dfrac{1}{x^2}$.

EXAMPLE 2 Graphing $y = \dfrac{1}{x^2}$

Analyze the graph of $H(x) = \dfrac{1}{x^2}$.

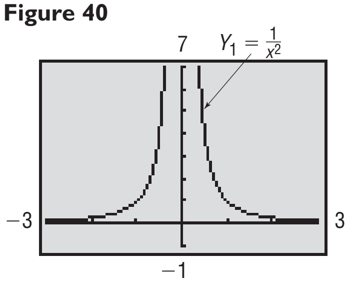
Solution The domain of $H(x) = \dfrac{1}{x^2}$ consists of all real numbers x except 0. The graph has no y-intercept, because x can never equal 0. The graph has no x-intercept because the equation $H(x) = 0$ has no solution. Therefore, the graph of H will not cross either coordinate axis.

Because

$$H(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = H(x)$$

H is an even function, so its graph is symmetric with respect to the y-axis.

Figure 40



See Figure 40. Notice that the graph confirms the conclusions just reached. But what happens to the graph as the values of x get closer and closer to 0? We use a TABLE to answer the question. See Table 9. The first four rows show that as x approaches 0 the values of $H(x)$ become larger and larger positive numbers. When this happens, we say that $H(x)$ is **unbounded in the positive direction**. We symbolize this by writing $H(x) \to \infty$ [read as "$H(x)$ **approaches infinity**"]. In calculus, **limits** are used to convey these ideas. There we use the symbolism $\lim_{x \to 0} H(x) = \infty$, read as "the limit of $H(x)$ as x approaches 0 is infinity" to mean that $H(x) \to \infty$ as $x \to 0$.

Table 9

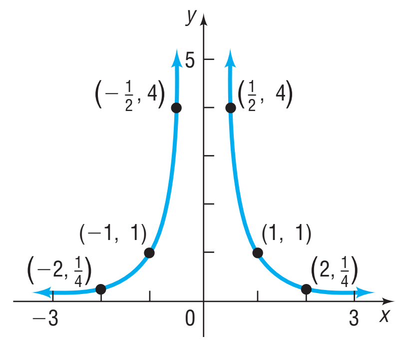
X	Y1
.1	100
.01	10000
.001	1E6
1E-4	1E8
10	.01
100	1E-4
1000	1E-6

Y1 = 1/X²

Look at the last three rows of Table 9. As $x \to \infty$, the values of $H(x)$ approach 0 (the end behavior of the graph). This is symbolized in calculus by writing $\lim_{x \to \infty} H(x) = 0$.

Figure 41 shows the graph of $H(x) = \frac{1}{x^2}$ drawn by hand.

Figure 41



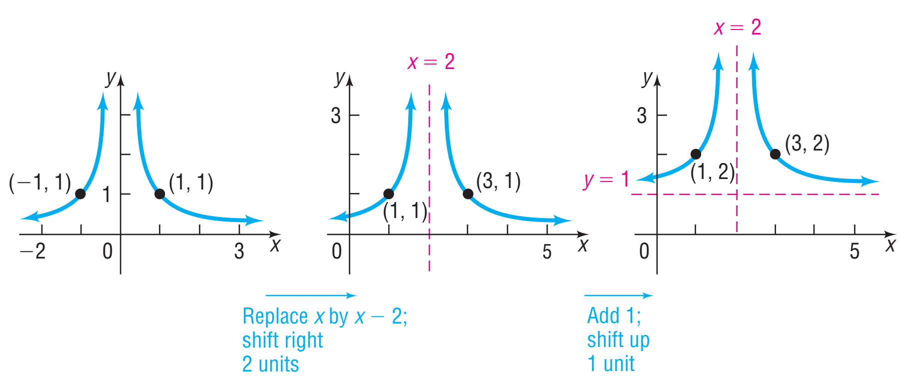
Sometimes transformations (shifting, compressing, stretching, and reflection) can be used to graph a rational function.

EXAMPLE 3 Using Transformations to Graph a Rational Function

Graph the rational function: $R(x) = \dfrac{1}{(x-2)^2} + 1$

Solution First, notice that the domain of R is the set of all real numbers except $x = 2$. To graph R, we start with the graph of $y = \frac{1}{x^2}$. See Figure 42 for the stages.

Figure 42



(a) $y = \frac{1}{x^2}$ (b) $y = \frac{1}{(x-2)^2}$ (c) $y = \frac{1}{(x-2)^2} + 1$

✔ **CHECK:** Graph $Y_1 = \dfrac{1}{(x-2)^2} + 1$ using a graphing utility to verify the graph obtained in Figure 42(c). ◀

NOW WORK PROBLEM 31.

Asymptotes

Notice that the y-axis in Figure 42(a) is transformed into the vertical line $x = 2$ in Figure 42(c), and the x-axis in Figure 42(a) is transformed into the horizontal line $y = 1$ in Figure 42(c). The **Exploration** that follows will help us analyze the role of these lines.

Exploration

(a) Using a graphing utility and the TABLE feature, evaluate the function $H(x) = \dfrac{1}{(x-2)^2} + 1$ at $x = 10, 100, 1000,$ and $10{,}000$. What happens to the values of H as x becomes unbounded in the positive direction, symbolized by $\lim_{x \to \infty} H(x)$?

(b) Evaluate H at $x = -10, -100, -1000,$ and $-10{,}000$. What happens to the values of H as x becomes unbounded in the negative direction, symbolized by $\lim_{x \to -\infty} H(x)$?

(c) Evaluate H at $x = 1.5, 1.9, 1.99, 1.999,$ and 1.9999. What happens to the values of H as x approaches 2, $x < 2$, symbolized by $\lim_{x \to 2^-} H(x)$?

(d) Evaluate H at $x = 2.5, 2.1, 2.01, 2.001,$ and 2.0001. What happens to the values of H as x approaches 2, $x > 2$, symbolized by $\lim_{x \to 2^+} H(x)$?

Result

(a) Table 10 shows the values of $Y_1 = H(x)$ as x approaches ∞. Notice that the values of H are approaching 1, so $\lim_{x \to \infty} H(x) = 1$.

(b) Table 11 shows the values of $Y_1 = H(x)$ as x approaches $-\infty$. Again the values of H are approaching 1, so $\lim_{x \to -\infty} H(x) = 1$.

(c) From Table 12 we see that, as x approaches 2, $x < 2$, the values of H are increasing without bound, so $\lim_{x \to 2^-} H(x) = \infty$.

(d) Finally, Table 13 reveals that, as x approaches 2, $x > 2$, the values of H are increasing without bound, so $\lim_{x \to 2^+} H(x) = \infty$.

Table 10

X	Y1
10	1.0156
100	1.0001
1000	1
10000	1

$Y_1 = 1/(X-2)^2+1$

Table 11

X	Y1
-10	1.0069
-100	1.0001
-1000	1
-10000	1

$Y_1 = 1/(X-2)^2+1$

Table 12

X	Y1
1.5	5
1.9	101
1.99	10001
1.999	1E6
1.9999	1E8

$Y_1 = 1/(X-2)^2+1$

Table 13

X	Y1
2.5	5
2.1	101
2.01	10001
2.001	1E6
2.0001	1E8

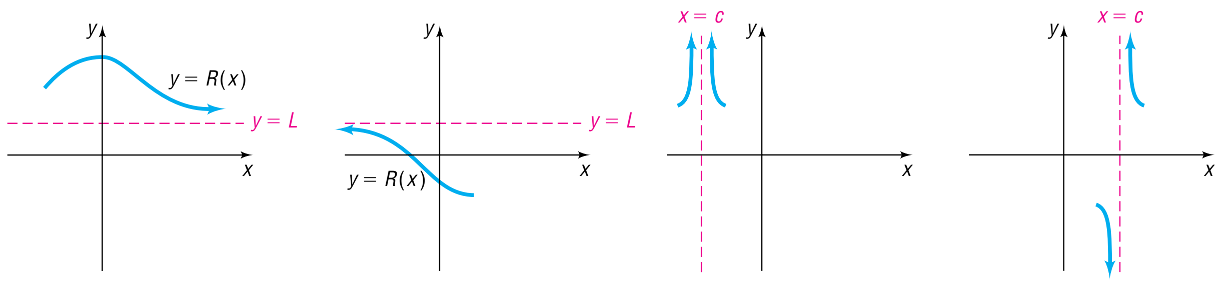
$Y_1 = 1/(X-2)^2+1$

The results of the Exploration reveal an important property of rational functions. The vertical line $x = 2$ and the horizontal line $y = 1$ are called *asymptotes* of the graph of H, which we define as follows:

Let R denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.

Even though the asymptotes of a function are not part of the graph of the function, they provide information about how the graph looks. Figure 43 illustrates some of the possibilities.

Figure 43



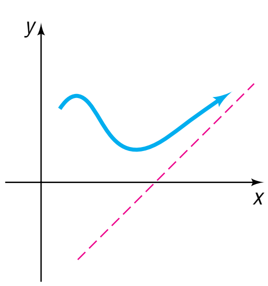
(a) End behavior: As $x \to \infty$, the values of $R(x)$ approach L [$\lim_{x\to\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(b) End behavior: As $x \to -\infty$, the values of $R(x)$ approach L [$\lim_{x\to-\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(c) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x\to c^-} R(x) = \infty$; $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

(d) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x\to c^-} R(x) = -\infty$; $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Figure 44
Oblique asymptote



A horizontal asymptote, when it occurs, describes a certain behavior of the graph as $x \to \infty$ or as $x \to -\infty$, that is, its end behavior. **The graph of a function may intersect a horizontal asymptote**.

A vertical asymptote, when it occurs, describes a certain behavior of the graph when x is close to some number c. **The graph of the function will never intersect a vertical asymptote**.

If an asymptote is neither horizontal nor vertical, it is called **oblique**. Figure 44 shows an oblique asymptote. An oblique asymptote, when it occurs, describes the end behavior of the graph. **The graph of a function may intersect an oblique asymptote**.

2 Find the Vertical Asymptotes of a Rational Function

The vertical asymptotes, if any, of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, are located at the zeros of the denominator of $q(x)$. Suppose that r is a zero, so $x - r$ is a factor. Now, as x approaches r, symbolized as $x \to r$, the values of $x - r$ approach 0, causing the ratio to become unbounded, that is, causing $|R(x)| \to \infty$. Based on the definition, we conclude that the line $x = r$ is a vertical asymptote.

Theorem

Locating Vertical Asymptotes

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, *in lowest terms*, will have a vertical asymptote $x = r$ if r is a real zero of the *denominator* q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, then R will have the vertical asymptote $x = r$.

WARNING
If a rational function is not in lowest terms, an application of this theorem will result in an incorrect listing of vertical asymptotes.

EXAMPLE 4 **Finding Vertical Asymptotes**

Find the vertical asymptotes, if any, of the graph of each rational function.

(a) $R(x) = \dfrac{x}{x^2 - 4}$ (b) $F(x) = \dfrac{x + 3}{x - 1}$

(c) $H(x) = \dfrac{x^2}{x^2 + 1}$ (d) $G(x) = \dfrac{x^2 - 9}{x^2 + 4x - 21}$

Solution (a) R is in lowest terms and the zeros of the denominator $x^2 - 4$ are -2 and 2. Hence, the lines $x = -2$ and $x = 2$ are the vertical asymptotes of the graph of R.

(b) F is in lowest terms and the only zero of the denominator is 1. The line $x = 1$ is the only vertical asymptote of the graph of F.

(c) H is in lowest terms and the denominator has no real zeros because the discriminant of $x^2 + 1$ is $b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot 1 = -4$. The graph of H has no vertical asymptotes.

(d) Factor $G(x)$ to determine if it is in lowest terms.

$$G(x) = \frac{x^2 - 9}{x^2 + 4x - 21} = \frac{(x + 3)(x - 3)}{(x + 7)(x - 3)} = \frac{x + 3}{x + 7}, \qquad x \neq 3$$

The only zero of the denominator of $G(x)$ in lowest terms is -7. Hence, the line $x = -7$ is the only vertical asymptote of the graph of G. ◀

WARNING
In Example 4(a), the vertical asymptotes are $x = -2$ and $x = 2$. Do not say that the vertical asymptotes are -2 and 2. ■

As Example 4 points out, rational functions can have no vertical asymptotes, one vertical asymptote, or more than one vertical asymptote. However, the graph of a rational function will never intersect any of its vertical asymptotes. (Do you know why?)

Exploration

Graph each of the following rational functions:

$$R(x) = \frac{1}{x - 1} \quad R(x) = \frac{1}{(x - 1)^2} \quad R(x) = \frac{1}{(x - 1)^3} \quad R(x) = \frac{1}{(x - 1)^4}$$

Each has the vertical asymptote $x = 1$. What happens to the value of $R(x)$ as x approaches 1 from the right side of the vertical asymptote; that is, what is $\lim_{x \to 1^+} R(x)$? What happens to the value of $R(x)$ as x approaches 1 from the left side of the vertical asymptote; that is, what is $\lim_{x \to 1^-} R(x)$? How does the multiplicity of the zero in the denominator affect the graph of R?

NOW WORK PROBLEM 45. **(Find the vertical asymptotes, if any.)**

3 Find the Horizontal or Oblique Asymptotes of a Rational Function

The procedure for finding horizontal and oblique asymptotes is somewhat more involved. To find such asymptotes, we need to know how the values of a function behave as $x \to -\infty$ or as $x \to \infty$.

If a rational function $R(x)$ is **proper**, that is, if the degree of the numerator is less than the degree of the denominator, then as $x \to -\infty$ or as $x \to \infty$ the value of $R(x)$ approaches 0. Consequently, the line $y = 0$ (the x-axis) is a horizontal asymptote of the graph.

Theorem

If a rational function is proper, the line $y = 0$ is a horizontal asymptote of its graph.

EXAMPLE 5 Finding Horizontal Asymptotes

Find the horizontal asymptotes, if any, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

Solution Since the degree of the numerator, 1, is less than the degree of the denominator, 2, the rational function R is proper. We conclude that the line $y = 0$ is a horizontal asymptote of the graph of R. ◀

To see why $y = 0$ is a horizontal asymptote of the function R in Example 5, we need to investigate the behavior of R as $x \to -\infty$ and $x \to \infty$. When $|x|$ is unbounded, the numerator of R, which is $x - 12$, can be approximated by the power function $y = x$, while the denominator of R, which is $4x^2 + x + 1$, can be approximated by the power function $y = 4x^2$. Applying these ideas to $R(x)$, we find

$$R(x) = \frac{x - 12}{4x^2 + x + 1} \underset{\substack{\uparrow \\ \text{For } |x| \text{ unbounded}}}{\approx} \frac{x}{4x^2} = \frac{1}{4x} \underset{\substack{\uparrow \\ \text{As } x \to -\infty \text{ or } x \to \infty}}{\to} 0$$

This shows that the line $y = 0$ is a horizontal asymptote of the graph of R.

We verify these results in Tables 14(a) and (b). Notice, as $x \to -\infty$ [Table 14(a)] or $x \to \infty$ [Table 14(b)], respectively, that the values of $R(x)$ approach 0.

Table 14

X	Y1
-10	-.0563
-100	-.0028
-1000	-3E-4
-10000	-3E-5
-1E5	-3E-6
-1E6	-3E-7
-1E7	-3E-8

Y1 = (X−12)/(4X²+...

(a)

X	Y1
10	-.0049
100	.00219
1000	2.5E-4
10000	2.5E-5
100000	2.5E-6
1E6	2.5E-7
1E7	2.5E-8

Y1 = (X−12)/(4X²+...

(b)

If a rational function $R(x) = \dfrac{p(x)}{q(x)}$ is **improper**, that is, if the degree of the numerator is greater than or equal to the degree of the denominator, we must use long division to write the rational function as the sum of a polynomial $f(x)$ plus a proper rational function $\dfrac{r(x)}{q(x)}$. That is, we write

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

where $f(x)$ is a polynomial and $\dfrac{r(x)}{q(x)}$ is a proper rational function. Since $\dfrac{r(x)}{q(x)}$ is proper, then $\dfrac{r(x)}{q(x)} \to 0$ as $x \to -\infty$ or as $x \to \infty$. As a result,

$$R(x) = \frac{p(x)}{q(x)} \to f(x), \qquad \text{as } x \to -\infty \text{ or as } x \to \infty$$

The possibilities are listed next.

1. If $f(x) = b$, a constant, then the line $y = b$ is a horizontal asymptote of the graph of R.
2. If $f(x) = ax + b$, $a \neq 0$, then the line $y = ax + b$ is an oblique asymptote of the graph of R.
3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

The following examples demonstrate these conclusions.

EXAMPLE 6 Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

Solution Since the degree of the numerator, 4, is larger than the degree of the denominator, 3, the rational function H is improper. To find any horizontal or oblique asymptotes, we use long division.

$$\begin{array}{r|l} & 3x + 3 \\ \hline x^3 - x^2 + 1 & 3x^4 \qquad\qquad - x^2 \\ & \underline{3x^4 - 3x^3 \qquad\quad + 3x} \\ & \qquad 3x^3 - x^2 - 3x \\ & \qquad \underline{3x^3 - 3x^2 \qquad + 3} \\ & \qquad\qquad 2x^2 - 3x - 3 \end{array}$$

As a result,

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1} = 3x + 3 + \frac{2x^2 - 3x - 3}{x^3 - x^2 + 1}$$

As $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 - 3x - 3}{x^3 - x^2 + 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $H(x) \to 3x + 3$. We conclude that the graph of the rational function H has an oblique asymptote $y = 3x + 3$.

We verify these results in Tables 15(a) and (b) with $Y_1 = H(x)$ and $Y_2 = 3x + 3$. As $x \to -\infty$ or $x \to \infty$, the difference in the values between Y_1 and Y_2 is indistinguishable.

Table 15

X	Y1	Y2
-10	-27.21	-27
-100	-297	-297
-1000	-2997	-2997
-10000	-29997	-29997
-1E5	-3E5	-3E5
-1E6	-3E6	-3E6
-1E7	-3E7	-3E7

Y1=(3X^4-X²)/(X...

(a)

X	Y1	Y2
10	33.185	33
100	303.02	303
1000	3003	3003
10000	30003	30003
100000	300003	300003
1E6	3E6	3E6
1E7	3E7	3E7

Y1=(3X^4-X²)/(X...

(b)

EXAMPLE 7 Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

Solution Since the degree of the numerator, 2, equals the degree of the denominator, 2, the rational function R is improper. To find any horizontal or oblique asymptotes, we use long division.

$$\begin{array}{r@{}l} & \;2 \\ 4x^2 - 1 & \overline{)\,8x^2 - x + 2} \\ & \,\underline{8x^2 \phantom{{}- x} - 2} \\ & \, - x + 4 \end{array}$$

As a result,

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1} = 2 + \frac{-x + 4}{4x^2 - 1}$$

Then, as $x \to -\infty$ or as $x \to \infty$,

$$\frac{-x + 4}{4x^2 - 1} \approx \frac{-x}{4x^2} = \frac{-1}{4x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $R(x) \to 2$. We conclude that $y = 2$ is a horizontal asymptote of the graph.

✔ **CHECK:** Verify the results of Example 7 by creating a TABLE with $Y_1 = R(x)$ and $Y_2 = 2$. ◀

In Example 7, we note that the quotient 2 obtained by long division is the quotient of the leading coefficients of the numerator polynomial and the denominator polynomial $\left(\frac{8}{4}\right)$. This means that we can avoid the long division process for rational functions whose numerator and denominator *are of the same degree* and conclude that the quotient of the leading coefficients will give us the horizontal asymptote.

NOW WORK PROBLEM 41.

EXAMPLE 8 Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

Solution Since the degree of the numerator, 5, is larger than the degree of the denominator, 3, the rational function G is improper. To find any horizontal or oblique asymptotes, we use long division.

$$\begin{array}{r@{}l} & \;2x^2 - 1 \\ x^3 - 1 & \overline{)\,2x^5 - x^3 \phantom{{}+ 2x^2} + 2} \\ & \,\underline{2x^5 \phantom{{}- x^3} - 2x^2 \phantom{{}+ 2}} \\ & \, -x^3 + 2x^2 + 2 \\ & \, \underline{-x^3 \phantom{{}+ 2x^2} + 1} \\ & \, 2x^2 + 1 \end{array}$$

As a result,

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1} = 2x^2 - 1 + \frac{2x^2 + 1}{x^3 - 1}$$

Then, as $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 + 1}{x^3 - 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $G(x) \to 2x^2 - 1$. We conclude that, for large values of $|x|$, the graph of G approaches the graph of $y = 2x^2 - 1$. That is, the graph of G will look like the graph of $y = 2x^2 - 1$ as $x \to -\infty$ or $x \to \infty$. Since $y = 2x^2 - 1$ is not a linear function, G has no horizontal or oblique asymptotes. ◀

We now summarize the procedure for finding horizontal and oblique asymptotes.

Summary

Finding Horizontal and Oblique Asymptotes of a Rational Function R

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$, then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).
2. If $n \geq m$, then R is improper. Here long division is used.
   (a) If $n = m$, the quotient obtained will be a number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.
   (b) If $n = m + 1$, the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.
   (c) If $n > m + 1$, the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for $|x|$ unbounded, the graph of R will behave like the graph of the quotient.

NOTE *The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote.* ■

3.3 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. *True or False:* The quotient of two polynomial expressions is a rational expression. (p. 971)

2. What is the quotient and remainder when $3x^3 - 6x^2 + 3x - 4$ is divided by $x^2 - 1$? (pp. 977–979)

3. Graph $y = \frac{1}{x}$. (p. 21)

4. Graph $y = 2(x + 1)^2 - 3$ using transformations. (pp. 118–126)

Concepts and Vocabulary

5. The line _____ is a horizontal asymptote of $R(x) = \dfrac{x^3 - 1}{x^3 + 1}$.

6. The line _____ is a vertical asymptote of $R(x) = \dfrac{x - 1}{x + 1}$.

7. For a rational function R, if the degree of the numerator is less than the degree of the denominator, then R is _____.

8. *True or False:* The domain of every rational function is the set of all real numbers.

9. *True or False:* If an asymptote is neither horizontal nor vertical, it is called oblique.

10. *True or False:* If the degree of the numerator of a rational function equals the degree of the denominator, then the ratio of the leading coefficients gives rise to the horizontal asymptote.

Skill Building

In Problems 11–22, find the domain of each rational function.

11. $R(x) = \dfrac{4x}{x - 3}$

12. $R(x) = \dfrac{5x^2}{3 + x}$

13. $H(x) = \dfrac{-4x^2}{(x - 2)(x + 4)}$

14. $G(x) = \dfrac{6}{(x + 3)(4 - x)}$

15. $F(x) = \dfrac{3x(x - 1)}{2x^2 - 5x - 3}$

16. $Q(x) = \dfrac{-x(1 - x)}{3x^2 + 5x - 2}$

17. $R(x) = \dfrac{x}{x^3 - 8}$

18. $R(x) = \dfrac{x}{x^4 - 1}$

19. $H(x) = \dfrac{3x^2 + x}{x^2 + 4}$

20. $G(x) = \dfrac{x - 3}{x^4 + 1}$

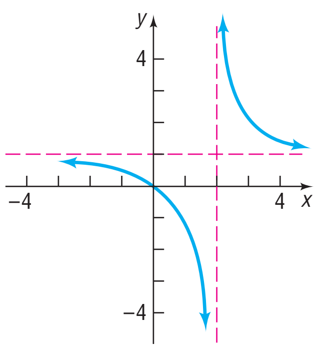
21. $R(x) = \dfrac{3(x^2 - x - 6)}{4(x^2 - 9)}$

22. $F(x) = \dfrac{-2(x^2 - 4)}{3(x^2 + 4x + 4)}$

In Problems 23–28, use the graph shown to find:

(a) *The domain and range of each function*
(b) *The intercepts, if any*
(c) *Horizontal asymptotes, if any*
(d) *Vertical asymptotes, if any*
(e) *Oblique asymptotes, if any*

23.



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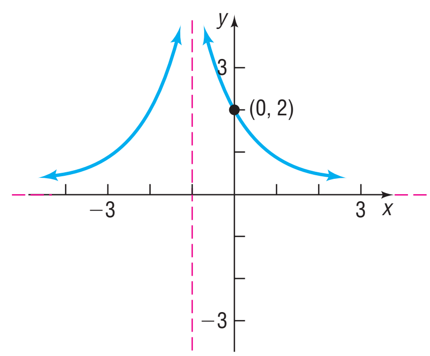

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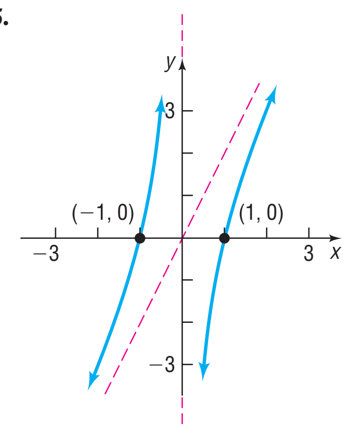

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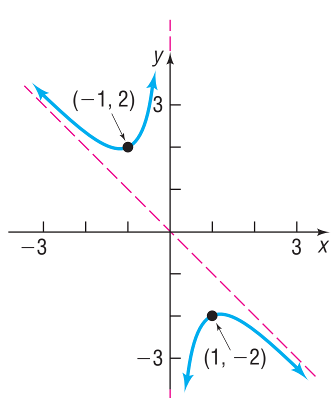

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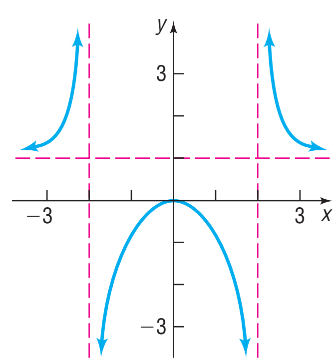
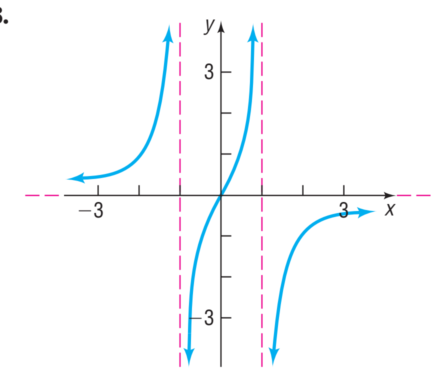


28.



In Problems 29–40, graph each rational function using transformations. Verify your result using a graphing utility.

29. $F(x) = 2 + \dfrac{1}{x}$

30. $Q(x) = 3 + \dfrac{1}{x^2}$

31. $R(x) = \dfrac{1}{(x - 1)^2}$

32. $R(x) = \dfrac{3}{x}$

33. $H(x) = \dfrac{-2}{x+1}$ **34.** $G(x) = \dfrac{2}{(x+2)^2}$ **35.** $R(x) = \dfrac{-1}{x^2+4x+4}$ **36.** $R(x) = \dfrac{1}{x-1} + 1$

37. $G(x) = 1 + \dfrac{2}{(x-3)^2}$ **38.** $F(x) = 2 - \dfrac{1}{x+1}$ **39.** $R(x) = \dfrac{x^2-4}{x^2}$ **40.** $R(x) = \dfrac{x-4}{x}$

In Problems 41–52, find the vertical, horizontal, and oblique asymptotes, if any, of each rational function.

41. $R(x) = \dfrac{3x}{x+4}$ **42.** $R(x) = \dfrac{3x+5}{x-6}$ **43.** $H(x) = \dfrac{x^3-8}{x^2-5x+6}$ **44.** $G(x) = \dfrac{-x^2+1}{x^2-5x+6}$

45. $T(x) = \dfrac{x^3}{x^4-1}$ **46.** $P(x) = \dfrac{4x^5}{x^3-1}$ **47.** $Q(x) = \dfrac{5-x^2}{3x^4}$ **48.** $F(x) = \dfrac{-2x^2+1}{2x^3+4x^2}$

49. $R(x) = \dfrac{3x^4+4}{x^3+3x}$ **50.** $R(x) = \dfrac{6x^2+x+12}{3x^2-5x-2}$ **51.** $G(x) = \dfrac{x^3-1}{x-x^2}$ **52.** $F(x) = \dfrac{x-1}{x-x^3}$

Applications and Extensions

53. Gravity In physics, it is established that the acceleration due to gravity, g (in meters/sec^2), at a height h meters above sea level is given by

$$g(h) = \frac{3.99 \times 10^{14}}{(6.374 \times 10^6 + h)^2}$$

where 6.374×10^6 is the radius of Earth in meters.

(a) What is the acceleration due to gravity at sea level?

(b) The Sears Tower in Chicago, Illinois, is 443 meters tall. What is the acceleration due to gravity at the top of the Sears Tower?

(c) The peak of Mount Everest is 8848 meters above sea level. What is the acceleration due to gravity on the peak of Mount Everest?

(d) Find the horizontal asymptote of $g(h)$.

(e) Solve $g(h) = 0$. How do you interpret your answer?

54. Population Model A rare species of insect was discovered in the Amazon Rain Forest. To protect the species, environmentalists declare the insect endangered and transplant the insect into a protected area. The population P of the insect t months after being transplanted is given below.

$$P(t) = \frac{50(1 + 0.5t)}{(2 + 0.01t)}$$

(a) How many insects were discovered? In other words, what was the population when $t = 0$?

(b) What will the population be after 5 years?

(c) Determine the horizontal asymptote of $P(t)$. What is the largest population that the protected area can sustain?

Discussion and Writing

55. If the graph of a rational function R has the vertical asymptote $x = 4$, then the factor $x - 4$ must be present in the denominator of R. Explain why.

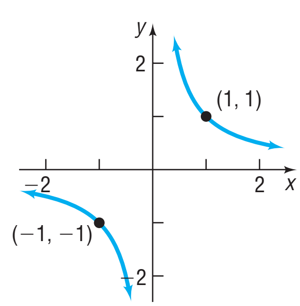
56. If the graph of a rational function R has the horizontal asymptote $y = 2$, then the degree of the numerator of R equals the degree of the denominator of R. Explain why.

57. Can the graph of a rational function have both a horizontal and an oblique asymptote? Explain.

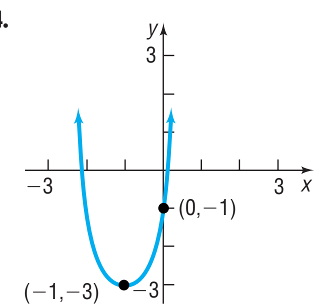
58. Make up a rational function that has $y = 2x + 1$ as an oblique asymptote. Explain the methodology that you used.

'Are You Prepared?' Answers

1. True **2.** Quotient: $3x - 6$; Remainder: $6x - 10$ **3.**



4.



3.4 The Graph of a Rational Function; Inverse and Joint Variation

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intercepts (Section 1.2, pp. 15–17)
- Even and Odd Functions (Section 2.3, pp. 80–82)

Now work the 'Are You Prepared?' problems on page 207.

OBJECTIVES
1. Analyze the Graph of a Rational Function
2. Solve Applied Problems Involving Rational Functions
3. Construct a Model Using Inverse Variation
4. Construct a Model Using Joint or Combined Variation

1 Analyze the Graph of a Rational Function

Graphing utilities make the task of graphing rational functions less time consuming. However, the results of algebraic analysis must be taken into account before drawing conclusions based on the graph provided by the utility. We will use the information collected in the last section in conjunction with the graphing utility to analyze the graph of a rational function $R(x) = \dfrac{p(x)}{q(x)}$. The analysis will require the following steps:

Analyzing the Graph of a Rational Function

STEP 1: Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph. The x-intercepts, if any, of $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms, satisfy the equation $p(x) = 0$. The y-intercept, if there is one, is $R(0)$.

STEP 4: Test for symmetry. Replace x by $-x$ in $R(x)$. If $R(-x) = R(x)$, there is symmetry with respect to the y-axis; if $R(-x) = -R(x)$, there is symmetry with respect to the origin.

STEP 5: Locate the vertical asymptotes. The vertical asymptotes, if any, of $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms are found by identifying the real zeros of $q(x)$. Each zero of the denominator gives rise to a vertical asymptote.

STEP 6: Locate the horizontal or oblique asymptotes, if any, using the procedure given in Section 3.3. Determine points, if any, at which the graph of R intersects these asymptotes.

STEP 7: Graph R using a graphing utility.

STEP 8: Use the results obtained in Steps 1 through 7 to graph R by hand.

EXAMPLE 1 **Analyzing the Graph of a Rational Function**

Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$

Solution First, we factor both the numerator and the denominator of R.

$$R(x) = \frac{x-1}{(x+2)(x-2)}$$

STEP 1: The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

STEP 2: R is in lowest terms.

STEP 3: We locate the x-intercepts by finding the zeros of the numerator. By inspection, 1 is the only x-intercept. The y-intercept is $R(0) = \frac{1}{4}$.

STEP 4: Because

$$R(-x) = \frac{-x-1}{x^2-4} = \frac{-(x+1)}{x^2-4}$$

we conclude that R is neither even nor odd. There is no symmetry with respect to the y-axis or the origin.

STEP 5: We locate the vertical asymptotes by finding the zeros of the denominator. Since R is in lowest terms, the graph of R has two vertical asymptotes: the lines $x = -2$ and $x = 2$.

STEP 6: The degree of the numerator is less than the degree of the denominator, so R is proper and the line $y = 0$ (the x-axis) is a horizontal asymptote of the graph. To determine if the graph of R intersects the horizontal asymptote, we solve the equation $R(x) = 0$:

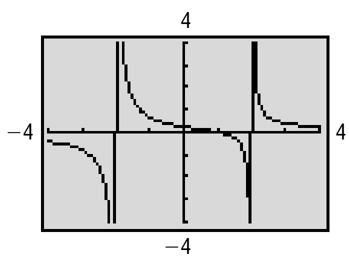
$$\frac{x-1}{x^2-4} = 0$$
$$x - 1 = 0$$
$$x = 1$$

The only solution is $x = 1$, so the graph of R intersects the horizontal asymptote at $(1, 0)$.

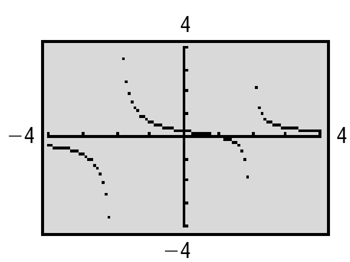
STEP 7: The analysis just completed helps us to set the viewing window to obtain a complete graph. Figure 45(a) shows the graph of $R(x) = \frac{x-1}{x^2-4}$ in connected mode, and Figure 45(b) shows it in dot mode. Notice in Figure 45(a) that the graph has vertical lines at $x = -2$ and $x = 2$. This is due to the fact that, when the graphing utility is in connected mode, it will "connect the dots" between consecutive pixels. We know that the graph of R does not cross the lines $x = -2$ and $x = 2$, since R is not defined at $x = -2$ or $x = 2$. When graphing rational functions, dot mode should be used if extraneous vertical lines appear. You should confirm that all the algebraic conclusions that we arrived at in Steps 1 through 6 are part of the graph. For example, the graph has vertical asymptotes at $x = -2$ and $x = 2$, and the graph has a horizontal asymptote at $y = 0$. The y-intercept is $\frac{1}{4}$ and the x-intercept is 1.

STEP 8: Using the information gathered in Steps 1 through 7, we obtain the graph of R shown in Figure 46.

Figure 45

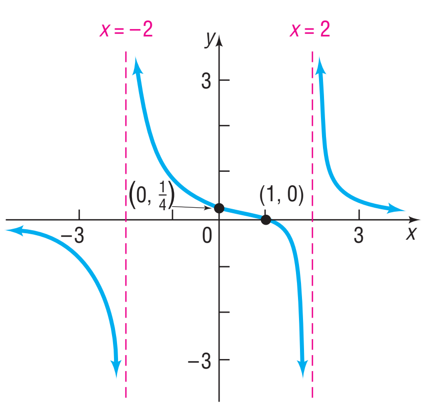


Connected mode
(a)



Dot mode
(b)

Figure 46



NOW WORK PROBLEM 7.

EXAMPLE 2 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 1}{x}$

Solution

Step 1: The domain of R is $\{x \mid x \neq 0\}$.

Step 2: R is in lowest terms.

Step 3: The graph has two x-intercepts: -1 and 1. There is no y-intercept, since x cannot equal 0.

Step 4: Since $R(-x) = -R(x)$, the function is odd and the graph is symmetric with respect to the origin.

Step 5: Since R is in lowest terms, the graph of $R(x)$ has the line $x = 0$ (the y-axis) as a vertical asymptote.

Step 6: Since the degree of the numerator, 2, is larger than the degree of the denominator, 1, the rational function R is improper. To find any horizontal or oblique asymptotes, we use long division.

$$\begin{array}{r} x \\ x\overline{)\,x^2 - 1} \\ \underline{x^2 \quad\;\;} \\ -1 \end{array}$$

The quotient is x, so the line $y = x$ is an oblique asymptote of the graph. To determine whether the graph of R intersects the asymptote $y = x$, we solve the equation $R(x) = x$.

$$\begin{aligned} R(x) = \frac{x^2 - 1}{x} &= x \\ x^2 - 1 &= x^2 \\ -1 &= 0 \quad \text{Impossible.} \end{aligned}$$

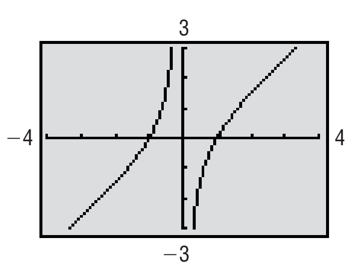
We conclude that the equation $\dfrac{x^2 - 1}{x} = x$ has no solution, so the graph of $R(x)$ does not intersect the line $y = x$.

NOTE
Because the determinator of the rational function is a monomial, we can also find the oblique asymptote as follows:

$$\frac{x^2 - 1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$$

Since $\frac{1}{x} \to 0$ as $x \to \infty$, $y = x$ is an oblique asymptote. ■

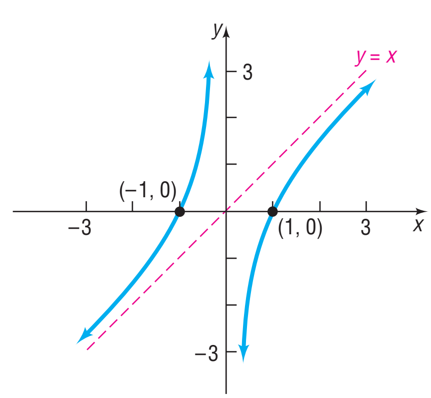
Figure 47



Step 7: See Figure 47. We see from the graph that there is no y-intercept and there are two x-intercepts, -1 and 1. The symmetry with respect to the origin is also evident. We can also see that there is a vertical asymptote at $x = 0$. Finally, it is not necessary to graph this function in dot mode since no extraneous vertical lines are present.

Step 8: Using the information gathered in Steps 1 through 7, we obtain the graph of R shown in Figure 48. Notice how the oblique asymptote is used as a guide in graphing the rational function by hand.

Figure 48



◀

NOW WORK PROBLEM 15.

EXAMPLE 3 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

Solution We factor R to get

$$R(x) = \frac{3x(x - 1)}{(x + 4)(x - 3)}$$

STEP 1: The domain of R is $\{x \mid x \neq -4, x \neq 3\}$.

STEP 2: R is in lowest terms.

STEP 3: The graph has two x-intercepts: 0 and 1. The y-intercept is $R(0) = 0$.

STEP 4: Because

$$R(-x) = \frac{-3x(-x - 1)}{(-x + 4)(-x - 3)} = \frac{3x(x + 1)}{(x - 4)(x + 3)}$$

we conclude that R is neither even nor odd. There is no symmetry with respect to the y-axis or the origin.

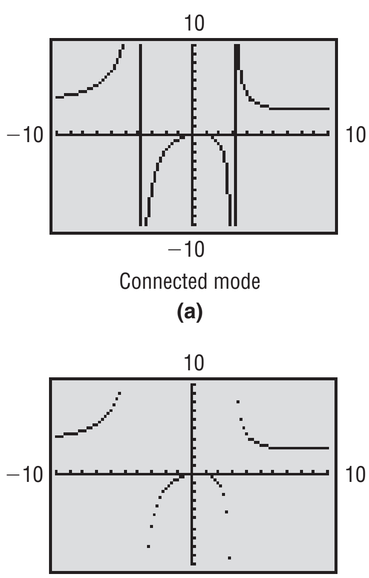
STEP 5: Since R is in lowest terms, the graph of R has two vertical asymptotes: $x = -4$ and $x = 3$.

STEP 6: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, we form the quotient of the leading coefficient of the numerator, 3, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 3$. To find out whether the graph of R intersects the asymptote, we solve the equation $R(x) = 3$.

$$R(x) = \frac{3x^2 - 3x}{x^2 + x - 12} = 3$$
$$3x^2 - 3x = 3x^2 + 3x - 36$$
$$-6x = -36$$
$$x = 6$$

The graph intersects the line $y = 3$ at $x = 6$, and $(6, 3)$ is a point on the graph of R.

Figure 49



Connected mode
(a)

Dot mode
(b)

STEP 7: Figure 49(a) shows the graph of R in connected mode. Notice the extraneous vertical lines at $x = -4$ and $x = 3$ (the vertical asymptotes). As a result, we also graph R in dot mode. See Figure 49(b).

STEP 8: Figure 49 does not display the graph between the two x-intercepts, 0 and 1. However, because the zeros in the numerator, 0 and 1, are of odd multiplicity (both are multiplicity 1), we know that the graph of R crosses the x-axis at 0 and 1. Therefore, the graph of R is above the x-axis for $0 < x < 1$. To see this part better, we graph R for $-1 \leq x \leq 2$ in Figure 50. Using MAXIMUM, we approximate the turning point to be $(0.52, 0.07)$, rounded to two decimal places.

Figure 50

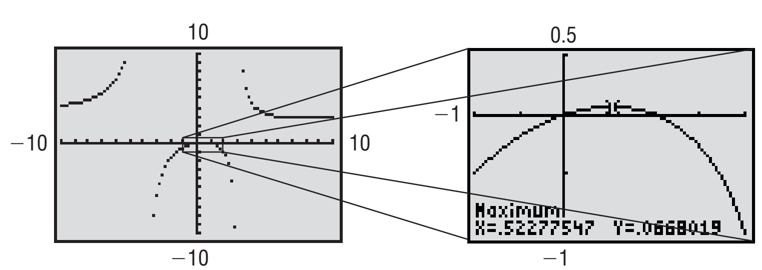
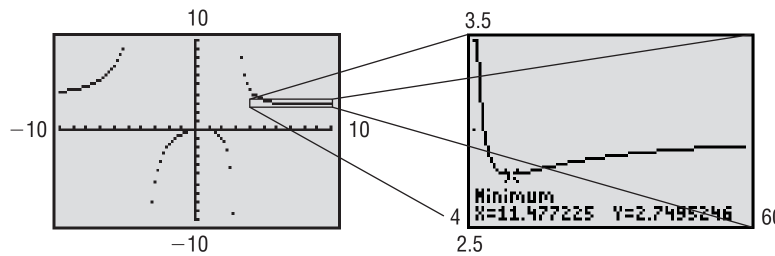


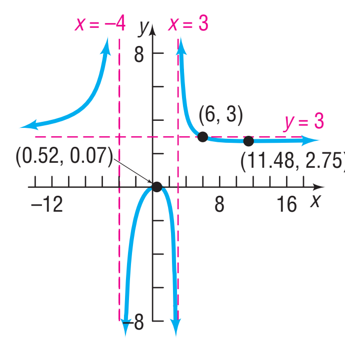
Figure 49 also does not display the graph of R crossing the horizontal asymptote at $(6, 3)$. To see this part better, we graph R for $4 \le x \le 60$ in Figure 51. Using MINIMUM, we approximate the turning point to be $(11.48, 2.75)$, rounded to two decimal places.

Figure 51



Using this information along with the information gathered in Steps 1 through 7, we obtain the graph of R shown in Figure 52.

Figure 52



EXAMPLE 4 Analyzing the Graph of a Rational Function with a Hole

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

Solution We factor R and obtain

$$R(x) = \frac{(2x - 1)(x - 2)}{(x + 2)(x - 2)}$$

Step 1: The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: In lowest terms,

$$R(x) = \frac{2x - 1}{x + 2}, \qquad x \neq 2$$

Step 3: The graph has one x-intercept: 0.5. The y-intercept is $R(0) = -0.5$.

Step 4: Because

$$R(-x) = \frac{2x^2 + 5x + 2}{x^2 - 4}$$

we conclude that R is neither even nor odd. There is no symmetry with respect to the y-axis or the origin.

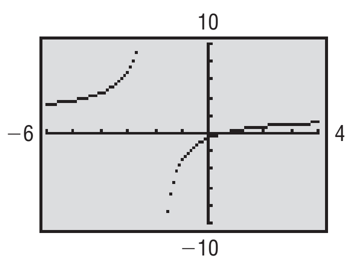
STEP 5: Since $x + 2$ is the only factor of the denominator of $R(x)$ *in lowest terms* the graph has one vertical asymptote, $x = -2$. However, the rational function is undefined at both $x = 2$ and $x = -2$.

STEP 6: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, we form the quotient of the leading coefficient of the numerator, 2, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 2$. To find whether the graph of R intersects the asymptote, we solve the equation $R(x) = 2$.

$$R(x) = \frac{2x - 1}{x + 2} = 2$$
$$2x - 1 = 2(x + 2)$$
$$2x - 1 = 2x + 4$$
$$-1 = 4 \qquad \text{Impossible}$$

The graph does not intersect the line $y = 2$.

Figure 53



STEP 7: Figure 53 shows the graph of $R(x)$. Notice that the graph has one vertical asymptote at $x = -2$. Also, the function appears to be continuous at $x = 2$.

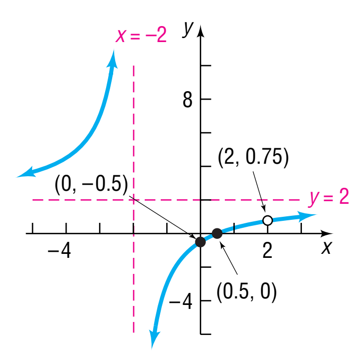
STEP 8: The analysis presented thus far does not explain the behavior of the graph at $x = 2$. We use the TABLE feature of our graphing utility to determine the behavior of the graph of R as x approaches 2. See Table 16. From the table, we conclude that the value of R approaches 0.75 as x approaches 2. This result is further verified by evaluating R in lowest terms at $x = 2$. We conclude that there is a hole in the graph at $(2, 0.75)$. Using the information gathered in Steps 1 through 7, we obtain the graph of R shown in Figure 54.

Table 16

X	Y1
1.99	.74687
1.999	.74969
1.9999	.74997
2	ERROR
2.0001	.75003
2.001	.75031
2.01	.75312

Y1 ■ (2X²−5X+2)/(…

Figure 54



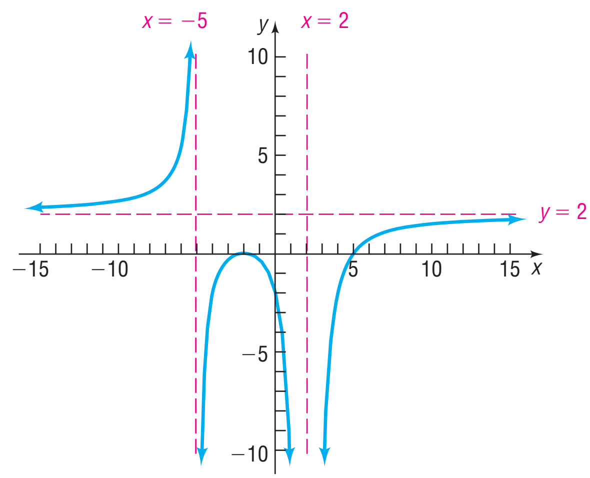
As Example 4 shows, the values excluded from the domain of a rational function give rise to either vertical asymptotes or holes.

NOW WORK PROBLEM 33.

EXAMPLE 5 Constructing a Rational Function from Its Graph

Make up a rational function that might have the graph shown in Figure 55.

Figure 55

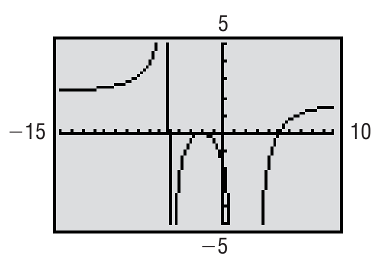


Solution The numerator of a rational function $R(x) = \frac{p(x)}{q(x)}$ in lowest terms determines the x-intercepts of its graph. The graph shown in Figure 55 has x-intercepts -2 (even multiplicity; graph touches the x-axis) and 5 (odd multiplicity; graph crosses the x-axis). So one possibility for the numerator is $p(x) = (x + 2)^2(x - 5)$.

The denominator of a rational function in lowest terms determines the vertical asymptotes of its graph. The vertical asymptotes of the graph are $x = -5$ and $x = 2$. Since $R(x)$ approaches ∞ from the left of $x = -5$ and $R(x)$ approaches $-\infty$ from the right of $x = -5$, we know that $(x + 5)$ is a factor of odd multiplicity in $q(x)$. Also, $R(x)$ approaches $-\infty$ from both sides of $x = 2$, so $(x - 2)$ is a factor of even multiplicity in $q(x)$. A possibility for the denominator is $q(x) = (x + 5)(x - 2)^2$.

So far we have $R(x) = \frac{(x + 2)^2(x - 5)}{(x + 5)(x - 2)^2}$. However, the horizontal asymptote of the graph given in Figure 55, is $y = 2$, so we know that the degree of the numerator must equal the degree in the denominator and the quotient of leading coefficients must be $\frac{2}{1}$. This leads to $R(x) = \frac{2(x + 2)^2(x - 5)}{(x + 5)(x - 2)^2}$. Figure 56 shows the graph of R drawn on a graphing utility. Since Figure 56 looks similar to Figure 55, we have found a rational function R for the graph in Figure 55.

Figure 56



NOW WORK PROBLEM 51.

2 Solve Applied Problems Involving Rational Functions

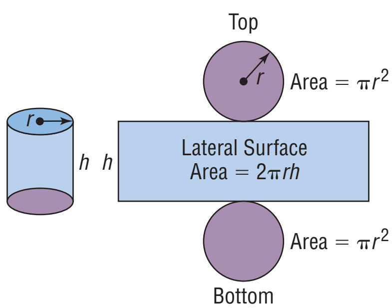
EXAMPLE 6 Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2} \text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius r of the can.
(b) Use a graphing utility to graph the function $C = C(r)$.
(c) What value of r will result in the least cost?
(d) What is this least cost?

Solution (a) Figure 57 illustrates the situation. Notice that the material required to produce a cylindrical can of height h and radius r consists of a rectangle of area $2\pi rh$ and two circles, each of area πr^2. The total cost C (in cents) of manufacturing the can is therefore

Figure 57



$$\begin{aligned} C &= \text{Cost of top and bottom} + \text{Cost of side} \\ &= \underbrace{(2\pi r^2 \text{ cm}^2)}_{\substack{\text{Total area}\\ \text{of top and}\\ \text{bottom}}} \underbrace{(0.05¢/\text{cm}^2)}_{\text{Cost/unit area}} + \underbrace{(2\pi rh \text{ cm}^2)}_{\substack{\text{Total area}\\ \text{of side}}} \underbrace{(0.02¢/\text{cm}^2)}_{\text{Cost/unit area}} \\ &= 0.10\pi r^2 + 0.04\pi rh \end{aligned}$$

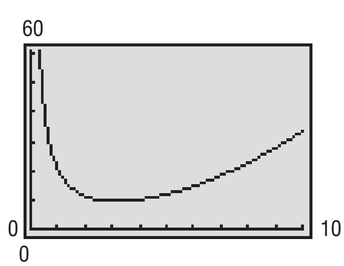
But we have the additional restriction that the height h and radius r must be chosen so that the volume V of the can is 500 cubic centimeters. Since $V = \pi r^2 h$, we have

$$500 = \pi r^2 h \quad \text{or} \quad h = \frac{500}{\pi r^2}$$

Substituting this expression for h, the cost C, in cents, as a function of the radius r, is

$$C(r) = 0.10\pi r^2 + 0.04\pi r \frac{500}{\pi r^2} = 0.10\pi r^2 + \frac{20}{r} = \frac{0.10\pi r^3 + 20}{r}$$

Figure 58



(b) See Figure 58 for the graph of $C(r)$.
(c) Using the MINIMUM command, the cost is least for a radius of about 3.17 centimeters.
(d) The least cost is $C(3.17) \approx 9.47¢$. ◀

NOW WORK PROBLEM 61.

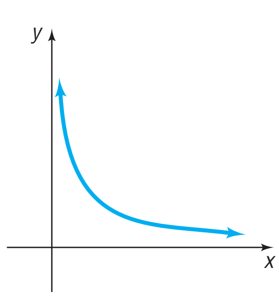
3 Construct a Model Using Inverse Variation

In Section 2.4, we discussed direct variation. We now study **inverse variation**.

Let x and y denote two quantities. Then y **varies inversely** with x, or y is **inversely proportional to** x, if there is a nonzero constant k such that

$$\boxed{y = \frac{k}{x}}$$

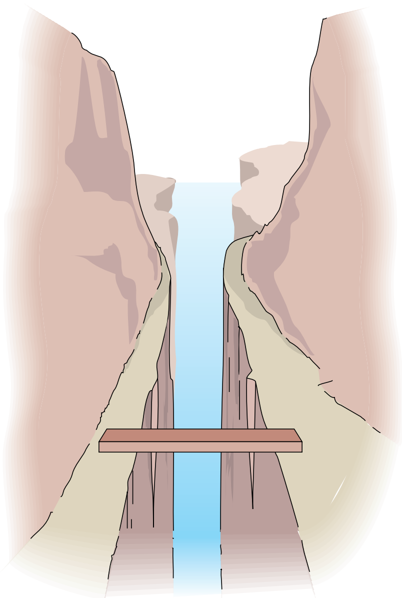
Figure 59
$y = \frac{k}{x}, k > 0, x > 0$



The graph in Figure 59 illustrates the relationship between y and x if y varies inversely with x and $k > 0$, $x > 0$.

EXAMPLE 7 Maximum Weight That Can Be Supported by a Piece of Pine

Figure 60



See Figure 60. The maximum weight W that can be safely supported by a 2-inch by 4-inch piece of pine varies inversely with its length l. Experiments indicate that the maximum weight that a 10-foot-long pine 2-by-4 can support is 500 pounds. Write a general formula relating the maximum weight W (in pounds) to the length l (in feet). Find the maximum weight W that can be safely supported by a length of 25 feet.

Solution Because W varies inversely with l, we know that

$$W = \frac{k}{l}$$

for some constant k. Because $W = 500$ when $l = 10$, we have

$$500 = \frac{k}{10}$$
$$k = 5000$$

Thus, in all cases,

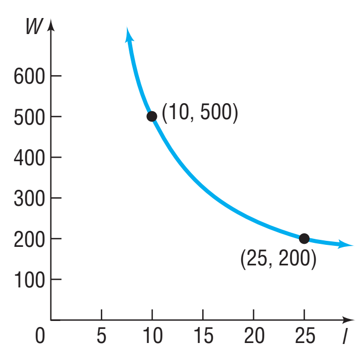
$$W(l) = \frac{5000}{l}$$

In particular, the maximum weight W that can be safely supported by a piece of pine 25 feet in length is

$$W(25) = \frac{5000}{25} = 200 \text{ pounds}$$

Figure 61 illustrates the relationship between the weight W and the length l. ◀

Figure 61



In direct or inverse variation, the quantities that vary may be raised to powers. For example, in the early seventeenth century, Johannes Kepler (1571–1630) discovered that the square of the period of revolution T of a planet around the Sun varies directly with the cube of its mean distance a from the Sun. That is, $T^2 = ka^3$, where k is the constant of proportionality.

NOW WORK PROBLEM 63.

4 Construct a Model Using Joint or Combined Variation

When a variable quantity Q is proportional to the product of two or more other variables, we say that Q **varies jointly** with these quantities. Finally, combinations of direct and/or inverse variation may occur. This is usually referred to as **combined variation**.

Let's look at an example.

EXAMPLE 8 Loss of Heat Through a Wall

The loss of heat through a wall varies jointly with the area of the wall and the difference between the inside and outside temperatures and varies inversely with the thickness of the wall. Write an equation that relates these quantities.

Solution We begin by assigning symbols to represent the quantities:

L = Heat loss $\quad$ T = Temperature difference

A = Area of wall $\quad$ d = Thickness of wall

Then

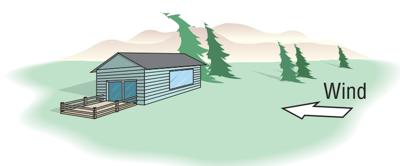
$$L = k\frac{AT}{d}$$

where k is the constant of proportionality. ◀

NOW WORK PROBLEM 69.

EXAMPLE 9 Force of the Wind on a Window

Figure 62



See Figure 62. The force F of the wind on a flat surface positioned at a right angle to the direction of the wind varies jointly with the area A of the surface and the square of the speed v of the wind. A wind of 30 miles per hour blowing on a window measuring 4 feet by 5 feet has a force of 150 pounds. What is the force on a window measuring 3 feet by 4 feet caused by a wind of 50 miles per hour?

Solution Since F varies jointly with A and v^2, we have

$$F = kAv^2$$

where k is the constant of proportionality. We are told that $F = 150$ when $A = 4 \cdot 5 = 20$ and $v = 30$. Then we have

$$150 = k(20)(900) \qquad F = kAv^2, F = 150, A = 20, v = 30$$

$$k = \frac{1}{120}$$

The general formula is therefore

$$F = \frac{1}{120}Av^2$$

For a wind of 50 miles per hour blowing on a window whose area is $A = 3 \cdot 4 = 12$ square feet, the force F is

$$F = \frac{1}{120}(12)(2500) = 250 \text{ pounds}$$ ◀

3.4 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find the intercepts of $-3x + 5y = 30$. (pp. 15–17)

2. Determine whether $f(x) = \dfrac{x^2+1}{x^4-1}$ is even, odd, or neither. Is the graph of f symmetric with respect to the y-axis or origin? (pp. 80–82)

Concepts and Vocabulary

3. If the numerator and the denominator of a rational function have no common factors, the rational function is _____.

4. *True or False:* The graph of a polynomial function sometimes has a hole.

5. *True or False:* The graph of a rational function never intersects a horizontal asymptote.

6. *True or False:* The graph of a rational function sometimes has a hole.

Skill Building

In Problems 7–44, follow Steps 1 through 8 on page 198 to analyze the graph of each function.

7. $R(x) = \dfrac{x+1}{x(x+4)}$

8. $R(x) = \dfrac{x}{(x-1)(x+2)}$

9. $R(x) = \dfrac{3x+3}{2x+4}$

10. $R(x) = \dfrac{2x+4}{x-1}$

11. $R(x) = \dfrac{3}{x^2-4}$

12. $R(x) = \dfrac{6}{x^2-x-6}$

13. $P(x) = \dfrac{x^4+x^2+1}{x^2-1}$

14. $Q(x) = \dfrac{x^4-1}{x^2-4}$

15. $H(x) = \dfrac{x^3-1}{x^2-9}$

16. $G(x) = \dfrac{x^3+1}{x^2+2x}$

17. $R(x) = \dfrac{x^2}{x^2+x-6}$

18. $R(x) = \dfrac{x^2+x-12}{x^2-4}$

19. $G(x) = \dfrac{x}{x^2-4}$

20. $G(x) = \dfrac{3x}{x^2-1}$

21. $R(x) = \dfrac{3}{(x-1)(x^2-4)}$

22. $R(x) = \dfrac{-4}{(x+1)(x^2-9)}$

23. $H(x) = \dfrac{4(x^2-1)}{x^4-16}$

24. $H(x) = \dfrac{x^2+4}{x^4-1}$

25. $F(x) = \dfrac{x^2-3x-4}{x+2}$

26. $F(x) = \dfrac{x^2+3x+2}{x-1}$

27. $R(x) = \dfrac{x^2+x-12}{x-4}$

28. $R(x) = \dfrac{x^2-x-12}{x+5}$

29. $F(x) = \dfrac{x^2+x-12}{x+2}$

30. $G(x) = \dfrac{x^2-x-12}{x+1}$

31. $R(x) = \dfrac{x(x-1)^2}{(x+3)^3}$

32. $R(x) = \dfrac{(x-1)(x+2)(x-3)}{x(x-4)^2}$

33. $R(x) = \dfrac{x^2+x-12}{x^2-x-6}$

34. $R(x) = \dfrac{x^2+3x-10}{x^2+8x+15}$

35. $R(x) = \dfrac{6x^2-7x-3}{2x^2-7x+6}$

36. $R(x) = \dfrac{8x^2+26x+15}{2x^2-x-15}$

37. $R(x) = \dfrac{x^2+5x+6}{x+3}$

38. $R(x) = \dfrac{x^2+x-30}{x+6}$

39. $f(x) = x + \dfrac{1}{x}$

40. $f(x) = 2x + \dfrac{9}{x}$

41. $f(x) = x^2 + \dfrac{1}{x}$

42. $f(x) = 2x^2 + \dfrac{9}{x}$

43. $f(x) = x + \dfrac{1}{x^3}$

44. $f(x) = 2x + \dfrac{9}{x^3}$

In Problems 45–50, graph each function and use MINIMUM to obtain the minimum value, rounded to two decimal places.

45. $f(x) = x + \dfrac{1}{x}, \quad x > 0$

46. $f(x) = 2x + \dfrac{9}{x}, \quad x > 0$

47. $f(x) = x^2 + \dfrac{1}{x}, \quad x > 0$

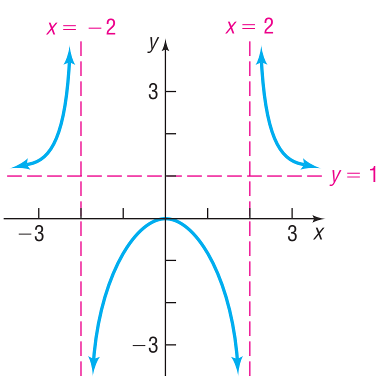
48. $f(x) = 2x^2 + \dfrac{9}{x}, \quad x > 0$

49. $f(x) = x + \dfrac{1}{x^3}, \quad x > 0$

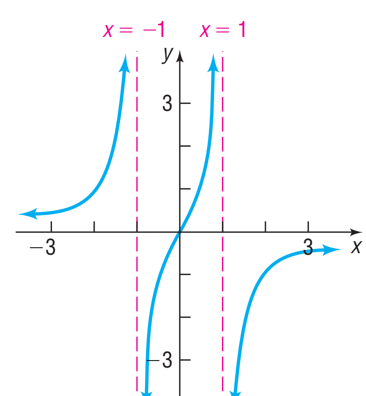
50. $f(x) = 2x + \dfrac{9}{x^3}, \quad x > 0$

In Problems 51–54, find a rational function that might have this graph. (More than one answer might be possible.)

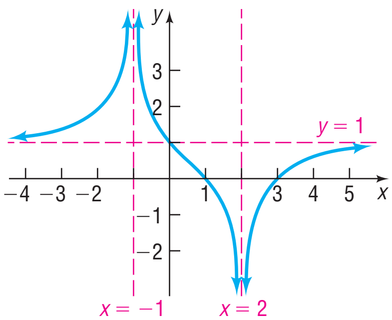
51.



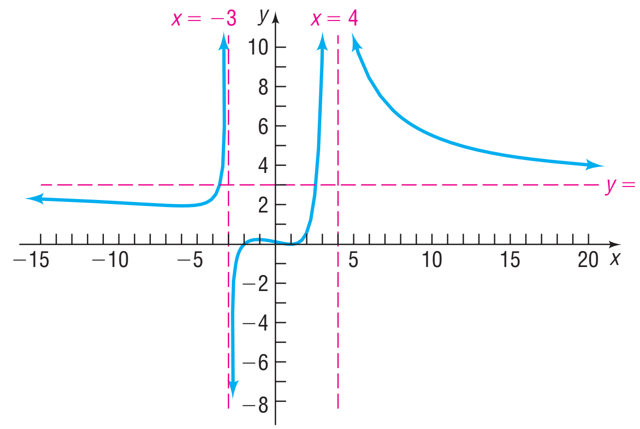
52.



53.



54.



Applications and Extensions

55. Drug Concentration The concentration C of a certain drug in a patient's bloodstream t hours after injection is given by

$$C(t) = \frac{t}{2t^2 + 1}$$

(a) Find the horizontal asymptote of $C(t)$. What happens to the concentration of the drug as t increases?
(b) Using a graphing utility, graph $C = C(t)$.
(c) Determine the time at which the concentration is highest.

56. Drug Concentration The concentration C of a certain drug in a patient's bloodstream t minutes after injection is given by

$$C(t) = \frac{50t}{t^2 + 25}$$

(a) Find the horizontal asymptote of $C(t)$. What happens to the concentration of the drug as t increases?
(b) Using a graphing utility, graph $C = C(t)$.
(c) Determine the time at which the concentration is highest.

57. Average Cost In Problem 95, Exercise 3.2, the cost function C (in thousands of dollars) for manufacturing x Chevy Cavaliers was determined to be

$$C(x) = 0.2x^3 - 2.3x^2 + 14.3x + 10.2$$

Economists define the **average cost function** as

$$\bar{C}(x) = \frac{C(x)}{x}$$

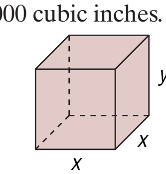
(a) Find the average cost function.
(b) What is the average cost of producing six Cavaliers?
(c) What is the average cost of producing nine Cavaliers?
(d) Using a graphing utility, graph the average cost function.
(e) Using a graphing utility, find the number of Cavaliers that should be produced to minimize the average cost.
(f) What is the minimum average cost?

58. Average Cost In Problem 96, Exercise 3.2, the cost function C (in thousands of dollars) for printing x textbooks (in thousands of units) was found to be

$$C(x) = 0.015x^3 - 0.595x^2 + 9.15x + 98.43$$

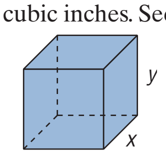
(a) Find the average cost function (refer to Problem 57).
(b) What is the average cost of printing 13,000 textbooks per week?
(c) What is the average cost of printing 25,000 textbooks per week?
(d) Using your graphing utility, graph the average cost function.
(e) Using your graphing utility, find the number of textbooks that should be printed to minimize the average cost.
(f) What is the minimum average cost?

59. Minimizing Surface Area United Parcel Service has contracted you to design a closed box with a square base that has a volume of 10,000 cubic inches. See the illustration.



(a) Find a function for the surface area of the box.
(b) Using a graphing utility, graph the function found in part (a).
(c) What is the minimum amount of cardboard that can be used to construct the box?
(d) What are the dimensions of the box that minimize the surface area?
(e) Why might UPS be interested in designing a box that minimizes the surface area?

60. Minimizing Surface Area United Parcel Service has contracted you to design a closed box with a square base that has a volume of 5000 cubic inches. See the illustration.



(a) Find a function for the surface area of the box.
(b) Using a graphing utility, graph the function found in part (a).
(c) What is the minimum amount of cardboard that can be used to construct the box?
(d) What are the dimensions of the box that minimize the surface area?
(e) Why might UPS be interested in designing a box that minimizes the surface area?

61. **Cost of a Can** A can in the shape of a right circular cylinder is required to have a volume of 500 cubic centimeters. The top and bottom are made of material that costs 6¢ per square centimeter, while the sides are made of material that costs 4¢ per square centimeter.
(a) Express the total cost C of the material as a function of the radius r of the cylinder. (Refer to Figure 57.)
(b) Using a graphing utility graph $C = C(r)$. For what value of r is the cost C a minimum?

62. **Material Needed to Make a Drum** A steel drum in the shape of a right circular cylinder is required to have a volume of 100 cubic feet.

(a) Express the amount A of material required to make the drum as a function of the radius r of the cylinder.
(b) How much material is required if the drum's radius is 3 feet?
(c) How much material is required if the drum's radius is 4 feet?
(d) How much material is required if the drum's radius is 5 feet?
(e) Using a graphing utility graph $A = A(r)$. For what value of r is A smallest?

63. **Demand** Suppose that the demand D for candy at the movie theater is inversely related to the price p.
(a) When the price of candy is \$2.75 per bag, the theater sells 156 bags of candy on a typical Saturday. Express the demand for candy as a function of its price.
(b) Determine the number of bags of candy that will be sold if the price is raised to \$3 a bag.

64. **Driving to School** The time t that it takes to get to school varies inversely with your average speed s.
(a) Suppose that it takes you 40 minutes to get to school when your average speed is 30 miles per hour. Express the driving time to school as a function of average speed.
(b) Suppose that your average speed to school is 40 miles per hour. How long will it take you to get to school?

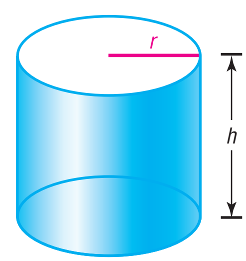
65. **Pressure** The volume of a gas V held at a constant temperature in a closed container varies inversely with its pressure P. If the volume of a gas is 600 cubic centimeters (cm^3) when the pressure is 150 millimeters of mercury (mm Hg), find the volume when the pressure is 200 mm Hg.

66. **Resistance** The current i in a circuit is inversely proportional to its resistance R measured in ohms. Suppose that when the current in a circuit is 30 amperes the resistance is 8 ohms. Find the current in the same circuit when the resistance is 10 ohms.

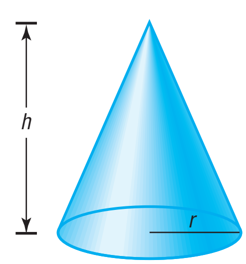
67. **Weight** The weight of an object above the surface of Earth varies inversely with the square of the distance from the center of Earth. If Maria weighs 125 pounds when she is on the surface of Earth (3960 miles from the center), determine Maria's weight if she is at the top of Mount McKinley (3.8 miles from the surface of Earth).

68. **Intensity of Light** The intensity I of light (measured in foot-candles) varies inversely with the square of the distance from the bulb. Suppose that the intensity of a 100-watt light bulb at a distance of 2 meters is 0.075 foot-candle. Determine the intensity of the bulb at a distance of 5 meters.

69. **Geometry** The volume V of a right circular cylinder varies jointly with the square of its radius r and its height h. The constant of proportionality is π. (See the figure.) Write an equation for V.



70. **Geometry** The volume V of a right circular cone varies jointly with the square of its radius r and its height h. The constant of proportionality is $\frac{\pi}{3}$. (See the figure.) Write an equation for V.



71. **Horsepower** The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute, rpm) and the cube of its diameter. If a shaft of a certain material 2 inches in diameter can transmit 36 hp at 75 rpm, what diameter must the shaft have to transmit 45 hp at 125 rpm?

72. Chemistry: Gas Laws The volume V of an ideal gas varies directly with the temperature T and inversely with the pressure P. Write an equation relating V, T, and P using k as the constant of proportionality. If a cylinder contains oxygen at a temperature of 300 K and a pressure of 15 atmospheres in a volume of 100 liters, what is the constant of proportionality k? If a piston is lowered into the cylinder, decreasing the volume occupied by the gas to 80 liters and raising the temperature to 310 K, what is the gas pressure?

73. Physics: Kinetic Energy The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity v. If an object weighing 25 pounds and moving with a velocity of 100 feet per second has a kinetic energy of 400 foot-pounds, find its kinetic energy when the velocity is 150 feet per second.

74. Electrical Resistance of a Wire The electrical resistance of a wire varies directly with the length of the wire and inversely with the square of the diameter of the wire. If a wire 432 feet long and 4 millimeters in diameter has a resistance of 1.24 ohms, find the length of a wire of the same material whose resistance is 1.44 ohms and whose diameter is 3 millimeters.

75. Measuring the Stress of Materials The stress in the material of a pipe subject to internal pressure varies jointly with the internal pressure and the internal diameter of the pipe and inversely with the thickness of the pipe. The stress is 100 pounds per square inch when the diameter is 5 inches, the thickness is 0.75 inch, and the internal pressure is 25 pounds per square inch. Find the stress when the internal pressure is 40 pounds per square inch if the diameter is 8 inches and the thickness is 0.50 inch.

76. Safe Load for a Beam The maximum safe load for a horizontal rectangular beam varies jointly with the width of the beam and the square of the thickness of the beam and inversely with its length. If an 8-foot beam will support up to 750 pounds when the beam is 4 inches wide and 2 inches thick, what is the maximum safe load in a similar beam 10 feet long, 6 inches wide, and 2 inches thick?

Discussion and Writing

77. Graph each of the following functions:

$$y = \frac{x^2 - 1}{x - 1} \qquad y = \frac{x^3 - 1}{x - 1}$$

$$y = \frac{x^4 - 1}{x - 1} \qquad y = \frac{x^5 - 1}{x - 1}$$

Is $x = 1$ a vertical asymptote? Why not? What is happening for $x = 1$? What do you conjecture about $y = \dfrac{x^n - 1}{x - 1}$, $n \geq 1$ an integer, for $x = 1$?

78. Graph each of the following functions:

$$y = \frac{x^2}{x - 1} \qquad y = \frac{x^4}{x - 1} \qquad y = \frac{x^6}{x - 1} \qquad y = \frac{x^8}{x - 1}$$

What similarities do you see? What differences?

79. Write a few paragraphs that provide a general strategy for graphing a rational function. Be sure to mention the following: proper, improper, intercepts, and asymptotes.

80. Create a rational function that has the following characteristics: crosses the x-axis at 2; touches the x-axis at -1; one vertical asymptote at $x = -5$ and another at $x = 6$; and one horizontal asymptote, $y = 3$. Compare yours to a fellow classmate's. How do they differ? What are their similarities?

81. Create a rational function that has the following characteristics: crosses the x-axis at 3; touches the x-axis at -2; one vertical asymptote, $x = 1$; and one horizontal asymptote, $y = 2$. Give your rational function to a fellow classmate and ask for a written critique of your rational function.

82. The formula on page 206 attributed to Johannes Kepler is one of the famous three Keplerian Laws of Planetary Motion. Go to the library and research these laws. Write a brief paper about these laws and Kepler's place in history.

83. Using a situation that has not been discussed in the text, write a real-world problem that you think involves two variables that vary directly. Exchange your problem with another student's to solve and critique.

84. Using a situation that has not been discussed in the text, write a real-world problem that you think involves two variables that vary inversely. Exchange your problem with another student's to solve and critique.

85. Using a situation that has not been discussed in the text, write a real-world problem that you think involves three variables that vary jointly. Exchange your problem with another student's to solve and critique.

'Are You Prepared?' Answers

1. $(-10, 0)$, $(0, 6)$ **2.** Even; the graph of f is symmetric with respect to the y-axis.

3.5 Polynomial and Rational Inequalities

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Inequalities (Appendix, Section A.8, pp. 1024–1025)

Now work the 'Are You Prepared?' problem on page 217.

OBJECTIVES 1 Solve Polynomial Inequalities Algebraically and Graphically
2 Solve Rational Inequalities Algebraically and Graphically

In this section we solve inequalities that involve polynomials of degree 2 and higher, as well as some that involve rational expressions. To solve such inequalities, we use the information obtained in the previous three sections about the graph of polynomial and rational functions. The general idea follows:

Suppose that the polynomial or rational inequality is in one of the forms

$$f(x) < 0 \qquad f(x) > 0 \qquad f(x) \leq 0 \qquad f(x) \geq 0$$

Locate the zeros of f if f is a polynomial function, and locate the zeros of the numerator and the denominator if f is a rational function. If we use these zeros to divide the real number line into intervals, then we know that on each interval the graph of f is either above the x-axis $[f(x) > 0]$ or below the x-axis $[f(x) < 0]$. In other words, we have found the solution of the inequality.

The following steps provide more detail.

Steps for Solving Polynomial and Rational Inequalities Algebraically

STEP 1: Write the inequality so that a polynomial or rational expression f is on the left side and zero is on the right side in one of the following forms:

$$f(x) > 0 \qquad f(x) \geq 0 \qquad f(x) < 0 \qquad f(x) \leq 0$$

For rational expressions, be sure that the left side is written as a single quotient.

STEP 2: Determine the numbers at which the expression f on the left side equals zero and, if the expression is rational, the numbers at which the expression f on the left side is undefined.

STEP 3: Use the numbers found in Step 2 to separate the real number line into intervals.

STEP 4: Select a number in each interval and evaluate f at the number.
(a) If the value of f is positive, then $f(x) > 0$ for all numbers x in the interval.
(b) If the value of f is negative, then $f(x) < 0$ for all numbers x in the interval.
If the inequality is not strict ($\geq$ or $\leq$), include the solutions of $f(x) = 0$ in the solution set, but be careful not to include values of x where the expression is undefined.

1 Solve Polynomial Inequalities Algebraically and Graphically

EXAMPLE 1 Solving a Polynomial Inequality

Solve the inequality $x^2 \leq 4x + 12$, and graph the solution set.

Algebraic Solution

STEP 1: Rearrange the inequality so that 0 is on the right side.

$$x^2 \leq 4x + 12$$

$$x^2 - 4x - 12 \leq 0 \qquad \text{Subtract } 4x + 12 \text{ from both sides of the inequality.}$$

This inequality is equivalent to the one we wish to solve.

STEP 2: Find the zeros of $f(x) = x^2 - 4x - 12$ by solving the equation $x^2 - 4x - 12 = 0$.

$$x^2 - 4x - 12 = 0$$

$$(x + 2)(x - 6) = 0 \qquad \text{Factor.}$$

$$x = -2 \quad \text{or} \quad x = 6$$

STEP 3: We use the zeros of f to separate the real number line into three intervals.

$$(-\infty, -2) \qquad (-2, 6) \qquad (6, \infty)$$

STEP 4: We select a number in each interval found in Step 3 and evaluate $f(x) = x^2 - 4x - 12$ at each number to determine if $f(x)$ is positive or negative. See Table 17.

Table 17

	−2	6	x
Interval	$(-\infty, -2)$	$(-2, 6)$	$(6, \infty)$
Number Chosen	−3	0	7
Value of f	$f(-3) = 9$	$f(0) = -12$	$f(7) = 9$
Conclusion	Positive	Negative	Positive

Since we want to know where $f(x)$ is negative, we conclude that $f(x) < 0$ for all x such that $-2 < x < 6$. However, because the original inequality is not strict, numbers x that satisfy the equation $x^2 = 4x + 12$ are also solutions of the inequality $x^2 \leq 4x + 12$. Thus, we include -2 and 6. The solution set of the given inequality is $\{x \mid -2 \leq x \leq 6\}$ or, using interval notation, $[-2, 6]$.

Graphing Solution

We graph $Y_1 = x^2$ and $Y_2 = 4x + 12$ on the same screen. See Figure 63. Using the INTERSECT command, we find that Y_1 and Y_2 intersect at $x = -2$ and at $x = 6$. The graph of Y_1 is below that of Y_2, $Y_1 < Y_2$, between the points of intersection. Since the inequality is not strict, the solution set is $\{x \mid -2 \leq x \leq 6\}$ or, using interval notation, $[-2, 6]$.

Figure 63

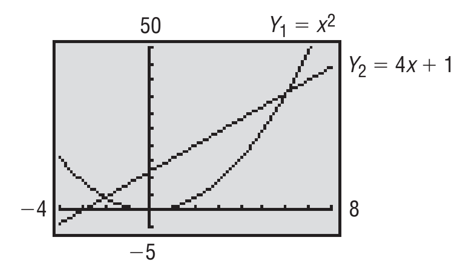


Figure 64

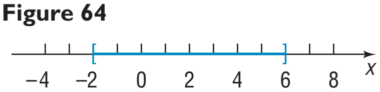


Figure 64 shows the graph of the solution set.

NOW WORK PROBLEMS 5 AND 9.

EXAMPLE 2 Solving a Polynomial Inequality

Solve the inequality $x^4 > x$, and graph the solution set.

Algebraic Solution

STEP 1: Rearrange the inequality so that 0 is on the right side.

$$x^4 > x$$

$$x^4 - x > 0 \quad \text{Subtract } x \text{ from both sides of the inequality.}$$

This inequality is equivalent to the one we wish to solve.

STEP 2: Find the zeros of $f(x) = x^4 - x$ by solving $x^4 - x = 0$.

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0 \quad \text{Factor out } x.$$

$$x(x - 1)(x^2 + x + 1) = 0 \quad \text{Factor the difference of two cubes.}$$

$$x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x^2 + x + 1 = 0 \quad \text{Set each factor equal to zero and solve.}$$

$$x = 0 \quad \text{or} \quad x = 1$$

The equation $x^2 + x + 1 = 0$ has no real solutions. (Do you see why?)

STEP 3: We use the zeros to separate the real number line into three intervals:

$$(-\infty, 0) \qquad (0, 1) \qquad (1, \infty)$$

STEP 4: We select a test number in each interval found in Step 3 and evaluate $f(x) = x^4 - x$ at each number to determine if $f(x)$ is positive or negative. See Table 18.

Table 18

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Number Chosen	-1	$\frac{1}{2}$	2
Value of f	$f(-1) = 2$	$f\left(\frac{1}{2}\right) = -\frac{7}{16}$	$f(2) = 14$
Conclusion	Positive	Negative	Positive

Since we want to know where $f(x)$ is positive, we conclude that $f(x) > 0$ for all numbers x for which $x < 0$ or $x > 1$. Because the original inequality is strict, numbers x that satisfy the equation $x^4 = x$ are not solutions. The solution set of the given inequality is $\{x|x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0)$ or $(1, \infty)$.

Graphing Solution

We graph $Y_1 = x^4$ and $Y_2 = x$ on the same screen. See Figure 65. Using the INTERSECT command, we find that Y_1 and Y_2 intersect at $x = 0$ and at $x = 1$. The graph of Y_1 is above that of Y_2, $Y_1 > Y_2$, to the left of $x = 0$ and to the right of $x = 1$. Since the inequality is strict, the solution set is $\{x|x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0)$ or $(1, \infty)$.

Figure 65

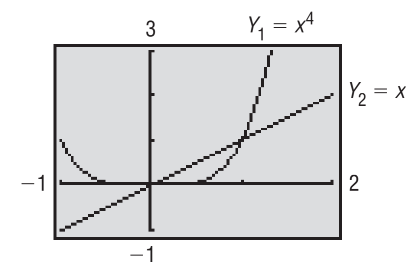


Figure 66

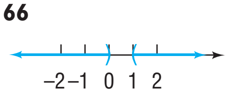


Figure 66 shows the graph of the solution set.

NOW WORK PROBLEM 21.

2 Solve Rational Inequalities Algebraically and Graphically

EXAMPLE 3 Solving a Rational Inequality

Solve the inequality $\dfrac{4x+5}{x+2} \geq 3$, and graph the solution set.

Algebraic Solution

STEP 1: We first note that the domain of the variable consists of all real numbers except -2. We rearrange terms so that 0 is on the right side.

$$\frac{4x+5}{x+2} - 3 \geq 0 \qquad \text{Subtract 3 from both sides of the inequality.}$$

STEP 2: For $f(x) = \dfrac{4x+5}{x+2} - 3$, find the zeros of f and the values of x at which f is undefined. To find these numbers, we must express f as a single quotient.

$$\begin{aligned} f(x) &= \frac{4x+5}{x+2} - 3 && \text{Least Common Denominator: } x+2 \\ &= \frac{4x+5}{x+2} - 3 \cdot \frac{x+2}{x+2} && \text{Multiply 3 by } \frac{x+2}{x+2} \\ &= \frac{4x+5-3x-6}{x+2} && \text{Write as a single quotient.} \\ &= \frac{x-1}{x+2} && \text{Combine like terms.} \end{aligned}$$

The zero of f is 1. Also, f is undefined for $x = -2$.

STEP 3: We use the numbers found in Step 2 to separate the real number line into three intervals.

$$(-\infty, -2) \qquad (-2, 1) \qquad (1, \infty)$$

STEP 4: We select a number in each interval found in Step 3 and evaluate $f(x) = \dfrac{4x+5}{x+2} - 3$ at each number to determine if $f(x)$ is positive or negative. See Table 19.

Table 19

Interval	$(\infty, -2)$	$(-2, 1)$	$(1, \infty)$
Number Chosen	-3	0	2
Value of f	$f(-3) = 4$	$f(0) = -\frac{1}{2}$	$f(2) = \frac{1}{4}$
Conclusion	Positive	Negative	Positive

We want to know where $f(x)$ is positive. We conclude that $f(x) > 0$ for all x such that $x < -2$ or $x > 1$. Because the original inequality is not strict, numbers x that satisfy the equation $\dfrac{x-1}{x+2} = 0$ are also solutions of the inequality. Since $\dfrac{x-1}{x+2} = 0$ only if $x = 1$, we conclude that the solution set is $\{x \mid x < -2 \text{ or } x \geq 1\}$ or, using interval notation, $(-\infty, -2)$ or $[1, \infty)$.

Figure 68

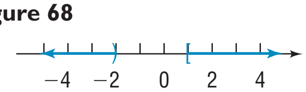
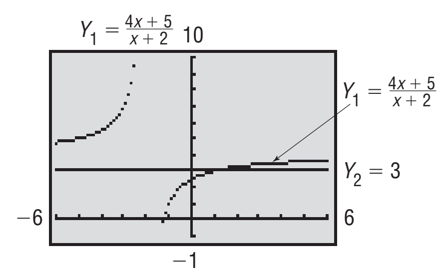


Figure 68 shows the graph of the solution set.

Graphing Solution

We first note that the domain of the variable consists of all real numbers except -2. We graph $Y_1 = \dfrac{4x+5}{x+2}$ and $Y_2 = 3$ on the same screen. See Figure 67. Using the INTERSECT command, we find that Y_1 and Y_2 intersect at $x = 1$. The graph of Y_1 is above that of Y_2, $Y_1 > Y_2$, to the left of $x = -2$ and to the right of $x = 1$. Since the inequality is not strict, the solution set is $\{x \mid x < -2 \text{ or } x \geq 1\}$. Using interval notation, the solution set is $(-\infty, -2)$ or $[1, \infty)$.

Figure 67



NOW WORK PROBLEM 35.

EXAMPLE 4 Minimum Sales Requirements

Table 20

Pounds of Cookies (in Hundreds), x	Profit, P
0	−20,000
50	−5990
75	412
120	10,932
200	26,583
270	36,948
340	44,381
420	49,638
525	49,225
610	44,381
700	34,220

Tami is considering leaving her \$30,000 a year job and buying a cookie company. According to the financial records of the firm, the relationship between pounds of cookies sold and profit is as exhibited by Table 20.

(a) Draw a scatter diagram of the data in Table 20 with the pounds of cookies sold as the independent variable.

(b) Use a graphing utility to find the quadratic function of best fit.

(c) Use the function found in part (b) to determine the number of pounds of cookies that Tami must sell for the profits to exceed \$30,000 a year and therefore make it worthwhile for her to quit her job.

(d) Using the function found in part (b), determine the number of pounds of cookies that Tami should sell to maximize profits.

(e) Using the function found in part (b), determine the maximum profit that Tami can expect to earn.

Solution

(a) Figure 69 shows the scatter diagram.

(b) Using a graphing utility, the quadratic function of best fit is

$$P(x) = -0.31x^2 + 295.86x - 20{,}042.52$$

where P is the profit achieved from selling x pounds (in hundreds) of cookies. See Figure 70.

(c) Since we want profit to exceed \$30,000 we need to solve the inequality

$$-0.31x^2 + 295.86x - 20{,}042.52 > 30{,}000$$

We graph

$$Y_1 = P(x) = -0.31x^2 + 295.86x - 20{,}042.52 \quad \text{and} \quad Y_2 = 30{,}000$$

on the same screen. See Figure 71.

Figure 69

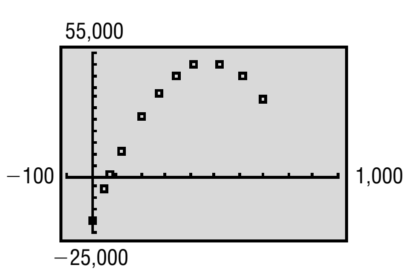


Figure 70

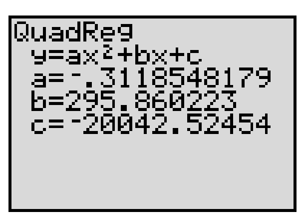
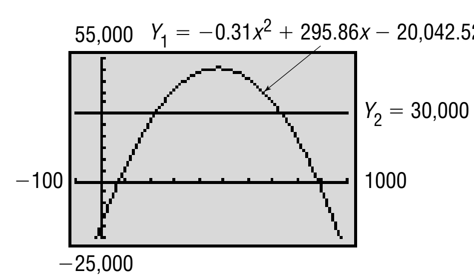


Figure 71



Using the INTERSECT command, we find that Y_1 and Y_2 intersect at $x = 220$ and at $x = 735$. The graph of Y_1 is above that of Y_2, $Y_1 > Y_2$, between the points of intersection. Tami must sell between 220 hundred or 22,000 and 735 hundred or 73,500 pounds of cookies to make it worthwhile to quit her job.

(d) The function $P(x)$ found in part (b) is a quadratic function whose graph opens down. The vertex is therefore the highest point. The x-coordinate of the vertex is

$$x = -\frac{b}{2a} = -\frac{295.86}{2(-0.31)} = 477$$

To maximize profit, 477 hundred (47,700) pounds of cookies must be sold.

(e) The maximum profit is

$$P(477) = -0.31(477)^2 + 295.86(477) - 20{,}042.52 = \$50{,}549$$ ◀

3.5 Assess Your Understanding

'Are You Prepared?'

Answer is given at the end of these exercises. If you get the wrong answer, read the pages listed in red.

1. Solve the inequality: $3 - 4x > 5$. Graph the solution set. (pp. 1024–1025)

Concepts and Vocabulary

2. *True or False:* A test number for the interval $-5 < x < 1$ is 0.

Skill Building

In Problems 3–56, solve each inequality algebraically. Verify your results using a graphing utility.

3. $(x - 5)(x + 2) < 0$

4. $(x - 5)(x + 2) > 0$

5. $x^2 - 4x > 0$

6. $x^2 + 8x > 0$

7. $x^2 - 9 < 0$

8. $x^2 - 1 < 0$

9. $x^2 + x \geq 12$

10. $x^2 + 7x \leq -12$

11. $2x^2 > 5x + 3$

12. $6x^2 > 6 + 5x$

13. $x(x - 7) < 8$

14. $x(x + 1) < 20$

15. $4x^2 + 9 < 6x$

16. $25x^2 + 16 < 40x$

17. $6(x^2 - 1 \geq 5x$

18. $2(2x^2 - 3x) \geq -9$

19. $(x - 1)(x^2 + x + 1) > 0$

20. $(x + 2)(x^2 - x + 1) > 0$

21. $(x - 1)(x - 2)(x - 3) < 0$

22. $(x + 1)(x + 2)(x + 3) < 0$

23. $x^3 - 2x^2 - 3x \geq 0$

24. $x^3 + 2x^2 - 3x \geq 0$

25. $x^4 > x^2$

26. $x^4 < 4x^2$

27. $x^3 > x^2$

28. $x^3 < 3x^2$

29. $x^4 > 1$

30. $x^3 > 1$

31. $x^2 - 7x - 8 < 0$

32. $x^2 + 12x + 32 \geq 0$

33. $x^4 - 3x^2 - 4 > 0$

34. $x^4 - 5x^2 + 6 < 0$

35. $\dfrac{x + 1}{x - 1} > 0$

36. $\dfrac{x - 3}{x + 1} > 0$

37. $\dfrac{(x - 1)(x + 1)}{x} < 0$

38. $\dfrac{(x - 3)(x + 2)}{x - 1} < 0$

39. $\dfrac{(x - 2)^2}{x^2 - 1} \geq 0$

40. $\dfrac{(x + 5)^2}{x^2 - 4} \geq 0$

41. $6x - 5 < \dfrac{6}{x}$

42. $x + \dfrac{12}{x} < 7$

43. $\dfrac{x + 4}{x - 2} \leq 1$

44. $\dfrac{x + 2}{x - 4} \geq 1$

45. $\dfrac{3x - 5}{x + 2} \leq 2$

46. $\dfrac{x - 4}{2x + 4} \geq 1$

47. $\dfrac{1}{x - 2} < \dfrac{2}{3x - 9}$

48. $\dfrac{5}{x - 3} > \dfrac{3}{x + 1}$

49. $\dfrac{2x + 5}{x + 1} > \dfrac{x + 1}{x - 1}$

50. $\dfrac{1}{x + 2} > \dfrac{3}{x + 1}$

51. $\dfrac{x^2(3 + x)(x + 4)}{(x + 5)(x - 1)} > 0$

52. $\dfrac{x(x^2 + 1)(x - 2)}{(x - 1)(x + 1)} > 0$

53. $\dfrac{2x^2 - x - 1}{x - 4} \leq 0$

54. $\dfrac{3x^2 + 2x - 1}{x + 2} > 0$

55. $\dfrac{x^2 + 3x - 1}{x + 3} > 0$

56. $\dfrac{x^2 - 5x + 3}{x - 5} < 0$

Applications and Extensions

57. For what positive numbers will the cube of a number exceed four times its square?

58. For what positive numbers will the square of a number exceed twice the number?

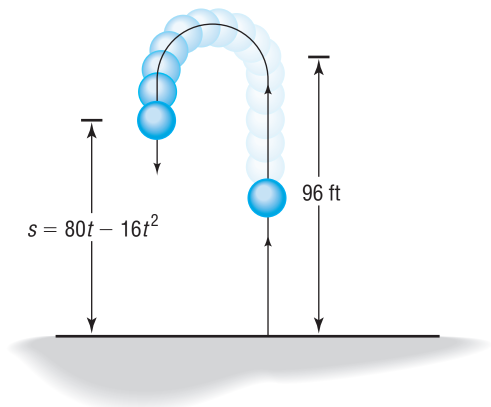
59. What is the domain of the function $f(x) = \sqrt{x^2 - 16}$?

60. What is the domain of the function $f(x) = \sqrt{x^3 - 3x^2}$?

61. What is the domain of the function $R(x) = \sqrt{\dfrac{x - 2}{x + 4}}$?

62. What is the domain of the function $R(x) = \sqrt{\dfrac{x - 1}{x + 4}}$?

63. Physics A ball is thrown vertically upward with an initial velocity of 80 feet per second. The distance s (in feet) of the ball from the ground after t seconds is $s = 80t - 16t^2$. See the figure.



(a) For what time interval is the ball more than 96 feet above the ground?
(b) Using a graphing utility, graph the relation between s and t.
(c) What is the maximum height of the ball?
(d) After how many seconds does the ball reach the maximum height?

64. Physics A ball is thrown vertically upward with an initial velocity of 96 feet per second. The distance s (in feet) of the ball from the ground after t seconds is $s = 96t - 16t^2$.
(a) For what time interval is the ball more than 112 feet above the ground?
(b) Using a graphing utility, graph the relation between s and t.
(c) What is the maximum height of the ball?
(d) After how many seconds does the ball reach the maximum height?

65. Business The monthly revenue achieved by selling x wristwatches is figured to be $x(40 - 0.2x)$ dollars. The wholesale cost of each watch is \$32.
(a) How many watches must be sold each month to achieve a profit (revenue $-$ cost) of at least \$50?
(b) Using a graphing utility, graph the revenue function.
(c) What is the maximum revenue that this firm could earn?
(d) How many wristwatches should the firm sell to maximize revenue?
(e) Using a graphing utility, graph the profit function.
(f) What is the maximum profit that this firm can earn?
(g) How many watches should the firm sell to maximize profit?
(h) Provide a reasonable explanation as to why the answers found in parts (d) and (g) differ. Is the shape of the revenue function reasonable in your opinion? Why?

66. Business The monthly revenue achieved by selling x boxes of candy is figured to be $x(5 - 0.05x)$ dollars. The wholesale cost of each box of candy is \$1.50.
(a) How many boxes must be sold each month to achieve a profit of at least \$60?
(b) Using a graphing utility, graph the revenue function.
(c) What is the maximum revenue that this firm could earn?
(d) How many boxes of candy should the firm sell to maximize revenue?
(e) Using a graphing utility, graph the profit function.
(f) What is the maximum profit that this firm can earn?
(g) How many boxes of candy should the firm sell to maximize profit?
(h) Provide a reasonable explanation as to why the answers found in parts (d) and (g) differ. Is the shape of the revenue function reasonable in your opinion? Why?

67. Cost of Manufacturing In Problem 95 of Section 3.2, a cubic function of best fit relating the cost C of manufacturing x Chevy Cavaliers was found. Budget constraints will not allow Chevy to spend more than \$97,000. Determine the number of Cavaliers that could be produced.

68. Cost of Printing In Problem 96 of Section 3.2, a cubic function of best fit relating the cost C of printing x textbooks in a week was found. Budget constraints will not allow the printer to spend more than \$170,000 per week. Determine the number of textbooks that could be printed in a week.

69. Prove that, if a, b are real numbers and $a \geq 0$, $b \geq 0$, then

$$a \leq b \quad \text{is equivalent to} \quad \sqrt{a} \leq \sqrt{b}$$

[Hint: $b - a = \left(\sqrt{b} - \sqrt{a}\right)\left(\sqrt{b} + \sqrt{a}\right)$.]

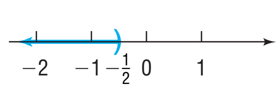
Discussion and Writing

70. Make up an inequality that has no solution. Make up one that has exactly one solution.

71. The inequality $x^2 + 1 < -5$ has no solution. Explain why.

'Are You Prepared?' Answer

1. $\left\{x \middle| x < -\dfrac{1}{2}\right\}$



3.6 The Real Zeros of a Polynomial Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Classification of Numbers (Appendix, Section A.1, p. 952)
- Factoring Polynomials (Appendix, Section A.3, pp. 969–971)
- Synthetic Division (Appendix, Section A.4, pp. 979–982)
- Polynomial Division (Appendix, Section A.4, pp. 977–979)
- Quadratic Formula (Appendix, Section A.5, pp. 992–995)

Now work the 'Are You Prepared?' problems on page 230.

OBJECTIVES
1 Use the Remainder and Factor Theorems
2 Use the Rational Zeros Theorem
3 Find the Real Zeros of a Polynomial Function
4 Solve Polynomial Equations
5 Use the Theorem for Bounds on Zeros
6 Use the Intermediate Value Theorem

In this section, we discuss techniques that can be used to find the real zeros of a polynomial function. Recall that if r is a real zero of a polynomial function f then $f(r) = 0$, r is an x-intercept of the graph of f, and r is a solution of the equation $f(x) = 0$. For polynomial and rational functions, we have seen the importance of the zeros for graphing. In most cases, however, the zeros of a polynomial function are difficult to find using algebraic methods. No nice formulas like the quadratic formula are available to help us find zeros for polynomials of degree 3 or higher. Formulas do exist for solving any third- or fourth-degree polynomial equation, but they are somewhat complicated. No general formulas exist for polynomial equations of degree 5 or higher. Refer to the Historical Feature at the end of this section for more information.

1 Use the Remainder and Factor Theorems

When we divide one polynomial (the dividend) by another (the divisor), we obtain a quotient polynomial and a remainder, the remainder being either the zero polynomial or a polynomial whose degree is less than the degree of the divisor. To check our work, we verify that

$$(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}$$

This checking routine is the basis for a famous theorem called the **division algorithm*** **for polynomials**, which we now state without proof.

Theorem

Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\boxed{\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\text{dividend}}{\underset{\uparrow}{f(x)}} = \underset{\text{quotient}}{\underset{\uparrow}{q(x)}}\,\underset{\text{divisor}}{\underset{\uparrow}{g(x)}} + \underset{\text{remainder}}{\underset{\uparrow}{r(x)}} \qquad \textbf{(1)}}$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

*A systematic process in which certain steps are repeated a finite number of times is called an **algorithm**. For example, long division is an algorithm.

In equation (1), $f(x)$ is the **dividend**, $g(x)$ is the **divisor**, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

If the divisor $g(x)$ is a first-degree polynomial of the form

$$g(x) = x - c, \qquad c \text{ a real number}$$

then the remainder $r(x)$ is either the zero polynomial or a polynomial of degree 0. As a result, for such divisors, the remainder is some number, say R, and we may write

$$f(x) = (x - c)q(x) + R \tag{2}$$

This equation is an identity in x and is true for all real numbers x. Suppose that $x = c$. Then equation (2) becomes

$$f(c) = (c - c)q(c) + R$$
$$f(c) = R$$

Substitute $f(c)$ for R in equation (2) to obtain

$$f(x) = (x - c)q(x) + f(c) \tag{3}$$

We have now proved the **Remainder Theorem**.

Remainder Theorem

Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

EXAMPLE 1 Using the Remainder Theorem

Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by

(a) $x - 3$ (b) $x + 2$

Solution

(a) We could use long division or synthetic division, but it is easier to use the Remainder Theorem, which says that the remainder is $f(3)$.

$$f(3) = (3)^3 - 4(3)^2 - 5 = 27 - 36 - 5 = -14$$

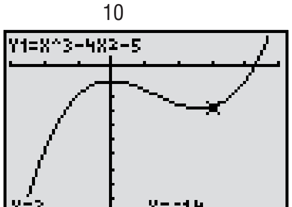
The remainder is -14.

(b) To find the remainder when $f(x)$ is divided by $x + 2 = x - (-2)$, we evaluate $f(-2)$.

$$f(-2) = (-2)^3 - 4(-2)^2 - 5 = -8 - 16 - 5 = -29$$

The remainder is -29. ◀

Figure 72

10


−3 5
−50

Compare the method used in Example 1(a) with the method used in Example 4 on page 981 (synthetic division). Which method do you prefer? Give reasons.

NOTE A graphing utility provides another way to find the value of a function, using the eVALUEate feature. Consult your manual for details. See Figure 72 for the results of Example 1(a). ■

An important and useful consequence of the Remainder Theorem is the **Factor Theorem**.

Factor Theorem

Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

The Factor Theorem actually consists of two separate statements:

1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

The proof requires two parts.

Proof

1. Suppose that $f(c) = 0$. Then, by equation (3), we have
$$\begin{aligned} f(x) &= (x - c)q(x) + f(c) \\ &= (x - c)q(x) + 0 \\ &= (x - c)q(x) \end{aligned}$$
for some polynomial $q(x)$. That is, $x - c$ is a factor of $f(x)$.
2. Suppose that $x - c$ is a factor of $f(x)$. Then there is a polynomial function q such that
$$f(x) = (x - c)q(x)$$
Replacing x by c, we find that
$$f(c) = (c - c)q(c) = 0 \cdot q(c) = 0$$
This completes the proof. ■

One use of the Factor Theorem is to determine whether a polynomial has a particular factor.

EXAMPLE 2 Using the Factor Theorem

Use the Factor Theorem to determine whether the function
$$f(x) = 2x^3 - x^2 + 2x - 3$$
has the factor

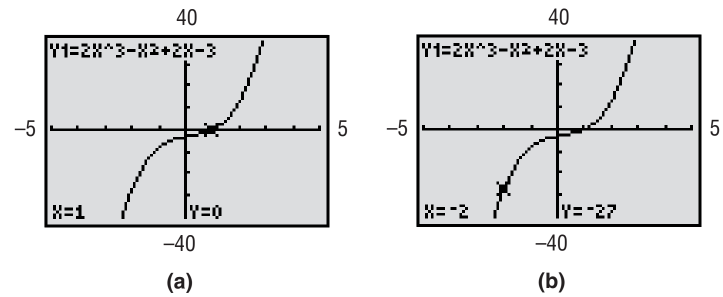
(a) $x - 1$ (b) $x + 2$

Solution The Factor Theorem states that if $f(c) = 0$ then $x - c$ is a factor.

(a) Because $x - 1$ is of the form $x - c$ with $c = 1$, we find the value of $f(1)$. We choose to use substitution.
$$f(1) = 2(1)^3 - (1)^2 + 2(1) - 3 = 2 - 1 + 2 - 3 = 0$$
See also Figure 73(a). By the Factor Theorem, $x - 1$ is a factor of $f(x)$.

(b) We first need to write $x + 2$ in the form $x - c$. Since $x + 2 = x - (-2)$, we find the value of $f(-2)$. See Figure 73(b). Because $f(-2) = -27 \neq 0$, we conclude from the Factor Theorem that $x - (-2) = x + 2$ is not a factor of $f(x)$.

Figure 73



◀

In Example 2, we found that $x - 1$ was a factor of f. To write f in factored form, we can use long division or synthetic division. Using synthetic division, we find that

$$\begin{array}{r|rrrr} 1 & 2 & -1 & 2 & -3 \\ & & 2 & 1 & 3 \\ \hline & 2 & 1 & 3 & 0 \end{array}$$

The quotient is $q(x) = 2x^2 + x + 3$ with a remainder of 0, as expected. We can write f in factored form as

$$f(x) = 2x^3 - x^2 + 2x - 3 = (x - 1)(2x^2 + x + 3)$$

NOW WORK PROBLEM 11.

The next theorem concerns the number of real zeros that a polynomial function may have. In counting the zeros of a polynomial, we count each zero as many times as its multiplicity.

Theorem

Number of Real Zeros

A polynomial function of degree n, $n \geq 1$, has at most n real zeros.

Proof The proof is based on the Factor Theorem. If r is a zero of a polynomial function f, then $f(r) = 0$ and, hence, $x - r$ is a factor of $f(x)$. Each zero corresponds to a factor of degree 1. Because f cannot have more first-degree factors than its degree, the result follows. ■

2 Use the Rational Zeros Theorem

The next result, called the **Rational Zeros Theorem**, provides information about the rational zeros of a polynomial *with integer coefficients*.

Theorem

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0, a_0 \neq 0}$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

EXAMPLE 3 Listing Potential Rational Zeros

List the potential rational zeros of

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

Solution Because f has integer coefficients, we may use the Rational Zeros Theorem. First, we list all the integers p that are factors of $a_0 = -6$ and all the integers q that are factors of the leading coefficient $a_3 = 2$.

$$\begin{aligned} p&: \quad \pm 1, \pm 2, \pm 3, \pm 6 \\ q&: \quad \pm 1, \pm 2 \end{aligned}$$

Now we form all possible ratios $\frac{p}{q}$.

$$\frac{p}{q}: \quad \pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

If f has a rational zero, it will be found in this list, which contains 12 possibilities. ◄

NOW WORK PROBLEM 21.

Be sure that you understand what the Rational Zeros Theorem says: For a polynomial with integer coefficients, *if* there is a rational zero, it is one of those listed. The Rational Zeros Theorem does not say that if a rational number is in the list of potential rational zeros, then it is a zero. It may be the case that the function does not have any rational zeros.

The Rational Zeros Theorem provides a list of potential rational zeros of a function f. If we graph f, we can get a better sense of the location of the x-intercepts and test to see if they are rational. We can also use the potential rational zeros to select our initial viewing window to graph f and then adjust the window based on the results. The graphs shown throughout the text will be those obtained after setting the final viewing window.

3 Find the Real Zeros of a Polynomial Function

EXAMPLE 4 Finding the Rational Zeros of a Polynomial Function

Continue working with Example 3 to find the rational zeros of

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

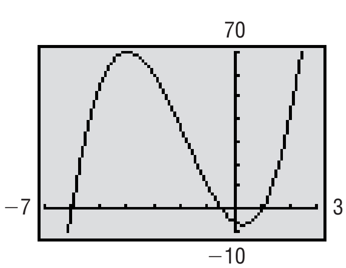
Solution We gather all the information that we can about the zeros.

STEP 1: Since f is a polynomial of degree 3, there are at most three real zeros.

STEP 2: We list the potential rational zeros obtained in Example 3:

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}.$$

Figure 74



STEP 3: We could, of course, use the Factor Theorem to test each potential rational zero to see if the value of f there is zero. This is not very efficient. The graph of f will tell us approximately where the real zeros are. So we only need to test those rational zeros that are nearby. Figure 74 shows the graph of f. We see that f has three zeros: one near -6, one between -1 and 0, and one near 1. From our original list of potential rational zeros, we will test -6, using eVALUEate.

STEP 4: Because $f(-6) = 0$, we know that -6 is a zero and $x + 6$ is a factor of f. We can use long division or synthetic division to factor f.

$$\begin{aligned} f(x) &= 2x^3 + 11x^2 - 7x - 6 \\ &= (x + 6)(2x^2 - x - 1) \end{aligned}$$

Now any solution of the equation $2x^2 - x - 1 = 0$ will be a zero of f. Because of this, we call the equation $2x^2 - x - 1 = 0$ a **depressed equation** of f. Since the degree of the depressed equation of f is less than that of the original polynomial, we work with the depressed equation to find the zeros of f.

The depressed equation $2x^2 - x - 1 = 0$ is a quadratic equation with discriminant $b^2 - 4ac = (-1)^2 - 4(2)(-1) = 9 > 0$. The equation has two real solutions, which can be found by factoring.

$$2x^2 - x - 1 = (2x + 1)(x - 1) = 0$$
$$2x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -\frac{1}{2} \quad \text{or} \quad x = 1$$

The zeros of f are $-6, -\frac{1}{2}$, and 1.

Because $f(x) = (x + 6)(2x^2 - x - 1)$, we completely factor f as follows:

$$f(x) = 2x^3 + 11x^2 - 7x - 6 = (x + 6)(2x^2 - x - 1) = (x + 6)(2x + 1)(x - 1)$$

Notice that the three zeros of f are among those given in the list of potential rational zeros in Example 3. ◀

Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of zeros.

STEP 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

STEP 3: Using a graphing utility, graph the polynomial function.

STEP 4: (a) Use eVALUEate, substitution, synthetic division, or long division to test a potential rational zero based on the graph.
(b) Each time that a zero (and thus a factor) is found, repeat Step 4 on the depressed equation. In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

EXAMPLE 5 Finding the Real Zeros of a Polynomial Function

Find the real zeros of $f(x) = x^5 - x^4 - 4x^3 + 8x^2 - 32x + 48$. Write f in factored form.

Solution

STEP 1: There are at most five real zeros.

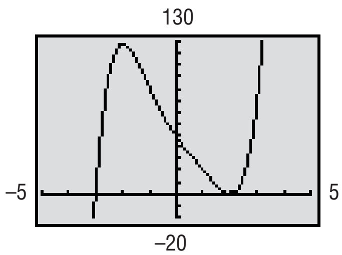
STEP 2: To obtain the list of potential rational zeros, we write the factors p of $a_0 = 48$ and the factors q of the leading coefficient $a_5 = 1$.

$$p: \quad \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, \pm24, \pm48$$
$$q: \quad \pm1$$

The potential rational zeros consist of all possible quotients $\frac{p}{q}$:

$$\frac{p}{q}: \quad \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, \pm24, \pm48$$

Figure 75



Step 3: Figure 75 shows the graph of f. The graph has the characteristics that we expect of the given polynomial of degree 5: no more than four turning points, y-intercept 48, and it behaves like $y = x^5$ for large $|x|$.

Step 4: Since -3 appears to be a zero and -3 is a potential rational zero, we eVALUEate f at -3 and find that $f(-3) = 0$. We use synthetic division to factor f.

$$\begin{array}{r|rrrrrr} -3 & 1 & -1 & -4 & 8 & -32 & 48 \\ & & -3 & 12 & -24 & 48 & -48 \\ \hline & 1 & -4 & 8 & -16 & 16 & 0 \end{array}$$

We can factor f as

$$f(x) = x^5 - x^4 - 4x^3 + 8x^2 - 32x + 48 = (x + 3)(x^4 - 4x^3 + 8x^2 - 16x + 16)$$

We now work with the first depressed equation:

$$q_1(x) = x^4 - 4x^3 + 8x^2 - 16x + 16 = 0$$

Repeat Step 4: In looking back at Figure 75, it appears that 2 might be a zero of even multiplicity. We check the potential rational zero 2 and find that $f(2) = 0$. Using synthetic division,

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 8 & -16 & 16 \\ & & 2 & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array}$$

Now we can factor f as

$$f(x) = (x + 3)(x - 2)(x^3 - 2x^2 + 4x - 8)$$

Repeat Step 4: The depressed equation $q_2(x) = x^3 - 2x^2 + 4x - 8 = 0$ can be factored by grouping.

$$x^3 - 2x^2 + 4x - 8 = (x^3 - 2x^2) + (4x - 8) = x^2(x - 2) + 4(x - 2) = (x - 2)(x^2 + 4) = 0$$

$$x - 2 = 0 \quad \text{or} \quad x^2 + 4 = 0$$

$$x = 2$$

Since $x^2 + 4 = 0$ has no real solutions, the real zeros of f are -3 and 2, with 2 being a zero of multiplicity 2. The factored form of f is

$$\begin{aligned} f(x) &= x^5 - x^4 - 4x^3 + 8x^2 - 32x + 48 \\ &= (x + 3)(x - 2)^2(x^2 + 4) \end{aligned}$$

◀

NOW WORK PROBLEM 39.

4 Solve Polynomial Equations

EXAMPLE 6 Solving a Polynomial Equation

Solve the equation: $x^5 - x^4 - 4x^3 + 8x^2 - 32x + 48 = 0$

Solution The solutions of this equation are the zeros of the polynomial function

$$f(x) = x^5 - x^4 - 4x^3 + 8x^2 - 32x + 48$$

Using the result of Example 5, the real zeros are -3 and 2. These are the real solutions of the equation $x^5 - x^4 - 4x^3 + 8x^2 - 32x + 48 = 0$. ◀

NOW WORK PROBLEM 63.

In Example 5, the quadratic factor $x^2 + 4$ that appears in the factored form of $f(x)$ is called *irreducible*, because the polynomial $x^2 + 4$ cannot be factored over the real numbers. In general, we say that a quadratic factor $ax^2 + bx + c$ is **irreducible** if it cannot be factored over the real numbers, that is, if it is prime over the real numbers.

Refer back to Examples 4 and 5. The polynomial function of Example 4 has three real zeros, and its factored form contains three linear factors. The polynomial function of Example 5 has two distinct real zeros, and its factored form contains two distinct linear factors and one irreducible quadratic factor.

Theorem

Every polynomial function (with real coefficients) can be uniquely factored into a product of linear factors and/or irreducible quadratic factors.

We shall prove this result in Section 3.7, and, in fact, we shall draw several additional conclusions about the zeros of a polynomial function. One conclusion is worth noting now. If a polynomial (with real coefficients) is of odd degree, then it must contain at least one linear factor. (Do you see why?) This means that it must have at least one real zero.

COROLLARY

A polynomial function (with real coefficients) of odd degree has at least one real zero.

5 Use the Theorem for Bounds on Zeros

One challenge in using a graphing utility is to set the viewing window so that a complete graph is obtained. The next theorem is a tool that can be used to find bounds on the zeros. This will assure that the function does not have any zeros outside these bounds. Then using these bounds to set Xmin and Xmax assures that all the x-intercepts appear in the viewing window.

A number M is a **bound** on the zeros of a polynomial if every zero r lies between $-M$ and M, inclusive. That is, M is a bound to the zeros of a polynomial f if

$$-M \leq \text{any zero of } f \leq M$$

Theorem

Bounds on Zeros

Let f denote a polynomial function whose leading coefficient is 1.

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

A bound M on the zeros of f is the smaller of the two numbers

$$\boxed{\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}, \quad 1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} \quad \textbf{(4)}}$$

where Max { } means "choose the largest entry in { }."

An example will help to make the theorem clear.

EXAMPLE 7 Using the Theorem for Finding Bounds on Zeros

Find a bound to the zeros of each polynomial.

(a) $f(x) = x^5 + 3x^3 - 9x^2 + 5$ (b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1$

Solution (a) The leading coefficient of f is 1.

$$f(x) = x^5 + 3x^3 - 9x^2 + 5 \qquad a_4 = 0, a_3 = 3, a_2 = -9, a_1 = 0, a_0 = 5$$

We evaluate the expressions in formula (4).

$$\begin{aligned} \text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} &= \text{Max}\{1, |5| + |0| + |-9| + |3| + |0|\} \\ &= \text{Max}\{1, 17\} = 17 \\ 1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} &= 1 + \text{Max}\{|5|, |0|, |-9|, |3|, |0|\} \\ &= 1 + 9 = 10 \end{aligned}$$

The smaller of the two numbers, 10, is the bound. Every zero of f lies between -10 and 10.

(b) First we write g so that its leading coefficient is 1.

$$g(x) = 4x^5 - 2x^3 + 2x^2 + 1 = 4\left(x^5 - \frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{4}\right)$$

Next we evaluate the two expressions in formula (4) with $a_4 = 0, a_3 = -\frac{1}{2}$, $a_2 = \frac{1}{2}$, $a_1 = 0$, and $a_0 = \frac{1}{4}$.

$$\begin{aligned} \text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} &= \text{Max}\left\{1, \left|\frac{1}{4}\right| + |0| + \left|\frac{1}{2}\right| + \left|-\frac{1}{2}\right| + |0|\right\} \\ &= \text{Max}\left\{1, \frac{5}{4}\right\} = \frac{5}{4} \\ 1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} &= 1 + \text{Max}\left\{\left|\frac{1}{4}\right|, |0|, \left|\frac{1}{2}\right|, \left|-\frac{1}{2}\right|, |0|\right\} \\ &= 1 + \frac{1}{2} = \frac{3}{2} \end{aligned}$$

The smaller of the two numbers, $\frac{5}{4}$, is the bound. Every zero of g lies between $-\frac{5}{4}$ and $\frac{5}{4}$.

◀

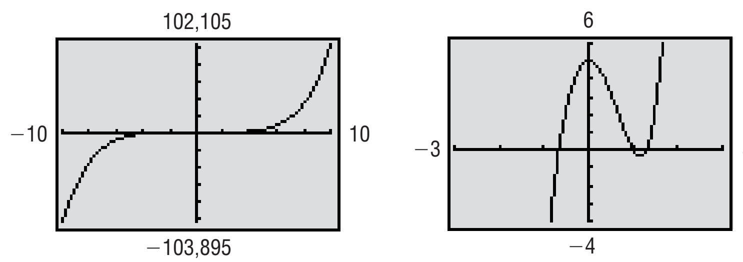
EXAMPLE 8 Obtaining Graphs Using Bounds on Zeros

Obtain a graph for each polynomial.

(a) $f(x) = x^5 + 3x^3 - 9x^2 + 5$ (b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1$

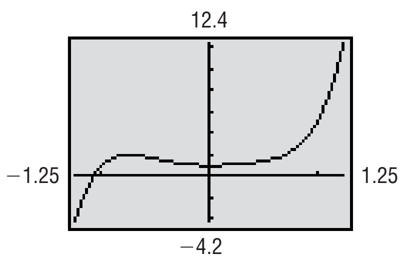
Solution (a) Based on Example 7(a), every zero lies between -10 and 10. Using $X\text{min} = -10$ and $X\text{max} = 10$, we graph $Y_1 = f(x) = x^5 + 3x^3 - 9x^2 + 5$. Figure 76(a) shows the graph obtained using ZOOM-FIT. Figure 76(b) shows the graph after adjusting the viewing window to improve the graph.

Figure 76



(a) **(b)**

Figure 77



(b) Based on Example 7(b), every zero lies between $-\frac{5}{4}$ and $\frac{5}{4}$. Using $X\text{min} = -\frac{5}{4}$ and $X\text{max} = \frac{5}{4}$, we graph $Y_1 = g(x) = 4x^5 - 2x^3 + 2x^2 + 1$. Figure 77 shows the graph after using ZOOM-FIT. Here no adjustment of the viewing window is needed. ◀

NOW WORK PROBLEM 33.

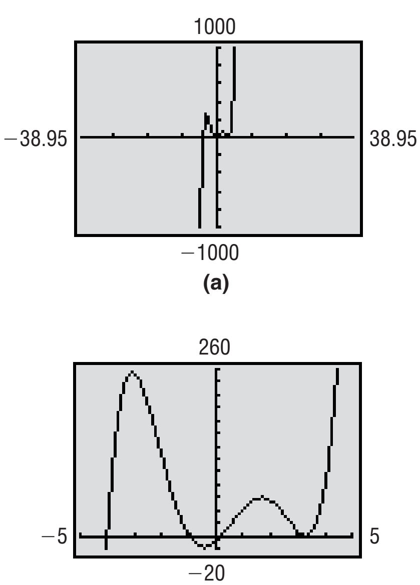
The next example shows how to proceed when some of the coefficients of the polynomial are not integers.

EXAMPLE 9 Finding the Zeros of a Polynomial

Find all the real zeros of the polynomial function

$$f(x) = x^5 - 1.8x^4 - 17.79x^3 + 31.672x^2 + 37.95x - 8.7121$$

Figure 78



(b)

Solution

Step 1: There are at most five real zeros.

Step 2: Since there are noninteger coefficients, the Rational Zeros Theorem does not apply.

Step 3: We determine the bounds of f. The leading coefficient of f is 1 with $a_4 = -1.8$, $a_3 = -17.79$, $a_2 = 31.672$, $a_1 = 37.95$, and $a_0 = -8.7121$. We evaluate the expressions using formula (4).

$$\begin{aligned}\text{Max}\{1, |-8.7121| + |37.95| + |31.672| + |-17.79| + |-1.8|\} &= \text{Max}\{1, 97.9241\}\\ &= 97.9241\\ 1 + \text{Max}\{|-8.7121|, |37.95|, |31.672|, |-17.79|, |-1.8|\} &= 1 + 37.95\\ &= 38.95\end{aligned}$$

The smaller of the two numbers, 38.95, is the bound. Every real zero of f lies between -38.95 and 38.95. Figure 78(a) shows the graph of f with $X\text{min} = -38.95$ and $X\text{max} = 38.95$. Figure 78(b) shows a graph of f after adjusting the viewing window to improve the graph.

Step 4: From Figure 78(b), we see that f appears to have four x-intercepts: one near -4, one near -1, one between 0 and 1, and one near 3. The x-intercept near 3 might be a zero of even multiplicity since the graph seems to touch the x-axis at that point.

We use the Factor Theorem to determine if -4 and -1 are zeros. Since $f(-4) = f(-1) = 0$, we know that -4 and -1 are zeros. Using ZERO (or ROOT), we find that the remaining zeros are 0.20 and 3.30, rounded to two decimal places.

There are no real zeros on the graph that have not already been identified. So, either 3.30 is a zero of multiplicity 2 or there are two distinct zeros, each of which is 3.30, rounded to two decimal places. (Example 10 will explain how to determine which is true.) ◀

6 Use the Intermediate Value Theorem

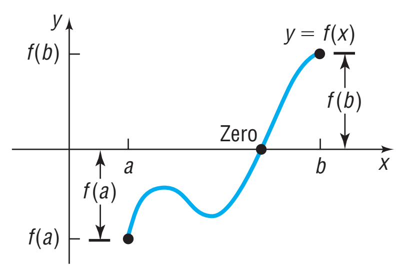
The Intermediate Value Theorem requires that the function be *continuous*. Although calculus is needed to explain the meaning precisely, the *idea* of a continuous function is easy to understand. Very basically, a function f is continuous when its graph can be drawn without lifting pencil from paper, that is, when the graph contains no holes or jumps or gaps. For example, every polynomial function is continuous.

Intermediate Value Theorem

Let f denote a continuous function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, then f has at least one zero between a and b.

Although the proof of this result requires advanced methods in calculus, it is easy to "see" why the result is true. Look at Figure 79.

Figure 79 If $f(a) < 0$ and $f(b) > 0$ and if f is continuous, there is at least one zero between a and b.



The Intermediate Value Theorem together with the TABLE feature of a graphing utility provides a basis for finding zeros.

EXAMPLE 10 Using the Intermediate Value Theorem and a Graphing Utility to Locate Zeros

Continue working with Example 9 to determine whether there is a repeated zero or two distinct zeros near 3.30.

Table 21

X	Y1
3.27	.08567
3.28	.03829
3.29	.00956
3.3	-1E-4
3.31	.0097
3.32	.03936
3.33	.08931

Y1=X^5-1.8X^4-1…

Solution We use the TABLE feature of a graphing utility. See Table 21. Since $f(3.29) = 0.00956 > 0$ and $f(3.30) = -0.0001 < 0$, by the Intermediate Value Theorem there is a zero between 3.29 and 3.30. Similarly, since $f(3.30) = -0.0001 < 0$ and $f(3.31) = 0.0097 > 0$, there is another zero between 3.30 and 3.31. Now we know that the five zeros of f are distinct. ◀

NOW WORK PROBLEM 75.

Historical Feature

Tartaglia
(1500–1557)

Formulas for the solution of third- and fourth-degree polynomial equations exist, and, while not very practical, they do have an interesting history.

In the 1500s in Italy, mathematical contests were a popular pastime, and persons possessing methods for solving problems kept them secret. (Solutions that were published were already common knowledge.) Niccolo of Brescia (1500–1557), commonly referred to as Tartaglia ("the stammerer"), had the secret for solving cubic (third-degree) equations, which gave him a decided advantage in the contests. See the Historical Problems.

Girolamo Cardano (1501–1576) found out that Tartaglia had the secret, and, being interested in cubics, he requested it from Tartaglia. The reluctant Tartaglia hesitated for some time, but finally, swearing Cardano to secrecy with midnight oaths by candlelight, told him the secret. Cardano then published the solution in his book *Ars Magna* (1545), giving Tartaglia the credit but rather compromising the secrecy. Tartaglia exploded into bitter recriminations, and each wrote pamphlets that reflected on the other's mathematics, moral character, and ancestry. See the Historical Problems.

The quartic (fourth-degree) equation was solved by Cardano's student Lodovico Ferrari, and this solution also was included, with credit and this time with permission, in the *Ars Magna*.

Attempts were made to solve the fifth-degree equation in similar ways, all of which failed. In the early 1800s, P. Ruffini, Niels Abel, and Evariste Galois all found ways to show that it is not possible to solve fifth-degree equations by formula, but the proofs required the introduction of new methods. Galois's methods eventually developed into a large part of modern algebra.

Historical Problems

Problems 1–8 develop the Tartaglia–Cardano solution of the cubic equation and show why it is not altogether practical.

1. Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - \frac{b}{3}$.
2. In the equation $x^3 + px + q = 0$, replace x by $H + K$. Let $3HK = -p$, and show that $H^3 + K^3 = -q$.
[**Hint:** $3H^2K + 3HK^2 = 3HKx$.]
3. Based on Problem 2, we have the two equations
$$3HK = -p \quad \text{and} \quad H^3 + K^3 = -q$$
Solve for K in $3HK = -p$ and substitute into $H^3 + K^3 = -q$. Then show that
$$H = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
[**Hint:** Look for an equation that is quadratic in form.]
4. Use the solution for H from Problem 3 and the equation $H^3 + K^3 = -q$ to show that
$$K = \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
5. Use the results from Problems 2–4 to show that the solution of $x^3 + px + q = 0$ is
$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
6. Use the result of Problem 5 to solve the equation $x^3 - 6x - 9 = 0$.
7. Use a calculator and the result of Problem 5 to solve the equation $x^3 + 3x - 14 = 0$.
8. Use the methods of this chapter to solve the equation $x^3 + 3x - 14 = 0$.

3.6 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. In the set $\left\{-2, -\sqrt{2}, 0, \frac{1}{2}, 4.5, \pi\right\}$, name the numbers that are integers. Which are rational numbers? (p. 952)
2. Factor the expression $6x^2 + x - 2$. (pp. 969–971)
3. Find the quotient and remainder if $3x^4 - 5x^3 + 7x - 4$ is divided by $x - 3$. (pp. 977–979 and pp. 979–982)
4. Solve the equation $x^2 + x - 3 = 0$. (pp. 992–995)

Concepts and Vocabulary

5. In the process of polynomial division, (Divisor)(Quotient) + _____ = _____.

6. When a polynomial function f is divided by $x - c$, the remainder is _____.

7. If a function f, whose domain is all real numbers, is even and if 4 is a zero of f, then _____ is also a zero.

8. *True or False:* Every polynomial function of degree 3 with real coefficients has exactly three real zeros.

9. *True or False:* The only potential rational zeros of $f(x) = 2x^5 - x^3 + x^2 - x + 1$ are $\pm 1, \pm 2$.

10. *True or False:* If f is a polynomial function of degree 4 and if $f(2) = 5$, then

$$\frac{f(x)}{x-2} = p(x) + \frac{5}{x-2}$$

where $p(x)$ is a polynomial of degree 3.

Skill Building

In Problems 11–20, use the Factor Theorem to determine whether $x - c$ is a factor of f. If it is, write f in factored form, that is, write f in the form $f(x) = (x - c)$ (quotient).

11. $f(x) = 4x^3 - 3x^2 - 8x + 4;\quad c = 3$

12. $f(x) = -4x^3 + 5x^2 + 8;\quad c = -2$

13. $f(x) = 3x^4 - 6x^3 - 5x + 10;\quad c = 1$

14. $f(x) = 4x^4 - 15x^2 - 4;\quad c = 2$

15. $f(x) = 3x^6 + 2x^3 - 176;\quad c = -2$

16. $f(x) = 2x^6 - 18x^4 + x^2 - 9;\quad c = -3$

17. $f(x) = 4x^6 - 64x^4 + x^2 - 16;\quad c = 4$

18. $f(x) = x^6 - 16x^4 + x^2 - 16;\quad c = -4$

19. $f(x) = 2x^4 - x^3 + 2x - 1;\quad c = -\frac{1}{2}$

20. $f(x) = 3x^4 + x^3 - 3x + 1;\quad c = -\frac{1}{3}$

In Problems 21–32, tell the maximum number of real zeros that each polynomial function may have. Then list the potential rational zeros of each polynomial function. Do not attempt to find the zeros.

21. $f(x) = 3x^4 - 3x^3 + x^2 - x + 1$

22. $f(x) = x^5 - x^4 + 2x^2 + 3$

23. $f(x) = x^5 - 6x^2 + 9x - 3$

24. $f(x) = 2x^5 - x^4 - x^2 + 1$

25. $f(x) = -4x^3 - x^2 + x + 2$

26. $f(x) = 6x^4 - x^2 + 2$

27. $f(x) = 3x^4 - x^2 + 2$

28. $f(x) = -4x^3 + x^2 + x + 2$

29. $f(x) = 2x^5 - x^3 + 2x^2 + 4$

30. $f(x) = 3x^5 - x^2 + 2x + 3$

31. $f(x) = 6x^4 + 2x^3 - x^2 + 2$

32. $f(x) = -6x^3 - x^2 + x + 3$

In Problems 33–38, find the bounds to the zeros of each polynomial function. Use the bounds to obtain a complete graph of f.

33. $f(x) = 2x^3 + x^2 - 1$

34. $f(x) = 3x^3 - 2x^2 + x + 4$

35. $f(x) = x^3 - 5x^2 - 11x + 11$

36. $f(x) = 2x^3 - x^2 - 11x - 6$

37. $f(x) = x^4 + 3x^3 - 5x^2 + 9$

38. $f(x) = 4x^4 - 12x^3 + 27x^2 - 54x + 81$

In Problems 39–56, find the real zeros of f. Use the real zeros to factor f.

39. $f(x) = x^3 + 2x^2 - 5x - 6$

40. $f(x) = x^3 + 8x^2 + 11x - 20$

41. $f(x) = 2x^3 - 13x^2 + 24x - 9$

42. $f(x) = 2x^3 - 5x^2 - 4x + 12$

43. $f(x) = 3x^3 + 4x^2 + 4x + 1$

44. $f(x) = 3x^3 - 7x^2 + 12x - 28$

45. $f(x) = x^3 - 8x^2 + 17x - 6$

46. $f(x) = x^3 + 6x^2 + 6x - 4$

47. $f(x) = x^4 + x^3 - 3x^2 - x + 2$

48. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

49. $f(x) = 2x^4 + 17x^3 + 35x^2 - 9x - 45$

50. $f(x) = 4x^4 - 15x^3 - 8x^2 + 15x + 4$

51. $f(x) = 2x^4 - 3x^3 - 21x^2 - 2x + 24$

52. $f(x) = 2x^4 + 11x^3 - 5x^2 - 43x + 35$

53. $f(x) = 4x^4 + 7x^2 - 2$

54. $f(x) = 4x^4 + 15x^2 - 4$

55. $f(x) = 4x^5 - 8x^4 - x + 2$

56. $f(x) = 4x^5 + 12x^4 - x - 3$

In Problems 57–62, find the real zeros of f rounded to two decimal places.

57. $f(x) = x^3 + 3.2x^2 - 16.83x - 5.31$

58. $f(x) = x^3 + 3.2x^2 - 7.25x - 6.3$

59. $f(x) = x^4 - 1.4x^3 - 33.71x^2 + 23.94x + 292.41$

60. $f(x) = x^4 + 1.2x^3 - 7.46x^2 - 4.692x + 15.2881$

61. $f(x) = x^3 + 19.5x^2 - 1021x + 1000.5$

62. $f(x) = x^3 + 42.2x^2 - 664.8x + 1490.4$

In Problems 63–72, find the real solutions of each equation.

63. $x^4 - x^3 + 2x^2 - 4x - 8 = 0$

64. $2x^3 + 3x^2 + 2x + 3 = 0$

65. $3x^3 + 4x^2 - 7x + 2 = 0$

66. $2x^3 - 3x^2 - 3x - 5 = 0$

67. $3x^3 - x^2 - 15x + 5 = 0$

68. $2x^3 - 11x^2 + 10x + 8 = 0$

69. $x^4 + 4x^3 + 2x^2 - x + 6 = 0$

70. $x^4 - 2x^3 + 10x^2 - 18x + 9 = 0$

71. $x^3 - \frac{2}{3}x^2 + \frac{8}{3}x + 1 = 0$

72. $x^3 - \frac{2}{3}x^2 + 3x - 2 = 0$

In Problems 73–78, use the Intermediate Value Theorem to show that each function has a zero in the given interval. Approximate the zero rounded to two decimal places.

73. $f(x) = 8x^4 - 2x^2 + 5x - 1;\quad [0, 1]$

74. $f(x) = x^4 + 8x^3 - x^2 + 2;\quad [-1, 0]$

75. $f(x) = 2x^3 + 6x^2 - 8x + 2;\quad [-5, -4]$

76. $f(x) = 3x^3 - 10x + 9;\quad [-3, -2]$

77. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 - 18x + 18;\quad [1.4, 1.5]$

78. $f(x) = x^5 - 3x^4 - 2x^3 + 6x^2 + x + 2;\quad [1.7, 1.8]$

Applications and Extensions

79. **Cost of Manufacturing** In Problem 95 of Section 3.2, you found the cost function for manufacturing Chevy Cavaliers. Use the methods learned in this section to determine how many Cavaliers can be manufactured at a total cost of \$3,000,000. In other words, solve the equation $C(x) = 3000$.

80. **Cost of Printing** In Problem 96 of Section 3.2, you found the cost function for printing textbooks. Use the methods learned in this section to determine how many textbooks can be printed at a total cost of \$200,000. In other words, solve the equation $C(x) = 200$.

81. Find k such that $f(x) = x^3 - kx^2 + kx + 2$ has the factor $x - 2$.

82. Find k such that $f(x) = x^4 - kx^3 + kx^2 + 1$ has the factor $x + 2$.

83. What is the remainder when $f(x) = 2x^{20} - 8x^{10} + x - 2$ is divided by $x - 1$?

84. What is the remainder when $f(x) = -3x^{17} + x^9 - x^5 + 2x$ is divided by $x + 1$?

85. One solution of the equation $x^3 - 8x^2 + 16x - 3 = 0$ is 3. Find the sum of the remaining solutions.

86. One solution of the equation $x^3 + 5x^2 + 5x - 2 = 0$ is -2. Find the sum of the remaining solutions.

87. What is the length of the edge of a cube if, after a slice 1 inch thick is cut from one side, the volume remaining is 294 cubic inches?

88. What is the length of the edge of a cube if its volume could be doubled by an increase of 6 centimeters in one edge, an increase of 12 centimeters in a second edge, and a decrease of 4 centimeters in the third edge?

89. Use the Factor Theorem to prove that $x - c$ is a factor of $x^n - c^n$ for any positive integer n.

90. Use the Factor Theorem to prove that $x + c$ is a factor of $x^n + c^n$ if $n \geq 1$ is an odd integer.

91. Let $f(x)$ be a polynomial function whose coefficients are integers. Suppose that r is a real zero of f and that the leading coefficient of f is 1. Use the Rational Zeros Theorem to show that r is either an integer or an irrational number.

92. Prove the Rational Zeros Theorem.
[**Hint:** Let $\frac{p}{q}$, where p and q have no common factors except 1 and -1, be a zero of the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, whose coefficients are all integers. Show that $a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n = 0$. Now, because p is a factor of the first n terms of this equation, p must also be a factor of the term $a_0 q^n$. Since p is not a factor of q (why?), p must be a factor of a_0. Similarly, q must be a factor of a_n.]

Discussion and Writing

93. Is $\frac{1}{3}$ a zero of $f(x) = 2x^3 + 3x^2 - 6x + 7$? Explain.

94. Is $\frac{1}{3}$ a zero of $f(x) = 4x^3 - 5x^2 - 3x + 1$? Explain.

95. Is $\frac{3}{5}$ a zero of $f(x) = 2x^6 - 5x^4 + x^3 - x + 1$? Explain.

96. Is $\frac{2}{3}$ a zero of $f(x) = x^7 + 6x^5 - x^4 + x + 2$? Explain.

'Are You Prepared?' Answers

1. Integers: $\{-2, 0\}$; Rational $\left\{-2, 0, \frac{1}{2}, 4.5\right\}$

2. $(3x + 2)(2x - 1)$

3. Quotient: $3x^3 + 4x^2 + 12x + 43$; Remainder: 125

4. $\left\{\frac{-1 - \sqrt{13}}{2}, \frac{-1 + \sqrt{13}}{2}\right\}$

3.7 Complex Zeros; Fundamental Theorem of Algebra

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Complex Numbers (Appendix, Section A.6, pp. 1000–1005)
- Quadratic Equations with a Negative Discriminant (Appendix, Section A.6, pp. 1005–1007)

Now work the 'Are You Prepared?' problems on page 237.

OBJECTIVES
1 Use the Conjugate Pairs Theorem
2 Find a Polynomial Function with Specified Zeros
3 Find the Complex Zeros of a Polynomial

In Section 3.6 we found the **real** zeros of a polynomial function. In this section we will find the **complex** zeros of a polynomial function. Finding the complex zeros of a function requires finding all zeros of the form $a + bi$. These zeros will be real if $b = 0$.

A variable in the complex number system is referred to as a **complex variable**.

A **complex polynomial function** f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers, $a_n \neq 0$, n is a nonnegative integer, and x is a complex variable. As before, a_n is called the **leading coefficient** of f. A complex number r is called a **(complex) zero** of f if $f(r) = 0$.

We have learned that some quadratic equations have no real solutions, but that in the complex number system every quadratic equation has a solution, either real or complex. The next result, proved by Karl Friedrich Gauss (1777–1855) when he was 22 years old,* gives an extension to complex polynomials. In fact, this result is so important and useful that it has become known as the **Fundamental Theorem of Algebra**.

Fundamental Theorem of Algebra

Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

We shall not prove this result, as the proof is beyond the scope of this book. However, using the Fundamental Theorem of Algebra and the Factor Theorem, we can prove the following result:

Theorem

Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n) \qquad \textbf{(2)}$$

where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n (not necessarily distinct) zeros.

*In all, Gauss gave four different proofs of this theorem, the first one in 1799 being the subject of his doctoral dissertation.

Proof Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

By the Fundamental Theorem of Algebra, f has at least one zero, say r_1. Then, by the Factor Theorem, $x - r_1$ is a factor, and

$$f(x) = (x - r_1)q_1(x)$$

where $q_1(x)$ is a complex polynomial of degree $n - 1$ whose leading coefficient is a_n. Again by the Fundamental Theorem of Algebra, the complex polynomial $q_1(x)$ has at least one zero, say r_2. By the Factor Theorem, $q_1(x)$ has the factor $x - r_2$, so

$$q_1(x) = (x - r_2)q_2(x)$$

where $q_2(x)$ is a complex polynomial of degree $n - 2$ whose leading coefficient is a_n. Consequently,

$$f(x) = (x - r_1)(x - r_2)q_2(x)$$

Repeating this argument n times, we finally arrive at

$$f(x) = (x - r_1)(x - r_2)\cdot\cdots\cdot(x - r_n)q_n(x)$$

where $q_n(x)$ is a complex polynomial of degree $n - n = 0$ whose leading coefficient is a_n. Thus, $q_n(x) = a_n x^0 = a_n$, and so

$$f(x) = a_n(x - r_1)(x - r_2)\cdot\cdots\cdot(x - r_n)$$

We conclude that every complex polynomial function $f(x)$ of degree $n \geq 1$ has exactly n (not necessarily distinct) zeros. ∎

1 Use the Conjugate Pairs Theorem

We can use the Fundamental Theorem of Algebra to obtain valuable information about the complex zeros of polynomials whose coefficients are real numbers.

Conjugate Pairs Theorem

Let $f(x)$ be a polynomial whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

In other words, for polynomials whose coefficients are real numbers, the zeros occur in conjugate pairs.

Proof Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and $a_n \neq 0$. If $r = a + bi$ is a zero of f, then $f(r) = f(a + bi) = 0$, so

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

We take the conjugate of both sides to get

$$\overline{a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0} = \bar{0}$$

$$\overline{a_n r^n} + \overline{a_{n-1}r^{n-1}} + \cdots + \overline{a_1 r} + \overline{a_0} = \bar{0}$$ The conjugate of a sum equals the sum of the conjugates (see the Appendix, Section A.6).

$$\overline{a_n}(\bar{r})^n + \overline{a_{n-1}}(\bar{r})^{n-1} + \cdots + \overline{a_1}\bar{r} + \overline{a_0} = \bar{0}$$ The conjugate of a product equals the product of the conjugates.

$$a_n(\bar{r})^n + a_{n-1}(\bar{r})^{n-1} + \cdots + a_1\bar{r} + a_0 = 0$$ The conjugate of a real number equals the real number.

This last equation states that $f(\bar{r}) = 0$; that is, $\bar{r} = a - bi$ is a zero of f. ∎

The value of this result should be clear. If we know that, $3 + 4i$ is a zero of a polynomial with real coefficients, then we know that $3 - 4i$ is also a zero. This result has an important corollary.

COROLLARY A polynomial f of odd degree with real coefficients has at least one real zero.

Proof Because complex zeros occur as conjugate pairs in a polynomial with real coefficients, there will always be an even number of zeros that are not real numbers. Consequently, since f is of odd degree, one of its zeros has to be a real number. ■

For example, the polynomial $f(x) = x^5 - 3x^4 + 4x^3 - 5$ has at least one zero that is a real number, since f is of degree 5 (odd) and has real coefficients.

EXAMPLE 1 Using the Conjugate Pairs Theorem

A polynomial f of degree 5 whose coefficients are real numbers has the zeros $1, 5i$, and $1 + i$. Find the remaining two zeros.

Solution Since complex zeros appear as conjugate pairs, it follows that $-5i$, the conjugate of $5i$, and $1 - i$, the conjugate of $1 + i$, are the two remaining zeros. ◀

NOW WORK PROBLEM 7.

2 Find a Polynomial Function with Specified Zeros

EXAMPLE 2 Finding a Polynomial Function Whose Zeros Are Given

(a) Find a polynomial f of degree 4 whose coefficients are real numbers and that has the zeros $1, 1$, and $-4 + i$.

(b) Graph the polynomial found in part (a) to verify your result.

Solution (a) Since $-4 + i$ is a zero, by the Conjugate Pairs Theorem, $-4 - i$ must also be a zero of f. Because of the Factor Theorem, if $f(c) = 0$, then $x - c$ is a factor of $f(x)$. So we can now write f as

$$f(x) = a(x - 1)(x - 1)[x - (-4 + i)][x - (-4 - i)]$$

where a is any real number. If we let $a = 1$, we obtain

$$\begin{aligned}
f(x) &= (x - 1)(x - 1)[x - (-4 + i)][x - (-4 - i)] \\
&= (x^2 - 2x + 1)[x^2 - (-4 + i)x - (-4 - i)x + (-4 + i)(-4 - i)] \\
&= (x^2 - 2x + 1)(x^2 + 4x - ix + 4x + ix + 16 + 4i - 4i - i^2) \\
&= (x^2 - 2x + 1)(x^2 + 8x + 17) \\
&= x^4 + 8x^3 + 17x^2 - 2x^3 - 16x^2 - 34x + x^2 + 8x + 17 \\
&= x^4 + 6x^3 + 2x^2 - 26x + 17
\end{aligned}$$

(b) A quick analysis of the polynomial f tells us what to expect:

At most three turning points.

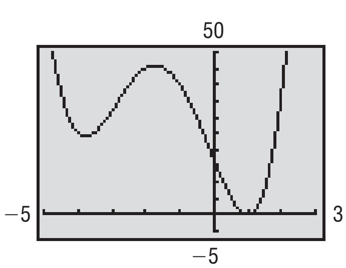
For large $|x|$, the graph will behave like $y = x^4$.

A repeated real zero at 1 so that the graph will touch the x-axis at 1.

The only x-intercept is at 1.

Figure 80 shows the complete graph. (Do you see why? The graph has exactly three turning points and the degree of the polynomial is 4.) ◀

Figure 80



Exploration

Graph the function found in Example 2 for $a = 2$ and $a = -1$. Does the value of a affect the zeros of f? How does the value of a affect the graph of f?

Now we can prove the theorem we conjectured earlier in Section 3.6.

Theorem Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible quadratic factors.

Proof Every complex polynomial f of degree n has exactly n zeros and can be factored into a product of n linear factors. If its coefficients are real, then those zeros that are complex numbers will always occur as conjugate pairs. As a result, if $r = a + bi$ is a complex zero, then so is $\bar{r} = a - bi$. Consequently, when the linear factors $x - r$ and $x - \bar{r}$ of f are multiplied, we have

$$(x - r)(x - \bar{r}) = x^2 - (r + \bar{r})x + r\bar{r} = x^2 - 2ax + a^2 + b^2$$

This second-degree polynomial has real coefficients and is irreducible (over the real numbers). Thus, the factors of f are either linear or irreducible quadratic factors. ■

3 Find the Complex Zeros of a Polynomial

EXAMPLE 3 Finding the Complex Zeros of a Polynomial

Find the complex zeros of the polynomial function

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$

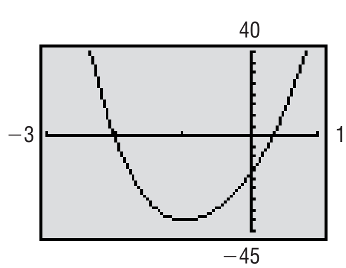
Solution

Step 1: The degree of f is 4. So f will have four complex zeros.

Step 2: The Rational Zeros Theorem provides information about the potential rational zeros of polynomials with integer coefficients. For this polynomial (which has integer coefficients), the potential rational zeros are

$$\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$$

Figure 81



Step 3: Figure 81 shows the graph of f. The graph has the characteristics that we expect of this polynomial of degree 4: It behaves like $y = 3x^4$ for large $|x|$ and has y-intercept -18. There are x-intercepts near -2 and between 0 and 1.

Step 4: Because $f(-2) = 0$, we known that -2 is a zero and $x + 2$ is a factor of f. We can use long division or synthetic division to factor f:

$$f(x) = (x + 2)(3x^3 - x^2 + 27x - 9)$$

From the graph of f and the list of potential rational zeros, it appears that $\frac{1}{3}$ may also be a zero of f. Since $f\left(\frac{1}{3}\right) = 0$, we know that $\frac{1}{3}$ is a zero of f. We use synthetic division on the depressed equation of f to factor.

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & 27 & -9 \\ & & 1 & 0 & 9 \\ \hline & 3 & 0 & 27 & 0 \end{array}$$

Using the bottom row of the synthetic division, we find

$$f(x) = (x + 2)\left(x - \frac{1}{3}\right)(3x^2 + 27) = 3(x + 2)\left(x - \frac{1}{3}\right)(x^2 + 9)$$

The factor $x^2 + 9$ does not have any real zeros; its complex zeros are $\pm 3i$. The complex zeros of $f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$ are $-2, \frac{1}{3}, 3i, -3i$. ◀

NOW WORK PROBLEM 33.

3.7 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find the sum and the product of the complex numbers $3 - 2i$ and $-3 + 5i$. (pp. 1001–1002)
2. In the complex number system, solve the equation $x^2 + 2x + 2 = 0$. (pp. 1005–1007)

Concepts and Vocabulary

3. Every polynomial function of odd degree with real coefficients will have at least _____ real zero(s).
4. If $3 + 4i$ is a zero of a polynomial function of degree 5 with real coefficients, then so is _____.
5. *True or False:* A polynomial function of degree n with real coefficients has exactly n complex zeros. At most n of them are real zeros.
6. *True or False:* A polynomial function of degree 4 with real coefficients could have $-3, 2 + i, 2 - i$, and $-3 + 5i$ as its zeros.

Skill Building

In Problems 7–16, information is given about a polynomial $f(x)$ whose coefficients are real numbers. Find the remaining zeros of f.

7. Degree 3; zeros: $3, 4 - i$
8. Degree 3; zeros: $4, 3 + i$
9. Degree 4; zeros: $i, 1 + i$
10. Degree 4; zeros: $1, 2, 2 + i$
11. Degree 5; zeros: $1, i, 2i$
12. Degree 5; zeros: $0, 1, 2, i$
13. Degree 4; zeros: $i, 2, -2$
14. Degree 4; zeros: $2 - i, -i$
15. Degree 6; zeros: $2, 2 + i, -3 - i, 0$
16. Degree 6; zeros: $i, 3 - 2i, -2 + i$

In Problems 17–22, form a polynomial $f(x)$ with real coefficients having the given degree and zeros.

17. Degree 4; zeros: $3 + 2i$; 4, multiplicity 2
18. Degree 4; zeros: $i, 1 + 2i$
19. Degree 5; zeros: $2; -i; 1 + i$
20. Degree 6; zeros: $i, 4 - i; 2 + i$
21. Degree 4; zeros: 3, multiplicity 2; $-i$
22. Degree 5; zeros: 1, multiplicity 3; $1 + i$

In Problems 23–30, use the given zero to find the remaining zeros of each function.

23. $f(x) = x^3 - 4x^2 + 4x - 16$; zero: $2i$
24. $g(x) = x^3 + 3x^2 + 25x + 75$; zero: $-5i$
25. $f(x) = 2x^4 + 5x^3 + 5x^2 + 20x - 12$; zero: $-2i$
26. $h(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$; zero: $3i$
27. $h(x) = x^4 - 9x^3 + 21x^2 + 21x - 130$; zero: $3 - 2i$
28. $f(x) = x^4 - 7x^3 + 14x^2 - 38x - 60$; zero: $1 + 3i$
29. $h(x) = 3x^5 + 2x^4 + 15x^3 + 10x^2 - 528x - 352$; zero: $-4i$
30. $g(x) = 2x^5 - 3x^4 - 5x^3 - 15x^2 - 207x + 108$; zero: $3i$

In Problems 31–40, find the complex zeros of each polynomial function. Write f in factored form.

31. $f(x) = x^3 - 1$
32. $f(x) = x^4 - 1$
33. $f(x) = x^3 - 8x^2 + 25x - 26$
34. $f(x) = x^3 + 13x^2 + 57x + 85$

35. $f(x) = x^4 + 5x^2 + 4$

36. $f(x) = x^4 + 13x^2 + 36$

37. $f(x) = x^4 + 2x^3 + 22x^2 + 50x - 75$

38. $f(x) = x^4 + 3x^3 - 19x^2 + 27x - 252$

39. $f(x) = 3x^4 - x^3 - 9x^2 + 159x - 52$

40. $f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65$

Discussion and Writing

In Problems 41 and 42, explain why the facts given are contradictory.

41. $f(x)$ is a polynomial of degree 3 whose coefficients are real numbers; its zeros are $4 + i$, $4 - i$, and $2 + i$.

42. $f(x)$ is a polynomial of degree 3 whose coefficients are real numbers; its zeros are 2, i, and $3 + i$.

43. $f(x)$ is a polynomial of degree 4 whose coefficients are real numbers; three of its zeros are 2, $1 + 2i$, and $1 - 2i$. Explain why the remaining zero must be a real number.

44. $f(x)$ is a polynomial of degree 4 whose coefficients are real numbers; two of its zeros are -3 and $4 - i$. Explain why one of the remaining zeros must be a real number. Write down one of the missing zeros.

'Are You Prepared?' Answers

1. Sum: $3i$; product: $1 + 21i$ **2.** $\{-1 - i, -1 + i\}$

Chapter Review

Things to Know

Quadratic function (pp. 150–157)

$f(x) = ax^2 + bx + c$

Graph is a parabola that opens up if $a > 0$ and opens down if $a < 0$.

Vertex: $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$

Axis of symmetry: $x = -\dfrac{b}{2a}$

y-intercept: $f(0)$

x-intercept(s): If any, found by finding the real solutions of the equation $ax^2 + bx + c = 0$.

Power function (pp. 171–174)

$f(x) = x^n, \quad n \ge 2$ even

Domain: all real numbers Range: nonnegative real numbers

Passes through $(-1, 1)$, $(0, 0)$, $(1, 1)$

Even function

Decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$

$f(x) = x^n, \quad n \ge 3$ odd

Domain: all real numbers Range: all real numbers

Passes through $(-1, -1)$, $(0, 0)$, $(1, 1)$

Odd function

Increasing on $(-\infty, \infty)$

Polynomial function (p. 170 and pp. 174–178)

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \ne 0$

Domain: all real numbers

At most $n - 1$ turning points

End behavior: Behaves like $y = a_n x^n$ for large $|x|$

Zeros of a polynomial function f (p. 175)

Numbers for which $f(x) = 0$; the real zeros of f are the x-intercepts of the graph of f.

Rational function (p. 186)	
$R(x) = \dfrac{p(x)}{q(x)}$	Domain: $\{x \mid q(x) \neq 0\}$
p, q are polynomial functions.	Vertical asymptotes: With $R(x)$ in lowest terms, if $q(r) = 0$, then $x = r$ is a vertical asymptote. Horizontal or oblique asymptotes: See the summary on page 198.
Inverse Variation (p. 205)	Let x and y denote two quantities. Then y varies inversely with x, or y is inversely proportional to x, if there is a nonzero constant k such that $y = \dfrac{k}{x}$.
Remainder Theorem (p. 220)	If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.
Factor Theorem (p. 220)	$x - c$ is a factor of a polynomial $f(x)$ if and only if $f(c) = 0$.
Rational Zeros Theorem (p. 222)	Let f be a polynomial function of degree 1 or higher of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0, a_0 \neq 0$ where each coefficient is an integer. If $\dfrac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0 and q must be a factor of a_n.
Intermediate Value Theorem (p. 229)	Let f denote a continuous function. If $a < b$ and $f(a)$ and $f(b)$ are of opposite sign, then f has at least one zero between a and b.
Fundamental Theorem of Algebra (p. 233)	Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.
Conjugate Pairs Theorem (p. 234)	Let $f(x)$ be a polynomial whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then its complex conjugate $\bar{r} = a - bi$ is also a zero of f.

Objectives

Section		You should be able to . . .	Review Exercises
3.1	1	Graph a quadratic function using transformations (p. 151)	1–6
	2	Identify the vertex and axis of symmetry of a quadratic function (p. 153)	7–16
	3	Graph a quadratic function using its vertex, axis, and intercepts (p. 154)	7–16
	4	Use the maximum or minimum value of a quadratic function to solve applied problems (p. 158)	115–122
	5	Use a graphing utility to find the quadratic function of best fit to data (p. 162)	125
3.2	1	Identify polynomial functions and their degree (p. 170)	23–26
	2	Graph polynomial functions using transformations (p. 171)	1–6, 27–32
	3	Identify the zeros of a polynomial function and their multiplicity (p. 174)	33–40
	4	Analyze the graph of a polynomial function (p. 179)	33–40
	5	Find the cubic function of best fit to data (p. 181)	126
3.3	1	Find the domain of a rational function (p. 187)	41–44
	2	Find the vertical asymptotes of a rational function (p. 190)	41–44
	3	Find the horizontal or oblique asymptotes of a rational function (p. 191)	41–44
3.4	1	Analyze the graph of a rational function (p. 198)	45–56
	2	Solve applied problems involving rational functions (p. 204)	127
	3	Construct a model using inverse variation (p. 205)	123

	4	Construct a model using joint or combined variation (p. 206)	124
3.5	1	Solve polynomial inequalities algebraically and graphically (p. 213)	57–58
	2	Solve rational inequalities algebraically and graphically (p. 215)	59–66
3.6	1	Use the Remainder and Factor Theorems (p. 219)	67–72
	2	Use the Rational Zeros Theorem (p. 222)	73–74
	3	Find the real zeros of a polynomial function (p. 223)	75–84
	4	Solve polynomial equations (p. 225)	85–88
	5	Use the Theorem for Bounds on Zeros (p. 226)	89–92
	6	Use the Intermediate Value Theorem (p. 229)	93–96
3.7	1	Use the Conjugate Pairs Theorem (p. 234)	97–100
	2	Find a polynomial function with specified zeros (p. 235)	97–100
	3	Find the complex zeros of a polynomial (p. 236)	101–114

Review Exercises

In Problems 1–6, graph each function using transformations (shifting, compressing, stretching, and reflection). Verify your result using a graphing utility.

1. $f(x) = (x-2)^2 + 2$ **2.** $f(x) = (x+1)^2 - 4$ **3.** $f(x) = -(x-4)^2$

4. $f(x) = (x-1)^2 - 3$ **5.** $f(x) = 2(x+1)^2 + 4$ **6.** $f(x) = -3(x+2)^2 + 1$

In Problems 7–16, graph each quadratic function by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.

7. $f(x) = (x-2)^2 + 2$ **8.** $f(x) = (x+1)^2 - 4$ **9.** $f(x) = \frac{1}{4}x^2 - 16$ **10.** $f(x) = -\frac{1}{2}x^2 + 2$

11. $f(x) = -4x^2 + 4x$ **12.** $f(x) = 9x^2 - 6x + 3$ **13.** $f(x) = \frac{9}{2}x^2 + 3x + 1$ **14.** $f(x) = -x^2 + x + \frac{1}{2}$

15. $f(x) = 3x^2 + 4x - 1$ **16.** $f(x) = -2x^2 - x + 4$

In Problems 17–22, determine whether the given quadratic function has a maximum value or a minimum value, and then find the value.

17. $f(x) = 3x^2 - 6x + 4$ **18.** $f(x) = 2x^2 + 8x + 5$ **19.** $f(x) = -x^2 + 8x - 4$

20. $f(x) = -x^2 - 10x - 3$ **21.** $f(x) = -3x^2 + 12x + 4$ **22.** $f(x) = -2x^2 + 4$

In Problems 23–26, determine which functions are polynomial functions. For those that are, state the degree. For those that are not, tell why not.

23. $f(x) = 4x^5 - 3x^2 + 5x - 2$ **24.** $f(x) = \frac{3x^5}{2x+1}$ **25.** $f(x) = 3x^2 + 5x^{1/2} - 1$ **26.** $f(x) = 3$

In Problems 27–32, graph each function using transformations (shifting, compressing, stretching, and reflection). Show all the stages. Verify your result using a graphing utility.

27. $f(x) = (x+2)^3$ **28.** $f(x) = -x^3 + 3$ **29.** $f(x) = -(x-1)^4$

30. $f(x) = (x-1)^4 - 2$ **31.** $f(x) = 2(x+1)^4 + 2$ **32.** $f(x) = (1-x)^3$

In Problems 33–40, for each polynomial function f:

(a) *Find the x- and y-intercepts of the graph of f.*
(b) *Determine whether the graph crosses or touches the x-axis at each x-intercept.*
(c) *End behavior: Find the power function that the graph of f resembles for large values of $|x|$.*
(d) *Use a graphing utility to graph f.*
(e) *Determine the number of turning points on the graph of f. Approximate the turning points if any exist, rounded to two decimal places.*
(f) *Use the information obtained in parts (a) to (e) to draw a complete graph of f by hand.*
(g) *Find the domain of f. Use the graph to find the range of f.*
(h) *Use the graph to determine where f is increasing and where f is decreasing.*

33. $f(x) = x(x+2)(x+4)$ **34.** $f(x) = x(x-2)(x-4)$ **35.** $f(x) = (x-2)^2(x+4)$

36. $f(x) = (x - 2)(x + 4)^2$ **37.** $f(x) = x^3 - 4x^2$ **38.** $f(x) = x^3 + 4x$

39. $f(x) = (x - 1)^2(x + 3)(x + 1)$ **40.** $f(x) = (x - 4)(x + 2)^2(x - 2)$

In Problems 41–44, find the domain of each rational function. Find any horizontal, vertical, or oblique asymptotes.

41. $R(x) = \dfrac{x + 2}{x^2 - 9}$ **42.** $R(x) = \dfrac{x^2 + 4}{x - 2}$ **43.** $R(x) = \dfrac{x^2 + 3x + 2}{(x + 2)^2}$ **44.** $R(x) = \dfrac{x^3}{x^3 - 1}$

In Problems 45–56, discuss each rational function following the eight steps on page 198.

45. $R(x) = \dfrac{2x - 6}{x}$ **46.** $R(x) = \dfrac{4 - x}{x}$ **47.** $H(x) = \dfrac{x + 2}{x(x - 2)}$ **48.** $H(x) = \dfrac{x}{x^2 - 1}$

49. $R(x) = \dfrac{x^2 + x - 6}{x^2 - x - 6}$ **50.** $R(x) = \dfrac{x^2 - 6x + 9}{x^2}$ **51.** $F(x) = \dfrac{x^3}{x^2 - 4}$ **52.** $F(x) = \dfrac{3x^3}{(x - 1)^2}$

53. $R(x) = \dfrac{2x^4}{(x - 1)^2}$ **54.** $R(x) = \dfrac{x^4}{x^2 - 9}$ **55.** $G(x) = \dfrac{x^2 - 4}{x^2 - x - 2}$ **56.** $F(x) = \dfrac{(x - 1)^2}{x^2 - 1}$

In Problems 57–66, solve each inequality (a) algebraically and (b) graphically.

57. $2x^2 + 5x - 12 < 0$ **58.** $3x^2 - 2x - 1 \geq 0$ **59.** $\dfrac{6}{x + 3} \geq 1$ **60.** $\dfrac{-2}{1 - 3x} < 1$

61. $\dfrac{2x - 6}{1 - x} < 2$ **62.** $\dfrac{3 - 2x}{2x + 5} \geq 2$ **63.** $\dfrac{(x - 2)(x - 1)}{x - 3} > 0$ **64.** $\dfrac{x + 1}{x(x - 5)} \leq 0$

65. $\dfrac{x^2 - 8x + 12}{x^2 - 16} > 0$ **66.** $\dfrac{x(x^2 + x - 2)}{x^2 + 9x + 20} \leq 0$

In Problems 67–70, find the remainder R when $f(x)$ is divided by $g(x)$. Is g a factor of f?

67. $f(x) = 8x^3 - 3x^2 + x + 4;\quad g(x) = x - 1$ **68.** $f(x) = 2x^3 + 8x^2 - 5x + 5;\quad g(x) = x - 2$

69. $f(x) = x^4 - 2x^3 + 15x - 2;\quad g(x) = x + 2$ **70.** $f(x) = x^4 - x^2 + 2x + 2;\quad g(x) = x + 1$

71. Find the value of $f(x) = 12x^6 - 8x^4 + 1$ at $x = 4$. **72.** Find the value of $f(x) = -16x^3 + 18x^2 - x + 2$ at $x = -2$.

In Problems 73 and 74, tell the maximum number of real zeros that each polynomial function may have. Then list the potential rational zeros of each polynomial function. Do not attempt to find the zeros.

73. $f(x) = 2x^8 - x^7 + 8x^4 - 2x^3 + x + 3$ **74.** $f(x) = -6x^5 + x^4 + 5x^3 + x + 1$

In Problems 75–80, find all the real zeros of each polynomial function.

75. $f(x) = x^3 - 3x^2 - 6x + 8$ **76.** $f(x) = x^3 - x^2 - 10x - 8$

77. $f(x) = 4x^3 + 4x^2 - 7x + 2$ **78.** $f(x) = 4x^3 - 4x^2 - 7x - 2$

79. $f(x) = x^4 - 4x^3 + 9x^2 - 20x + 20$ **80.** $f(x) = x^4 + 6x^3 + 11x^2 + 12x + 18$

In Problems 81–84, determine the real zeros of the polynomial function. Approximate all irrational zeros rounded to two decimal places.

81. $f(x) = 2x^3 - 11.84x^2 - 9.116x + 82.46$ **82.** $f(x) = 12x^3 + 39.8x^2 - 4.4x - 3.4$

83. $g(x) = 15x^4 - 21.5x^3 - 1718.3x^2 + 5308x + 3796.8$ **84.** $g(x) = 3x^4 + 67.93x^3 + 486.265x^2 + 1121.32x + 412.195$

In Problems 85–88, find the real solutions of each equation.

85. $2x^4 + 2x^3 - 11x^2 + x - 6 = 0$ **86.** $3x^4 + 3x^3 - 17x^2 + x - 6 = 0$

87. $2x^4 + 7x^3 + x^2 - 7x - 3 = 0$ **88.** $2x^4 + 7x^3 - 5x^2 - 28x - 12 = 0$

In Problems 89–92, find bounds to the zeros of each polynomial function. Obtain a complete graph of f.

89. $f(x) = x^3 - x^2 - 4x + 2$ **90.** $f(x) = x^3 + x^2 - 10x - 5$

91. $f(x) = 2x^3 - 7x^2 - 10x + 35$ **92.** $f(x) = 3x^3 - 7x^2 - 6x + 14$

In Problems 93–96, use the Intermediate Value Theorem to show that each polynomial has a zero in the given interval. Approximate the zero rounded to two decimal places.

93. $f(x) = 3x^3 - x - 1$; $[0, 1]$

94. $f(x) = 2x^3 - x^2 - 3$; $[1, 2]$

95. $f(x) = 8x^4 - 4x^3 - 2x - 1$; $[0, 1]$

96. $f(x) = 3x^4 + 4x^3 - 8x - 2$; $[1, 2]$

In Problems 97–100, information is given about a complex polynomial $f(x)$ whose coefficients are real numbers. Find the remaining zeros of f. Write a polynomial function whose zeros are given.

97. Degree 3; zeros: $4 + i, 6$

98. Degree 3; zeros: $3 + 4i, 5$

99. Degree 4; zeros: $i, 1 + i$

100. Degree 4; zeros: $1, 2, 1 + i$

In Problems 101–114, solve each equation in the complex number system.

101. $x^2 + x + 1 = 0$

102. $x^2 - x + 1 = 0$

103. $2x^2 + x - 2 = 0$

104. $3x^2 - 2x - 1 = 0$

105. $x^2 + 3 = x$

106. $2x^2 + 1 = 2x$

107. $x(1 - x) = 6$

108. $x(1 + x) = 2$

109. $x^4 + 2x^2 - 8 = 0$

110. $x^4 + 8x^2 - 9 = 0$

111. $x^3 - x^2 - 8x + 12 = 0$

112. $x^3 - 3x^2 - 4x + 12 = 0$

113. $3x^4 - 4x^3 + 4x^2 - 4x + 1 = 0$

114. $x^4 + 4x^3 + 2x^2 - 8x - 8 = 0$

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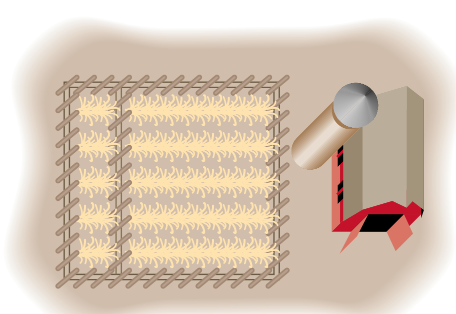
115. Find the point on the line $y = x$ that is closest to the point $(3, 1)$.

[**Hint:** Find the minimum value of the function $f(x) = d^2$, where d is the distance from $(3, 1)$ to a point on the line.]

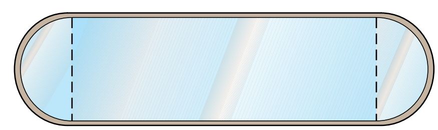
116. Landscaping A landscape engineer has 200 feet of border to enclose a rectangular pond. What dimensions will result in the largest pond?

117. Enclosing the Most Area with a Fence A farmer with 10,000 meters of fencing wants to enclose a rectangular field and then divide it into two plots with a fence parallel to one of the sides (see the figure). What is the largest area that can be enclosed?

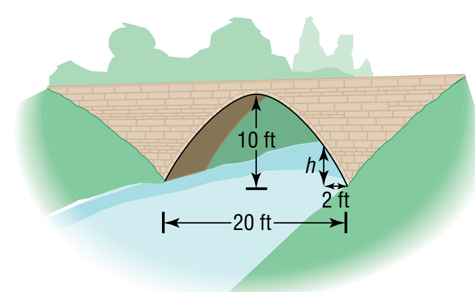


118. A rectangle has one vertex on the line $y = 8 - 2x$, $x > 0$, another at the origin, one on the positive x-axis, and one on the positive y-axis. Find the largest area A that can be enclosed by the rectangle.

119. Architecture A special window in the shape of a rectangle with semicircles at each end is to be constructed so that the outside dimensions are 100 feet in length. See the illustration. Find the dimensions that maximizes the area of the rectangle.



120. Parabolic Arch Bridges A horizontal bridge is in the shape of a parabolic arch. Given the information shown in the figure, what is the height h of the arch 2 feet from shore?



121. Minimizing Marginal Cost The marginal cost of a product can be thought of as the cost of producing one additional unit of output. For example, if the marginal cost of producing the 50th product is \$6.20, then it cost \$6.20 to increase production from 49 to 50 units of output. Callaway Golf Company has determined that the marginal cost C of manufacturing x Big Bertha golf clubs may be expressed by the quadratic function

$$C(x) = 4.9x^2 - 617.4x + 19{,}600$$

(a) How many clubs should be manufactured to minimize the marginal cost?

(b) At this level of production, what is the marginal cost?

122. Violent Crimes The function

$$V(t) = -10.0t^2 + 39.2t + 1862.6$$

models the number V (in thousands) of violent crimes committed in the United States t years after 1990. So $t = 0$ represents 1990, $t = 1$ represents 1991, and so on.

(a) Determine the year in which the most violent crimes were committed.
(b) Approximately how many violent crimes were committed during this year?
(c) Using a graphing utility, graph $V = V(t)$. Were the number of violent crimes increasing or decreasing during the years 1994 to 1998?

SOURCE: Based on data obtained from the Federal Bureau of Investigation.

123. Weight of a Body The weight of a body varies inversely with the square of its distance from the center of Earth. Assuming that the radius of Earth is 3960 miles, how much would a man weigh at an altitude of 1 mile above Earth's surface if he weighs 200 pounds on Earth's surface?

124. Resistance due to a Conductor The resistance (in ohms) of a circular conductor varies directly with the length of the conductor and inversely with the square of the radius of the conductor. If 50 feet of wire with a radius of 6×10^{-3} inch has a resistance of 10 ohms, what would be the resistance of 100 feet of the same wire if the radius is increased to 7×10^{-3} inch?

125. Advertising A small manufacturing firm collected the following data on advertising expenditures A (in thousands of dollars) and total revenue R (in thousands of dollars).

Advertising	Total Revenue
20	\$6101
22	\$6222
25	\$6350
25	\$6378
27	\$6453
28	\$6423
29	\$6360
31	\$6231

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Use a graphing utility to find the quadratic function of best fit to these data.
(c) Use the function found in part (b) to determine the optimal level of advertising for this firm.
(d) Use the function found in part (b) to find the revenue that the firm can expect if it uses the optimal level of advertising.
(e) With a graphing utility, graph the quadratic function of best fit on the scatter diagram.

126. AIDS Cases in the United States The following data represent the cumulative number of reported AIDS cases in the United States for 1990–1997.

Year, t	Number of AIDS Cases, A
1990, 1	193,878
1991, 2	251,638
1992, 3	326,648
1993, 4	399,613
1994, 5	457,280
1995, 6	528,215
1996, 7	594,760
1997, 8	653,253

SOURCE: U.S. Center for Disease Control and Prevention

(a) Draw a scatter diagram of the data.
(b) The cubic function of best fit to these data is

$$A(t) = -212t^3 + 2429t^2 + 59{,}569t + 130{,}003$$

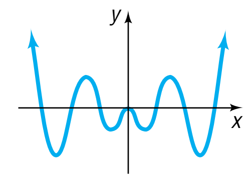
Use this function to predict the cumulative number of AIDS cases reported in the United States in 2000.
(c) Use a graphing utility to verify that the function given in part (b) is the cubic function of best fit.
(d) With a graphing utility, draw a scatter diagram of the data and then graph the cubic function of best fit on the scatter diagram.
(e) Do you think the function given in part (b) will be useful in predicting the number of AIDS cases in 2005?

127. Making a Can A can in the shape of a right circular cylinder is required to have a volume of 250 cubic centimeters.
(a) Express the amount A of material to make the can as a function of the radius r of the cylinder.
(b) How much material is required if the can is of radius 3 centimeters?
(c) How much material is required if the can is of radius 5 centimeters?
(d) Graph $A = A(r)$. For what value of r is A smallest?

128. Design a polynomial function with the following characteristics: degree 6; four real zeros, one of multiplicity 3; y-intercept 3; behaves like $y = -5x^6$ for large values of $|x|$. Is this polynomial unique? Compare your polynomial with those of other students. What terms will be the same as everyone else's? Add some more characteristics, such as symmetry or naming the real zeros. How does this modify the polynomial?

129. Design a rational function with the following characteristics: three real zeros, one of multiplicity 2; y-intercept 1; vertical asymptotes $x = -2$ and $x = 3$; oblique asymptote $y = 2x + 1$. Is this rational function unique? Compare yours with those of other students. What will be the same as everyone else's? Add some more characteristics, such as symmetry or naming the real zeros. How does this modify the rational function?

130. The illustration shows the graph of a polynomial function.
(a) Is the degree of the polynomial even or odd?
(b) Is the leading coefficient positive or negative?
(c) Is the function even, odd, or neither?
(d) Why is x^2 necessarily a factor of the polynomial?
(e) What is the minimum degree of the polynomial?
(f) Formulate five different polynomials whose graphs could look like the one shown. Compare yours to those of other students. What similarities do you see? What differences?



Chapter Test

1. Graph $f(x) = (x - 3)^4 - 2$.

2. $f(x) = 3x^2 - 12x + 4$
(a) Determine if the function has a maximum or a minimum and explain how you know.
(b) Algebraically, determine the vertex of the graph.
(c) Determine the axis of symmetry of the graph.
(d) Algebraically, determine the intercepts of the graph.
(e) Graph the function by hand.

3. For the polynomial function $g(x) = 2x^3 + 5x^2 - 28x - 15$,
(a) Determine the maximum number of real zeros that the function may have.
(b) Find bounds to the zeros of the function.
(c) List the potential rational zeros.
(d) Determine the real zeros of g. Factor g over the reals.

4. Find the complex zeros of $f(x) = x^3 - 4x^2 + 25x - 100$.

5. Solve $3x^3 + 2x - 1 = 8x^2 - 4$ in the complex number system.

In Problems 6 and 7, find the domain of each function. Find any horizontal, vertical, or oblique asymptotes.

6. $g(x) = \dfrac{2x^2 - 14x + 24}{x^2 + 6x - 40}$

7. $r(x) = \dfrac{x^2 + 2x - 3}{x + 1}$

8. Sketch the graph of the function in Problem 7. Label all intercepts, vertical asymptotes, horizontal asymptotes, and oblique asymptotes. Verify your results using a graphing utility.

In Problems 9 and 10, write a function that meets the given conditions.

9. Fourth-degree polynomial with real coefficients; zeros: $-2, 0, 3 + i$

10. Rational function; asymptotes: $y = 2, x = 4$; domain: $\{x | x \neq 4, x \neq 9\}$

11. Use the Intermediate Value Theorem to show that the function $f(x) = -2x^2 - 3x + 8$ has at least one real zero on the interval $[0, 4]$.

In Problems 12 and 13, solve each inequality algebraically. Write the solution set in interval notation. Verify your results using a graphing utility.

12. $3x^2 - x - 4 \geq x^2 - 3x + 8$

13. $\dfrac{x + 2}{x - 3} < 2$

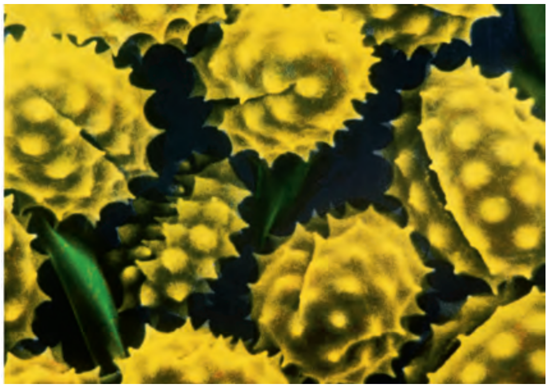
14. Given the polynomial function $f(x) = -2(x - 1)^2(x + 2)$, do the following:
(a) Find the x- and y-intercepts of the graph of f.
(b) Determine whether the graph crosses or touches the x-axis at each x-intercept.
(c) Find the power function that the graph of f resembles for large values of $|x|$.
(d) Using results of parts (a)–(c), describe what the graph of f should look like. Verify this with a graphing utility.
(e) Approximate the turning points of f rounded to two decimal places.
(f) By hand, use the information in parts (a)–(e) to draw a complete graph of f.

15. The table gives the average home game attendance A for the St. Louis Cardinals (in thousands) during the years 1995–2003, where x is the number of years since 1995.
SOURCE: www.baseball-almanac.com

x	0	1	2	3	4	5	6	7	8
A	24.570	32.774	32.519	39.453	40.197	41.191	38.390	37.182	35.930

(a) Use a graphing utility to find the quadratic function of best fit to the data. Round coefficients to three decimal places.
(b) Use the function found in part (a) to estimate the average home game attendance for the St. Louis Cardinals in 2004.

Chapter Projects



Date	Grains per cubic meter	Date	Grains per cubic meter
8/4	18	9/8	45
8/9	14	9/10	93
8/12	12	9/13	14
8/13	16	9/15	10
8/17	37	9/17	11
8/18	30	9/20	5
8/23	53	9/22	3
8/24	22	9/27	4
8/25	47	9/29	6
8/26	7	10/1	4
8/31	214	10/4	4
9/2	31	10/5	7
9/3	44	10/6	7
9/6	50		

SOURCE: National Allergy Bureau at the American Academy of Allergy, Asthma and Immunology website (*www.aaaai.org*)

1. **Weed Pollen** Given in the table below is the Weed pollen count for Lexington, Kentucky, from August 4, 2004, to October 6, 2004, the prime season for weed pollen.

   (a) Let the independent variable D represent the date, where $D = 1$ on 8/4, $D = 6$ on 8/9, $D = 9$ on 8/12 . . . and $D = 64$ on 10/6. Let the dependent variable P represent the number of grains per cubic meter of weed pollen. Draw a scatter diagram of the data using your graphing utility, and by hand on graph paper.

   (b) In the hand-drawn scatter diagram, sketch a smooth curve that fits the data. Try to have the curve pass through as many of the points as closely as possible. How many turning points are there? What does this tell you about the degree of the polynomial that could be fit to the data? If the ends of the graph on the left and on the right are extended downward, how many real zeros are indicated by the graph? How many complex? Why?

   (c) Use a graphing utility to determine the quartic function of the best fit. Graph it on your graphing utility. Does it look like what you expected from your hand-drawn graph? Explain.

   (d) Use a graphing utility to determine the cubic function of best fit. Graph it on your graphing utility. Does it look like what you expected from your hand-drawn graph? Explain.

   (e) Use a graphing utility to determine the quadratic functon of best fit. Graph it on your graphing utility. Does it look like what you expected from your hand-drawn graph? Explain.

   (f) Which of the three functions seems to fit the best? Explain your reasoning.

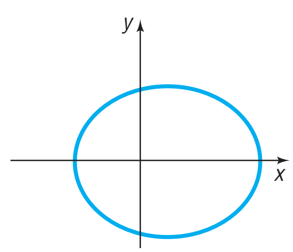
   (g) Investigate weed pollen counts at other times of the year. (Go to the National Allergy Bureau at www.aaaai.org.) Would the polynomial you found in part (c) work for other 2-month periods? Why or why not? What about for the same period of time in other years?

The following projects are available at the Instructor's Resource Center (IRC):

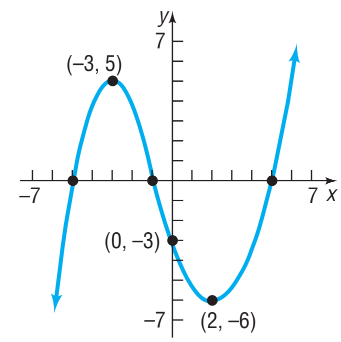
2. **Project at Motorola** ***How Many Cellphones Can I Make?***
3. **First and Second Differences**
4. **Cannons**
5. **Maclaurin Series**
6. **Theory of Equations**
7. **CBL Experiment**

Cumulative Review

1. Find the distance between the points $P = (1, 3)$ and $Q = (-4, 2)$.
2. Solve the inequality $x^2 \geq x$ and graph the solution set.
3. Solve the inequality $x^2 - 3x < 4$ and graph the solution set.
4. Find a linear function with slope -3 that contains the point $(-1, 4)$. Graph the function.
5. Find the equation of the line parallel to the line $y = 2x + 1$ and containing the point $(3, 5)$. Express your answer in slope–intercept form and graph the line.
6. Graph the equation $y = x^3$.
7. Does the relation $\{(3, 6), (1, 3), (2, 5), (3, 8)\}$ represent a function? Why or why not?
8. Solve the equation $x^3 - 6x^2 + 8x = 0$.
9. Solve the inequality $3x + 2 \leq 5x - 1$ and graph the solution set.
10. Find the center and radius of the circle $x^2 + 4x + y^2 - 2y - 4 = 0$. Graph the circle.
11. For the equation $y = x^3 - 9x$, determine the intercepts and test for symmetry.
12. Find an equation of the line perpendicular to $3x - 2y = 7$ that contains the point $(1, 5)$.
13. Is the following graph the graph of a function? Why or why not?



14. For the function $f(x) = x^2 + 5x - 2$, find
    (a) $f(3)$
    (b) $f(-x)$
    (c) $-f(x)$
    (d) $f(3x)$
    (e) $\dfrac{f(x+h) - f(x)}{h}, h \neq 0$
15. Answer the following questions regarding the function $f(x) = \dfrac{x+5}{x-1}$.
    (a) What is the domain of f?
    (b) Is the point $(2, 6)$ on the graph of f?
    (c) If $x = 3$, what is $f(x)$? What point is on the graph of f?
    (d) If $f(x) = 9$, what is x? What point is on the graph of f?
16. Graph the function $f(x) = -3x + 7$.
17. Graph $f(x) = 2x^2 - 4x + 1$ by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.
18. Find the average rate of change of $f(x) = x^2 + 3x + 1$ from 1 to x. Use this result to find the slope of the secant line containing $(1, f(1))$ and $(2, f(2))$.
19. In parts (a) to (f) use the following graph.



    (a) Determine the intercepts.
    (b) Based on the graph, tell whether the graph is symmetric with respect to the x-axis, the y-axis, and/or the origin.
    (c) Based on the graph, tell whether the function is even, odd, or neither.
    (d) List the intervals on which f is increasing. List the intervals on which f is decreasing.
    (e) List the numbers, if any, at which f has a local maximum. What are these local maxima?
    (f) List the numbers, if any, at which f has a local minimum. What are these local minima?
20. Determine algebraically whether the function $f(x) = \dfrac{5x}{x^2 - 9}$ is even, odd, or neither.
21. For the function $f(x) = \begin{cases} 2x + 1 & \text{if } -3 < x < 2 \\ -3x + 4 & \text{if } x \geq 2 \end{cases}$
    (a) Find the domain of f.
    (b) Locate any intercepts.
    (c) Graph the function.
    (d) Based on the graph, find the range.
22. Graph the function $f(x) = -3(x+1)^2 + 5$ using transformations.
23. Suppose that $f(x) = x^2 - 5x + 1$ and $g(x) = -4x - 7$.
    (a) Find $f + g$ and state its domain.
    (b) Find $\dfrac{f}{g}$ and state its domain.
24. **Demand Equation** The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{10}x + 150, \qquad 0 \leq x \leq 1500$$

    (a) Express the revenue R as a function of x.
    (b) What is the revenue if 100 units are sold?
    (c) What quantity x maximizes revenue? What is the maximum revenue?
    (d) What price should the company charge to maximize revenue?