Applications of Trigonometric Functions

New maps pinpoint Lewis and Clark's journey through Missouri

KANSAS CITY, Mo.—Nearly two centuries ago, Congress commissioned Meriwether Lewis and William Clark to explore trade routes to the West.

Their long-documented journey took them through Missouri, but their exact route has been up for debate.

Now, modern computer technology combined with 19th century land surveys may provide a precise picture. The latest computer-generated maps of Lewis and Clark's expedition were unveiled here this week by Missouri Secretary of State Matt Blunt.

The maps combine the terrain of the early 19th century with contemporary geographical markers. They are the most precise to date of Lewis and Clark's journeys through Missouri in 1804 and 1806, said the project's lead researcher, Jim Harlan.

"We've known what they've done, but not with this much certainty," said Harlan, geographic resource project director at the University of Missouri-Columbia. "I think we need more information and less speculation. That's what this is all about."

SOURCE: Sophia Maines, "New Maps Pinpoint Lewis and Clark's Journey Through Missouri," *Kansas City Star*, August 1, 2001. Distributed by Knight Ridder/Tribune Information Services.

—*See Chapter Project 1.*

7

A LOOK BACK In Chapter 5 we defined the six trigonometric functions using the unit circle and then extended this definition to include circles of radius r. In particular, we learned to evaluate the trigonometric functions. We also learned how to graph sinusoidal functions.

A LOOK AHEAD In this chapter, we define the trigonometric functions using right triangles and use the trigonometric functions to solve applied problems. The first four sections deal with applications involving right triangles and *oblique triangles*, triangles that do not have a right angle. To solve problems involving oblique triangles, we will develop the Law of Sines and the Law of Cosines. We will also develop formulas for finding the area of a triangle.

The final section deals with applications of sinusoidal functions involving simple harmonic motion and damped motion.

OUTLINE

7.1 Right Triangle Trigonometry; Applications

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Pythagorean Theorem (Appendix A, Section A.2, pp. 961–962)
- Trigonometric Equations (I) (Section 6.7, pp. 496–500)

Now work the 'Are You Prepared?' problems on page 526.

OBJECTIVES
1 Find the Value of Trigonometric Functions of Acute Angles
2 Use the Complementary Angle Theorem
3 Solve Right Triangles
4 Solve Applied Problems

1 Find the Value of Trigonometric Functions of Acute Angles

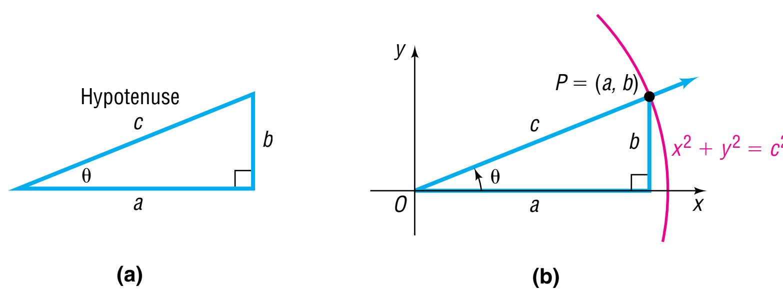
A triangle in which one angle is a right angle (90°) is called a **right triangle.** Recall that the side opposite the right angle is called the **hypotenuse,** and the remaining two sides are called the **legs** of the triangle. In Figure 1(a), we have labeled the hypotenuse as c to indicate that its length is c units, and, in a like manner, we have labeled the legs as a and b. Because the triangle is a right triangle, the Pythagorean Theorem tells us that

$$a^2 + b^2 = c^2$$

In Figure 1(a), we also show the angle θ. The angle θ is an **acute angle:** that is, $0^\circ < \theta < 90^\circ$ for θ measured in degrees and $0 < \theta < \frac{\pi}{2}$ for θ measured in radians.

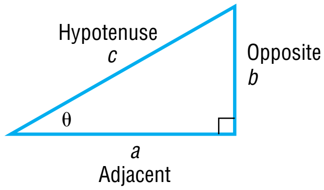
Place θ in standard position, as shown in Figure 1(b). Then the coordinates of the point P are (a, b). Also, P is a point on the terminal side of θ that is on the circle $x^2 + y^2 = c^2$. (Do you see why?)

Figure 1



Now use the theorem on page 382. By referring to the lengths of the sides of the triangle by the names hypotenuse (c), opposite (b), and adjacent (a), as indicated in Figure 2, we can express the trigonometric functions of θ as ratios of the sides of a right triangle.

Figure 2



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{b}{c} \qquad \csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{c}{b}$$
$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{a}{c} \qquad \sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{c}{a} \qquad \textbf{(1)}$$
$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{b}{a} \qquad \cot\theta = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{a}{b}$$

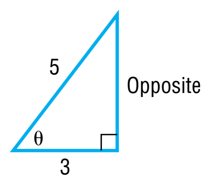
Notice that each of the trigonometric functions of the acute angle θ is positive.

EXAMPLE 1 Finding the Value of Trigonometric Functions from a Right Triangle

Find the exact value of the six trigonometric functions of the angle θ in Figure 3.

Solution We see in Figure 3 that the two given sides of the triangle are

Figure 3



$$c = \text{Hypotenuse} = 5, \quad a = \text{Adjacent} = 3$$

To find the length of the opposite side, we use the Pythagorean Theorem.

$$\begin{aligned}(\text{Adjacent})^2 + (\text{Opposite})^2 &= (\text{Hypotenuse})^2 \\ 3^2 + (\text{Opposite})^2 &= 5^2 \\ (\text{Opposite})^2 &= 25 - 9 = 16 \\ \text{Opposite} &= 4\end{aligned}$$

Now that we know the lengths of the three sides, we use the ratios in equations (1) to find the value of each of the six trigonometric functions.

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{4}{5} \qquad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{3}{5} \qquad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{4}{3}$$

$$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{5}{4} \qquad \sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{5}{3} \qquad \cot\theta = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{3}{4}$$

◀

NOW WORK PROBLEM 9.

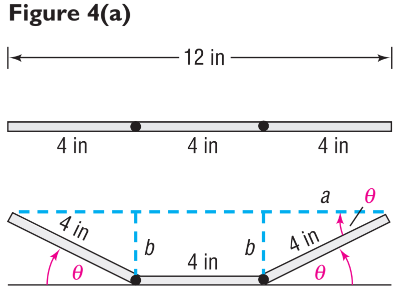
The values of the trigonometric functions of an acute angle are ratios of the lengths of the sides of a right triangle. This way of viewing the trigonometric functions leads to many applications and, in fact, was the point of view used by early mathematicians (before calculus) in studying the subject of trigonometry.

We look at one such application next.

EXAMPLE 2 Constructing a Rain Gutter

A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, this length is bent up at an angle θ. See Figure 4(a).

Figure 4(a)



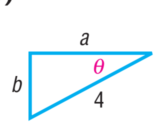
(a) Express the area A of the opening as a function of θ.
[**Hint:** Let b denote the vertical height of the bend.]

(b) Graph $A = A(\theta)$. Find the angle θ that makes A largest. (This bend will allow the most water to flow through the gutter.)

Solution

(a) Look again at Figure 4(a). The area A of the opening is the sum of the areas of two congruent right triangles and one rectangle. Look at Figure 4(b), showing the triangle in Figure 4(a) redrawn. We see that

Figure 4(b)



$$\cos\theta = \frac{a}{4} \quad \text{so} \quad a = 4\cos\theta \qquad \sin\theta = \frac{b}{4} \quad \text{so} \quad b = 4\sin\theta$$

The area of the triangle is

$$\text{area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}ab = \frac{1}{2}(4\cos\theta)(4\sin\theta) = 8\sin\theta\cos\theta$$

So the area of the two triangles is $16\sin\theta\cos\theta$.

The rectangle has length 4 and height b, so its area is

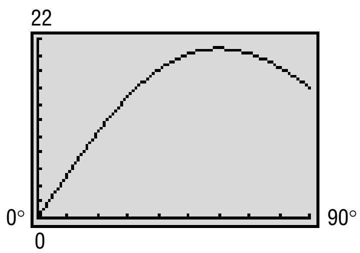
$$4b = 4(4\sin\theta) = 16\sin\theta$$

The area A of the opening is

$$A = \text{area of the two triangles} + \text{area of the rectangle}$$
$$A(\theta) = 16\sin\theta\cos\theta + 16\sin\theta = 16\sin\theta(\cos\theta + 1)$$

(b) Figure 5 shows the graph of $A = A(\theta)$. Using MAXIMUM, the angle θ that makes A largest is 60°. ◀

Figure 5

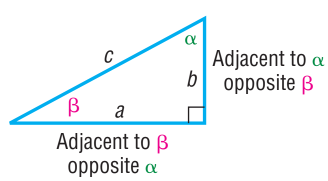


2 Use the Complementary Angle Theorem

Two acute angles are called **complementary** if their sum is a right angle. Because the sum of the angles of any triangle is 180°, it follows that, for a right triangle, the two acute angles are complementary.

Refer now to Figure 6. We have labeled the angle opposite side b as β and the angle opposite side a as α. Notice that side b is adjacent to angle α and side a is adjacent to angle β. As a result,

Figure 6



$$\begin{aligned} \sin\beta &= \frac{b}{c} = \cos\alpha & \cos\beta &= \frac{a}{c} = \sin\alpha & \tan\beta &= \frac{b}{a} = \cot\alpha \\ \csc\beta &= \frac{c}{b} = \sec\alpha & \sec\beta &= \frac{c}{a} = \csc\alpha & \cot\beta &= \frac{a}{b} = \tan\alpha \end{aligned} \qquad \textbf{(2)}$$

Because of these relationships, the functions sine and cosine, tangent and cotangent, and secant and cosecant are called **cofunctions** of each other. The identities (2) may be expressed in words as follows:

Complementary Angle Theorem

Cofunctions of complementary angles are equal.

Examples of this theorem are given next:

Complementary angles
$\sin 30° = \cos 60°$
Cofunctions

Complementary angles
$\tan 40° = \cot 50°$
Cofunctions

Complementary angles
$\sec 80° = \csc 10°$
Cofunctions

EXAMPLE 3 Using the Complementary Angle Theorem

(a) $\sin 62° = \cos(90° - 62°) = \cos 28°$

(b) $\tan\dfrac{\pi}{12} = \cot\left(\dfrac{\pi}{2} - \dfrac{\pi}{12}\right) = \cot\dfrac{5\pi}{12}$

(c) $\sin^2 40° + \sin^2 50° = \sin^2 40° + \cos^2 40° = 1$

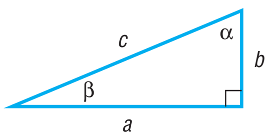
$\sin 50° = \cos 40°$ ◀

NOW WORK PROBLEM 19.

3 Solve Right Triangles

In the discussion that follows, we will always label a right triangle so that side a is opposite angle α, side b is opposite angle β, and side c is the hypotenuse, as shown in Figure 7. **To solve a right triangle** means to find the missing lengths of its sides and the measurements of its angles. We shall follow the practice of expressing the lengths of the sides rounded to two decimal places and expressing angles in degrees rounded to one decimal place. (Be sure that your calculator is in degree mode.)

Figure 7



To solve a right triangle, we need to know one of the acute angles α or β and a side, or else two sides. Then we make use of the Pythagorean Theorem and the fact that the sum of the angles of a triangle is 180°. The sum of the angles α and β in a right triangle is therefore 90°.

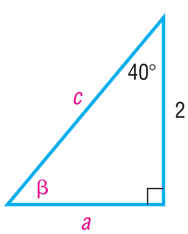
For the right triangle shown in Figure 7, we have

$$\boxed{c^2 = a^2 + b^2, \qquad \alpha + \beta = 90^\circ}$$

EXAMPLE 4 Solving a Right Triangle

Use Figure 8. If $b = 2$ and $\alpha = 40°$, find a, c, and β.

Figure 8



Solution Since $\alpha = 40°$ and $\alpha + \beta = 90°$, we find that $\beta = 50°$. To find the sides a and c, we use the facts that

$$\tan 40^\circ = \frac{a}{2} \quad \text{and} \quad \cos 40^\circ = \frac{2}{c}$$

Now solve for a and c.

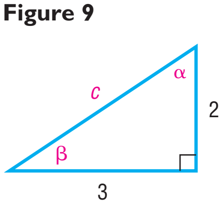
$$a = 2\tan 40^\circ \approx 1.68 \quad \text{and} \quad c = \frac{2}{\cos 40^\circ} \approx 2.61$$

NOW WORK PROBLEM 29.

EXAMPLE 5 Solving a Right Triangle

Use Figure 9. If $a = 3$ and $b = 2$, find c, α, and β.

Figure 9



Solution Since $a = 3$ and $b = 2$, then, by the Pythagorean Theorem, we have

$$c^2 = a^2 + b^2 = 3^2 + 2^2 = 9 + 4 = 13$$

$$c = \sqrt{13} \approx 3.61$$

To find angle α, we use the fact that

$$\tan \alpha = \frac{3}{2} \quad \text{so} \quad \alpha = \tan^{-1}\frac{3}{2}$$

Set the mode on your calculator to degrees. Then, rounded to one decimal place, we find that $\alpha = 56.3°$. Since $\alpha + \beta = 90°$, we find that $\beta = 33.7°$. ◀

NOTE To avoid round-off errors when using a calculator, we will store unrounded values in memory for use in subsequent calculations. ■

NOW WORK PROBLEM 39.

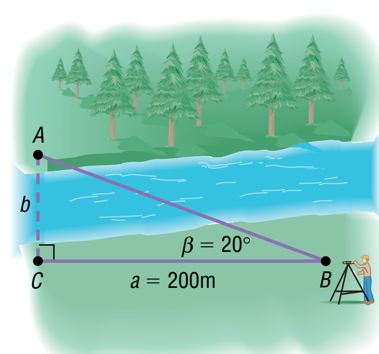
4 Solve Applied Problems

One common use for trigonometry is to measure heights and distances that are either awkward or impossible to measure by ordinary means.

EXAMPLE 6 Finding the Width of a River

A surveyor can measure the width of a river by setting up a transit* at a point C on one side of the river and taking a sighting of a point A on the other side. Refer to Figure 10. After turning through an angle of 90° at C, the surveyor walks a distance of 200 meters to point B. Using the transit at B, the angle β is measured and found to be 20°. What is the width of the river rounded to the nearest meter?

Figure 10



Solution We seek the length of side b. We know a and β, so we use the fact that

$$\tan \beta = \frac{b}{a}$$

to get

$$\tan 20° = \frac{b}{200}$$

$$b = 200 \tan 20° \approx 72.79 \text{ meters}$$

The width of the river is 73 meters, rounded to the nearest meter. ◀

NOW WORK PROBLEM 49.

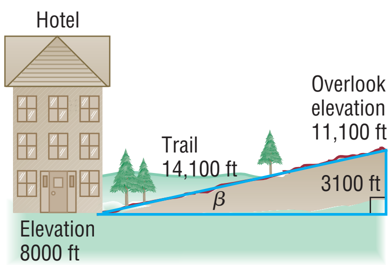
*An instrument used in surveying to measure angles.

EXAMPLE 7 Finding the Inclination of a Mountain Trail

A straight trail leads from the Alpine Hotel, elevation 8000 feet, to a scenic overlook, elevation 11,100 feet. The length of the trail is 14,100 feet. What is the inclination (grade) of the trail? That is, what is the angle β in Figure 11?

Solution As Figure 11 illustrates, the angle β obeys the equation

Figure 11



$$\sin \beta = \frac{3100}{14{,}100}$$

Using a calculator,

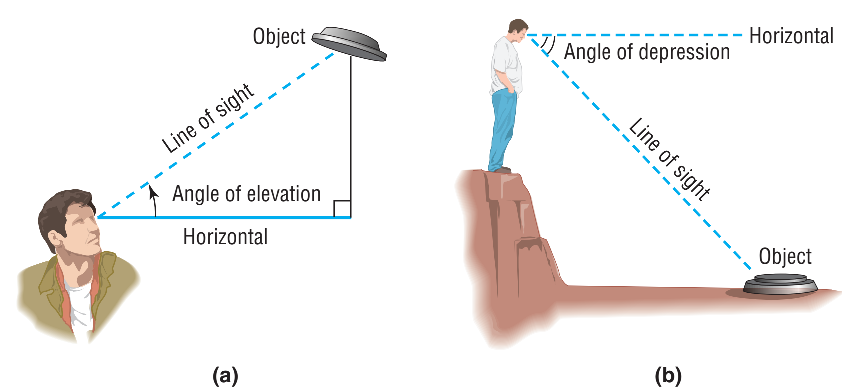
$$\beta = \sin^{-1}\frac{3100}{14{,}100} \approx 12.7°$$

The inclination (grade) of the trail is approximately 12.7°. ◀

Vertical heights can sometimes be measured using either the *angle of elevation* or the *angle of depression*. If a person is looking up at an object, the acute angle measured from the horizontal to a line-of-sight observation of the object is called the **angle of elevation.** See Figure 12(a).

If a person is looking down at an object, the acute angle made by the line-of-sight observation of the object and the horizontal is called the **angle of depression.** See Figure 12(b).

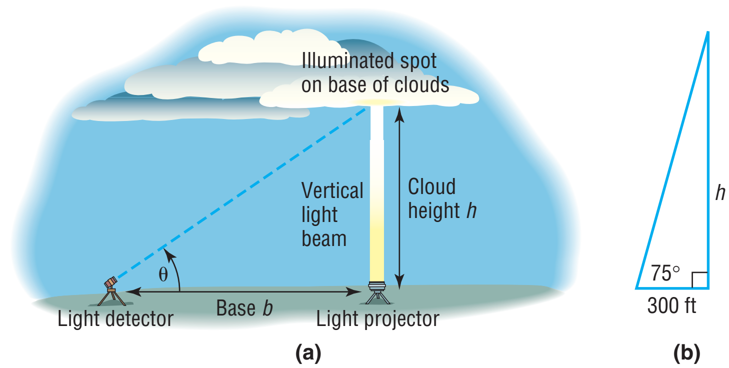
Figure 12



EXAMPLE 8 Finding the Height of a Cloud

Meteorologists find the height of a cloud using an instrument called a **ceilometer.** A ceilometer consists of a **light projector** that directs a vertical light beam up to the cloud base and a **light detector** that scans the cloud to detect the light beam. See Figure 13(a). On December 1, 2004, at Midway Airport in Chicago, a ceilometer with a base of 300 feet was employed to find the height of the cloud cover. If the angle of elevation of the light detector is 75°, what is the height of the cloud cover? Round the answer to the nearest foot.

Figure 13



Solution Figure 13(b) illustrates the situation. To find the height h, we use the fact that $\tan 75° = \frac{h}{300}$, so

$$h = 300 \tan 75° \approx 1120 \text{ feet}$$

The ceiling (height to the base of the cloud cover) is approximately 1120 feet. ◀

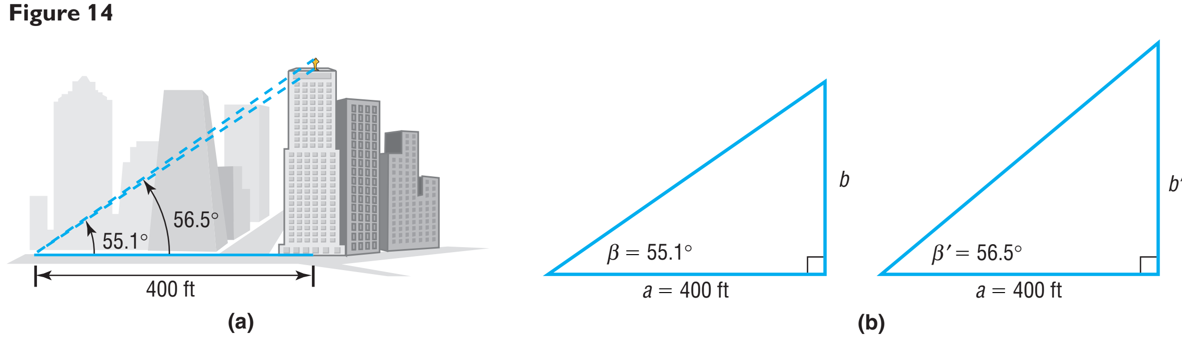
NOW WORK PROBLEM 51.

The idea behind Example 8 can also be used to find the height of an object with a base that is not accessible to the horizontal.

EXAMPLE 9 Finding the Height of a Statue on a Building

Adorning the top of the Board of Trade building in Chicago is a statue of Ceres, the Roman goddess of wheat. From street level, two observations are taken 400 feet from the center of the building. The angle of elevation to the base of the statue is found to be 55.1°; the angle of elevation to the top of the statue is 56.5°. See Figure 14(a). What is the height of the statue? Round the answer to the nearest foot.

Figure 14



Solution Figure 14(b) shows two triangles that replicate Figure 14(a). The height of the statue of Ceres will be $b' - b$. To find b and b', we refer to Figure 14(b).

$$\tan 55.1° = \frac{b}{400} \qquad \tan 56.5° = \frac{b'}{400}$$

$$b = 400 \tan 55.1° \approx 573 \qquad b' = 400 \tan 56.5° \approx 604$$

The height of the statue is approximately $604 - 573 = 31$ feet. ◀

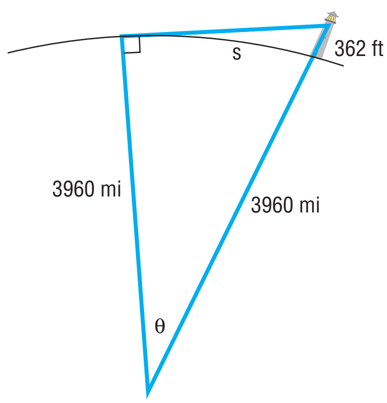
NOW WORK PROBLEM 59.

EXAMPLE 10 The Gibb's Hill Lighthouse, Southampton, Bermuda

In operation since 1846, the Gibb's Hill Lighthouse stands 117 feet high on a hill 245 feet high, so its beam of light is 362 feet above sea level. A brochure states that the light can be seen on the horizon about 26 miles distant. Verify the accuracy of this statement.

Solution Figure 15 illustrates the situation. The central angle θ, positioned at the center of Earth, radius 3960 miles, obeys the equation

Figure 15



$$\cos\theta = \frac{3960}{3960 + \dfrac{362}{5280}} \approx 0.999982687 \qquad 1 \text{ mile} = 5280 \text{ feet}$$

Solving for θ, we find

$$\theta \approx 0.33715° \approx 20.23'$$

The brochure does not indicate whether the distance is measured in nautical miles or statute miles. Let's calculate both distances.

The distance s in nautical miles (refer to Problem 114, p. 369) is the measure of angle θ in minutes, so $s = 20.23$ nautical miles.

The distance s in statute miles is given by the formula $s = r\theta$, where θ is measured in radians. Since

$$\theta = 20.23' \underset{1' = \frac{1}{60}°}{\approx} 0.33715° \underset{1° = \frac{\pi}{180}\text{ radian}}{\approx} 0.00588 \text{ radian}$$

we find that

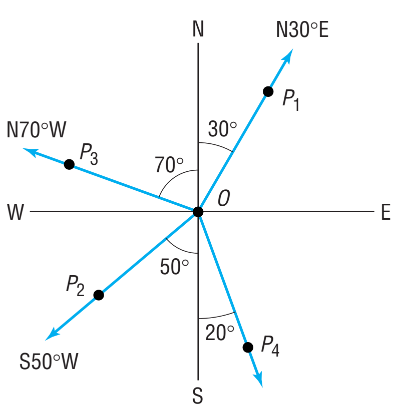
$$s = r\theta = (3960)(0.00588) = 23.3 \text{ miles}$$

In either case, it would seem that the brochure overstated the distance somewhat. ◀

In navigation and surveying, the **direction** or **bearing** from a point O to a point P equals the acute angle θ between the ray OP and the vertical line through O, the north–south line.

Figure 16 illustrates some bearings. Notice that the bearing from O to P_1 is denoted by the symbolism N30°E, indicating that the bearing is 30° east of north. In writing the bearing from O to P, the direction north or south always appears first, followed by an acute angle, followed by east or west. In Figure 16, the bearing from O to P_2 is S50°W, and from O to P_3 it is N70°W.

Figure 16



EXAMPLE 11 Finding the Bearing of an Object

In Figure 16, what is the bearing from O to an object at P_4?

Solution The acute angle between the ray OP_4 and the north–south line through O is given as 20°. The bearing from O to P_4 is S20°E. ◀

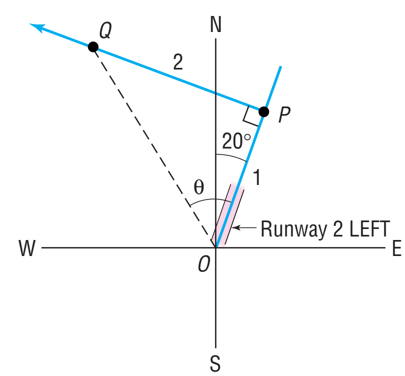
EXAMPLE 12 Finding the Bearing of an Airplane

A Boeing 777 aircraft takes off from O'Hare Airport on runway 2 LEFT, which has a bearing of N20°E.* After flying for 1 mile, the pilot of the aircraft requests permission to turn 90° and head toward the northwest. The request is granted. After the plane goes 2 miles in this direction, what bearing should the control tower use to locate the aircraft?

Solution Figure 17 illustrates the situation. After flying 1 mile from the airport O (the control tower), the aircraft is at P. After turning 90° toward the northwest and flying 2 miles, the aircraft is at the point Q. In triangle OPQ, the angle θ obeys the equation

$$\tan\theta = \frac{2}{1} = 2 \quad \text{so} \quad \theta = \tan^{-1} 2 \approx 63.4°$$

Figure 17



The acute angle between north and the ray OQ is $63.4° - 20° = 43.4°$. The bearing of the aircraft from O to Q is N43.4°W.

NOW WORK PROBLEM 67.

*In air navigation, the term **azimuth** is employed to denote the positive angle measured clockwise from the north (N) to a ray OP. In Figure 16, the azimuth from O to P_1 is 30°; the azimuth from O to P_2 is 230°; the azimuth from O to P_3 is 290°. In naming runways, the units digit is left off the azimuth. Runway 2 LEFT means the left runway with a direction of azimuth 20° (bearing N20°E). Runway 23 is the runway with azimuth 230° and bearing S50°W.

7.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. In a right triangle, if the length of the hypotenuse is 5 and the length of one of the other sides is 3, what is the length of the third side? (p. 962)

2. If θ is an angle, $0° < \theta < 90°$, solve the equation $\tan\theta = \frac{1}{2}$. (pp. 496–500)

Concepts and Vocabulary

3. *True or False:* The angles 52° and 48° are complementary.
4. *True or False:* In a right triangle, one of the angles is 90° and the sum of the other two angles is 90°.
5. When you look up at an object, the acute angle measured from the horizontal to a line-of-sight observation of the object is called the __________ __________ __________.
6. When you look down at an object, the acute angle described in Problem 5 is called the __________ __________ __________.
7. *True or False:* In a right triangle, if two sides are known, we can solve the triangle.
8. *True or False:* In a right triangle, if we know the two acute angles, we can solve the triangle.

Skill Building

In Problems 9–18, find the exact value of the six trigonometric functions of the angle θ in each figure.

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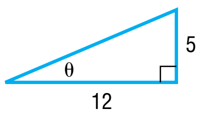

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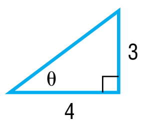

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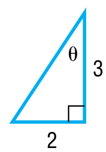

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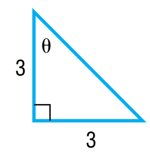
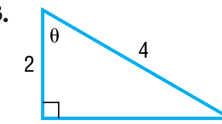


13.


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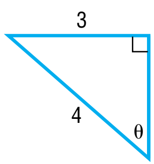

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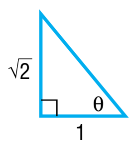

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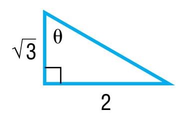

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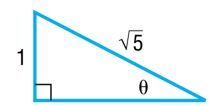

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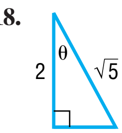
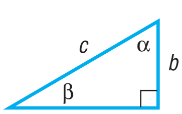


In Problems 19–28, find the exact value of each expression. Do not use a calculator.

19. $\sin 38° - \cos 52°$
20. $\tan 12° - \cot 78°$
21. $\dfrac{\cos 10°}{\sin 80°}$
22. $\dfrac{\cos 40°}{\sin 50°}$
23. $1 - \cos^2 20° - \cos^2 70°$
24. $1 + \tan^2 5° - \csc^2 85°$
25. $\tan 20° - \dfrac{\cos 70°}{\cos 20°}$
26. $\cot 40° - \dfrac{\sin 50°}{\sin 40°}$
27. $\cos 35° \sin 55° + \sin 35° \cos 55°$
28. $\sec 35° \csc 55° - \tan 35° \cot 55°$

In Problems 29–42, use the right triangle shown in the margin. Then, using the given information, solve the triangle.



29. $b = 5$, $\beta = 20°$; find a, c, and α
30. $b = 4$, $\beta = 10°$; find a, c, and α
31. $a = 6$, $\beta = 40°$; find b, c, and α
32. $a = 7$, $\beta = 50°$; find b, c, and α
33. $b = 4$, $\alpha = 10°$; find a, c, and β
34. $b = 6$, $\alpha = 20°$; find a, c, and β
35. $a = 5$, $\alpha = 25°$; find b, c, and β
36. $a = 6$, $\alpha = 40°$; find b, c, and β
37. $c = 9$, $\beta = 20°$; find b, a, and α
38. $c = 10$, $\alpha = 40°$; find b, a, and β
39. $a = 5$, $b = 3$; find c, α, and β
40. $a = 2$, $b = 8$; find c, α, and β
41. $a = 2$, $c = 5$; find b, α, and β
42. $b = 4$, $c = 6$; find a, α, and β

Applications and Extensions

43. Geometry A right triangle has a hypotenuse of length 8 inches. If one angle is 35°, find the length of each leg.

44. Geometry A right triangle has a hypotenuse of length 10 centimeters. If one angle is 40°, find the length of each leg.

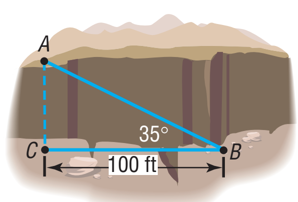
45. Geometry A right triangle contains a 25° angle. If one leg is of length 5 inches, what is the length of the hypotenuse? [**Hint:** Two answers are possible.]

46. Geometry A right triangle contains an angle of $\frac{\pi}{8}$ radian. If one leg is of length 3 meters, what is the length of the hypotenuse?
[**Hint:** Two answers are possible.]

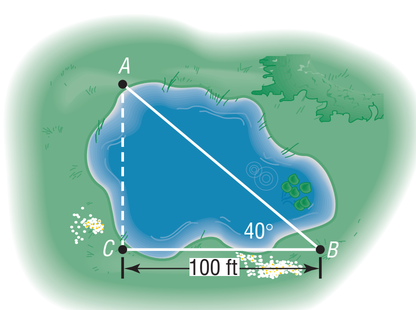
47. Geometry The hypotenuse of a right triangle is 5 inches. If one leg is 2 inches, find the degree measure of each angle.

48. Geometry The hypotenuse of a right triangle is 3 feet. If one leg is 1 foot, find the degree measure of each angle.

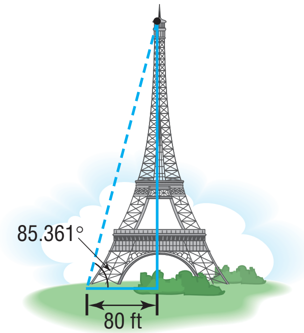
49. Finding the Width of a Gorge Find the distance from A to C across the gorge illustrated in the figure.



50. Finding the Distance across a Pond Find the distance from A to C across the pond illustrated in the figure.



51. The Eiffel Tower The tallest tower built before the era of television masts, the Eiffel Tower was completed on March 31, 1889. Find the height of the Eiffel Tower (before a television mast was added to the top) using the information given in the illustration.



52. Finding the Distance of a Ship from Shore A ship, offshore from a vertical cliff known to be 100 feet in height, takes a sighting of the top of the cliff. If the angle of elevation is found to be 25°, how far offshore is the ship?

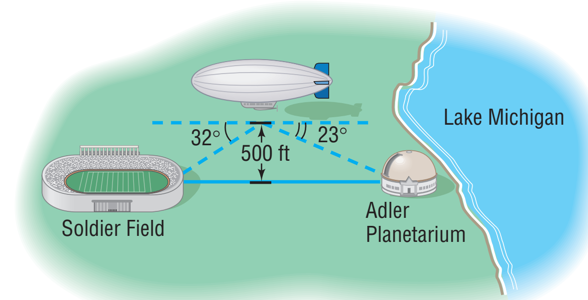
53. Finding the Distance to a Plateau Suppose that you are headed toward a plateau 50 meters high. If the angle of elevation to the top of the plateau is 20°, how far are you from the base of the plateau?

54. Statue of Liberty A ship is just offshore of New York City. A sighting is taken of the Statue of Liberty, which is about 305 feet tall. If the angle of elevation to the top of the statue is 20°, how far is the ship from the base of the statue?

55. Finding the Reach of a Ladder A 22-foot extension ladder leaning against a building makes a 70° angle with the ground. How far up the building does the ladder touch?

56. Finding the Height of a Building To measure the height of a building, two sightings are taken a distance of 50 feet apart. If the first angle of elevation is 40° and the second is 32°, what is the height of the building?

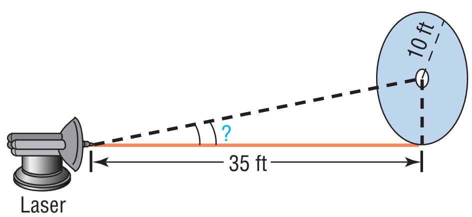
57. Finding the Distance between Two Objects A blimp, suspended in the air at a height of 500 feet, lies directly over a line from Soldier Field to the Adler Planetarium on Lake Michigan (see the figure). If the angle of depression from the blimp to the stadium is 32° and from the blimp to the planetarium is 23°, find the distance between Soldier Field and the Adler Planetarium.



58. Finding the Angle of Elevation of the Sun At 10 AM on April 26, 2005, a building 300 feet high casts a shadow 50 feet long. What is the angle of elevation of the Sun?

59. **Mt. Rushmore** To measure the height of Lincoln's caricature on Mt. Rushmore, two sightings 800 feet from the base of the mountain are taken. If the angle of elevation to the bottom of Lincoln's face is 32° and the angle of elevation to the top is 35°, what is the height of Lincoln's face?

60. **Directing a Laser Beam** A laser beam is to be directed through a small hole in the center of a circle of radius 10 feet. The origin of the beam is 35 feet from the circle (see the figure). At what angle of elevation should the beam be aimed to ensure that it goes through the hole?



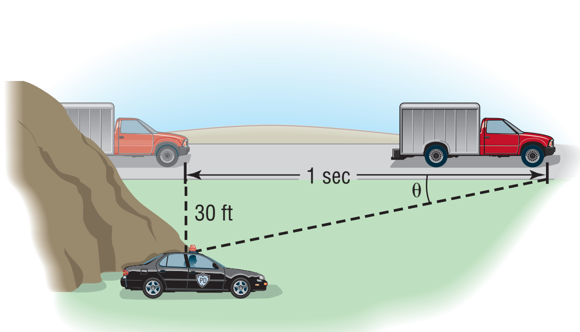
61. **Finding the Length of a Guy Wire** A radio transmission tower is 200 feet high. How long should a guy wire be if it is to be attached to the tower 10 feet from the top and is to make an angle of 21° with the ground?

62. **Finding the Height of a Tower** A guy wire 80 feet long is attached to the top of a radio transmission tower, making an angle of 25° with the ground. How high is the tower?

63. **Washington Monument** The angle of elevation of the Sun is 35.1° at the instant it casts a shadow 789 feet long of the Washington Monument. Use this information to calculate the height of the monument.

64. **Finding the Length of a Mountain Trail** A straight trail with an inclination of 17° leads from a hotel at an elevation of 9000 feet to a mountain lake at an elevation of 11,200 feet. What is the length of the trail?

65. **Finding the Speed of a Truck** A state trooper is hidden 30 feet from a highway. One second after a truck passes, the angle θ between the highway and the line of observation from the patrol car to the truck is measured. See the illustration.



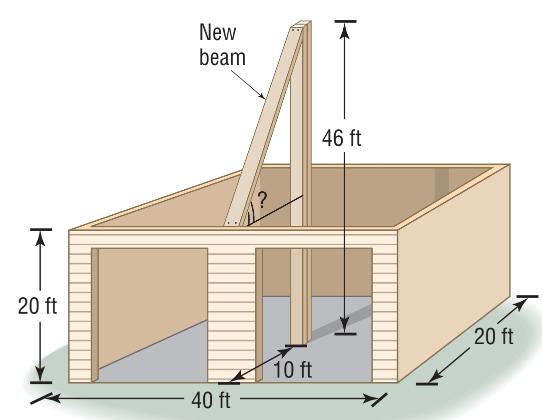
(a) If the angle measures 15°, how fast is the truck traveling? Express the answer in feet per second and in miles per hour.
(b) If the angle measures 20°, how fast is the truck traveling? Express the answer in feet per second and in miles per hour.
(c) If the speed limit is 55 miles per hour and a speeding ticket is issued for speeds of 5 miles per hour or more over the limit, for what angles should the trooper issue a ticket?

66. **Security** A security camera in a neighborhood bank is mounted on a wall 9 feet above the floor. What angle of depression should be used if the camera is to be directed to a spot 6 feet above the floor and 12 feet from the wall?

67. **Finding the Bearing of an Aircraft** A DC-9 aircraft leaves Midway Airport from runway 4 RIGHT, whose bearing is N40°E. After flying for $\frac{1}{2}$ mile, the pilot requests permission to turn 90° and head toward the southeast. The permission is granted. After the airplane goes 1 mile in this direction, what bearing should the control tower use to locate the aircraft?

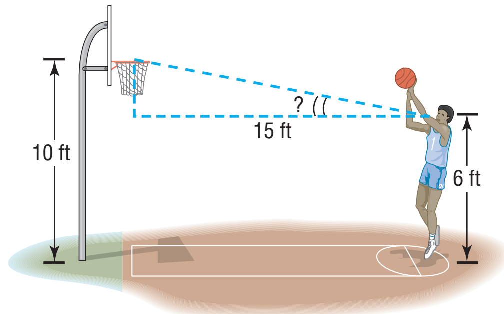
68. **Finding the Bearing of a Ship** A ship leaves the port of Miami with a bearing of S80°E and a speed of 15 knots. After 1 hour, the ship turns 90° toward the south. After 2 hours, maintaining the same speed, what is the bearing to the ship from port?

69. **Finding the Pitch of a Roof** A carpenter is preparing to put a roof on a garage that is 20 feet by 40 feet by 20 feet. A steel support beam 46 feet in height is positioned in the center of the garage. To support the roof, another beam will be attached to the top of the center beam (see the figure). At what angle of elevation is the new beam? In other words, what is the pitch of the roof?

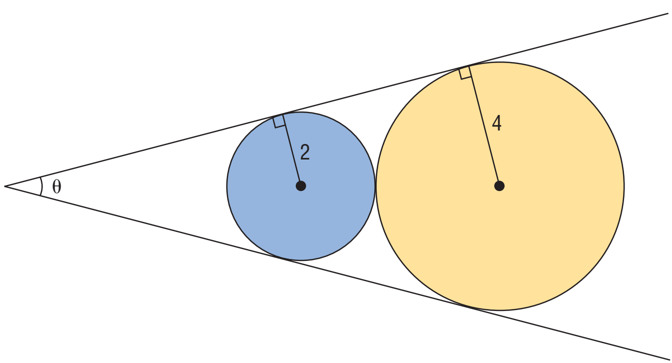


70. **Shooting Free Throws in Basketball** The eyes of a basketball player are 6 feet above the floor. The player is at the free-throw line, which is 15 feet from the center of the basket rim.

See the figure. What is the angle of elevation from the player's eyes to the center of the rim?
[**Hint:** The rim is 10 feet above the floor.]

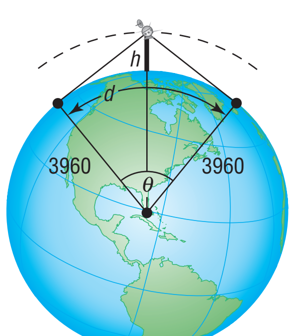


71. Geometry Find the value of the angle θ in degrees rounded to the nearest tenth of a degree.

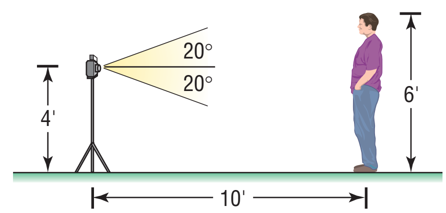


72. Surveillance Satellites A surveillance satellite circles Earth at a height of h miles above the surface. Suppose that d is the distance, in miles, on the surface of Earth that can be observed from the satellite. See the illustration.

(a) Find an equation that relates the central angle θ to the height h.
(b) Find an equation that relates the observable distance d and θ.
(c) Find an equation that relates d and h.
(d) If d is to be 2500 miles, how high must the satellite orbit above Earth?
(e) If the satellite orbits at a height of 300 miles, what distance d on the surface can be observed?

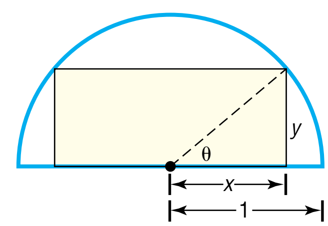


73. Photography A camera is mounted on a tripod 4 feet high at a distance of 10 feet from George, who is 6 feet tall. See the illustration. If the camera lens has an angle of depression and an angle of elevation of 20°, will George's feet and head be seen by the lens? If not, how far back will the camera need to be moved to include George's feet and head?



74. Construction A ramp for wheelchair accessibility is to be constructed with an angle of elevation of 15° and a final height of 5 feet. How long is the ramp?

75. Geometry A rectangle is inscribed in a semicircle of radius 1. See the illustration.

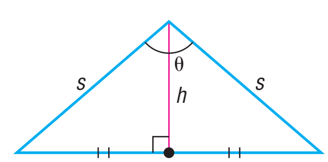


(a) Express the area A of the rectangle as a function of the angle θ shown in the illustration.
(b) Show that $A = \sin(2\theta)$.
(c) Find the angle θ that results in the largest area A.
(d) Find the dimensions of this largest rectangle.

76. Area of an Isosceles Triangle Show that the area A of an isosceles triangle, whose equal sides are of length s and the angle between them is θ, is

$$A = \frac{1}{2}s^2 \sin\theta$$

[**Hint:** See the illustration. The height h bisects the angle θ and is the perpendicular bisector of the base.]



Discussion and Writing

77. The Gibb's Hill Lighthouse, Southampton, Bermuda In operation since 1846, the Gibb's Hill Lighthouse stands 117 feet high on a hill 245 feet high, so its beam of light is 362 feet above sea level. A brochure states that ships 40 miles away can see the light and planes flying at 10,000 feet can see it 120 miles away. Verify the accuracy of these statements. What assumption did the brochure make about the height of the ship?

78. Explain how you would measure the width of the Grand Canyon from a point on its ridge.

79. Explain how you would measure the height of a TV tower that is on the roof of a tall building.

'Are You Prepared?' Answers

1. 4 **2.** 26.6°

7.2 The Law of Sines

PREPARING FOR THIS SECTION *Before getting started, review the following:*

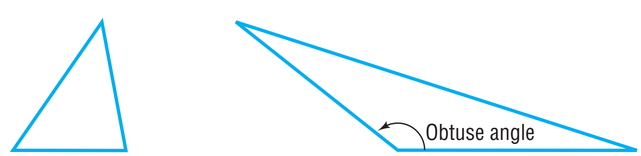
- Trigonometric Equations (I) (Section 6.7, pp. 496–500)
- Difference Formula for Sines (Section 6.4, p. 476)

Now work the 'Are You Prepared?' problems on page 538.

OBJECTIVES
1 Solve SAA or ASA Triangles
2 Solve SSA Triangles
3 Solve Applied Problems

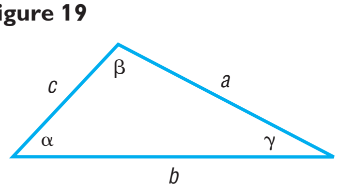
If none of the angles of a triangle is a right angle, the triangle is called **oblique**. An oblique triangle will have either three acute angles or two acute angles and one obtuse angle (an angle between 90° and 180°). See Figure 18.

Figure 18



(a) All angles are acute **(b)** Two acute angles and one obtuse angle

Figure 19



In the discussion that follows, we will always label an oblique triangle so that side a is opposite angle α, side b is opposite angle β, and side c is opposite angle γ, as shown in Figure 19.

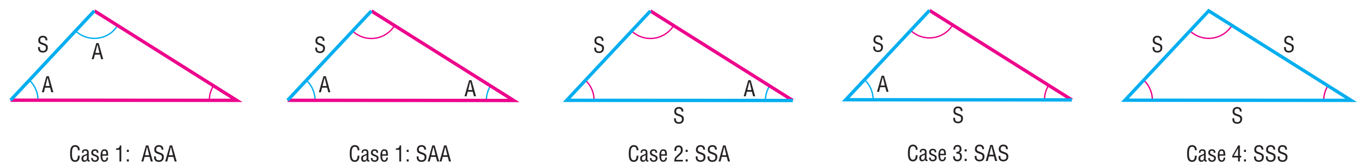
To **solve an oblique triangle** means to find the lengths of its sides and the measurements of its angles. To do this, we shall need to know the length of one side* along with (i) two angles; (ii) one angle and one other side; or (iii) the other two sides. There are four possibilities to consider:

*The reason we need to know the length of one side is that, if we only know the angles, this will result in a family of *similar triangles*.

CASE 1: One side and two angles are known (ASA or SAA).
CASE 2: Two sides and the angle opposite one of them are known (SSA).
CASE 3: Two sides and the included angle are known (SAS).
CASE 4: Three sides are known (SSS).

Figure 20 illustrates the four cases.

Figure 20



WARNING
Oblique triangles cannot be solved using the methods of Section 7.1. Do you know why?

The **Law of Sines** is used to solve triangles for which Case 1 or 2 holds. Cases 3 and 4 are considered when we study the Law of Cosines in the next section.

Theorem

Law of Sines

For a triangle with sides a, b, c and opposite angles α, β, γ, respectively,

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} \qquad (1)$$

A proof of the Law of Sines is given at the end of this section.

In applying the Law of Sines to solve triangles, we use the fact that the sum of the angles of any triangle equals 180°; that is,

$$\alpha + \beta + \gamma = 180° \qquad (2)$$

1 Solve SAA or ASA Triangles

Our first two examples show how to solve a triangle when one side and two angles are known (Case 1: SAA or ASA).

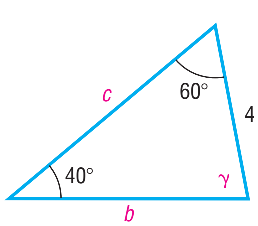
EXAMPLE 1 Using the Law of Sines to Solve a SAA Triangle

Solve the triangle: $\alpha = 40°, \beta = 60°, a = 4$

Solution

Figure 21 shows the triangle that we want to solve. The third angle γ is found using equation (2).

Figure 21



$$\begin{aligned} \alpha + \beta + \gamma &= 180° \\ 40° + 60° + \gamma &= 180° \\ \gamma &= 80° \end{aligned}$$

Now we use the Law of Sines (twice) to find the unknown sides b and c.

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} \qquad \frac{\sin \alpha}{a} = \frac{\sin \gamma}{c}$$

NOTE

Although not a check, we can verify the reasonableness of our answer by determining if the longest side is opposite the largest angle and the shortest side is opposite the smallest angle.

Because $a = 4$, $\alpha = 40°$, $\beta = 60°$, and $\gamma = 80°$, we have

$$\frac{\sin 40°}{4} = \frac{\sin 60°}{b} \qquad \frac{\sin 40°}{4} = \frac{\sin 80°}{c}$$

Solving for b and c, we find that

$$b = \frac{4 \sin 60°}{\sin 40°} \approx 5.39 \qquad c = \frac{4 \sin 80°}{\sin 40°} \approx 6.13$$

Notice in Example 1 that we found b and c by working with the given side a. This is better than finding b first and working with a rounded value of b to find c.

NOW WORK PROBLEM 9.

EXAMPLE 2 Using the Law of Sines to Solve an ASA Triangle

Solve the triangle: $\alpha = 35°$, $\beta = 15°$, $c = 5$

Solution

Figure 22

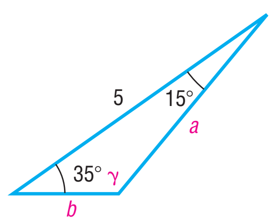


Figure 22 illustrates the triangle that we want to solve. Because we know two angles ($\alpha = 35°$ and $\beta = 15°$), we find the third angle using equation (2).

$$\begin{aligned} \alpha + \beta + \gamma &= 180° \\ 35° + 15° + \gamma &= 180° \\ \gamma &= 130° \end{aligned}$$

Now we know the three angles and one side ($c = 5$) of the triangle. To find the remaining two sides a and b, we use the Law of Sines (twice).

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c} \qquad \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

$$\frac{\sin 35°}{a} = \frac{\sin 130°}{5} \qquad \frac{\sin 15°}{b} = \frac{\sin 130°}{5}$$

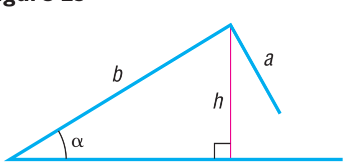
$$a = \frac{5 \sin 35°}{\sin 130°} \approx 3.74 \qquad b = \frac{5 \sin 15°}{\sin 130°} \approx 1.69$$

NOW WORK PROBLEM 23.

2 Solve SSA Triangles

Case 2 (SSA), which applies to triangles for which two sides and the angle opposite one of them are known, is referred to as the **ambiguous case**, because the known information may result in one triangle, two triangles, or no triangle at all. Suppose that we are given sides a and b and angle α, as illustrated in Figure 23. The key to determining the possible triangles, if any, that may be formed from the given information lies primarily with the height h and the fact that $h = b \sin \alpha$.

Figure 23



No Triangle If $a < h = b \sin \alpha$, then side a is not sufficiently long to form a triangle. See Figure 24.

One Right Triangle If $a = h = b \sin \alpha$, then side a is just long enough to form a right triangle. See Figure 25.

Figure 24
$a < h = b \sin \alpha$

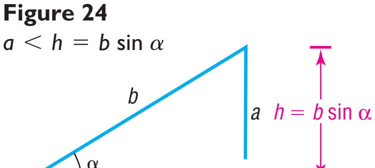
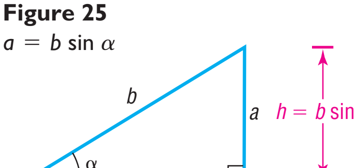


Figure 25
$a = b \sin \alpha$



Two Triangles If $a < b$ and $h = b \sin \alpha < a$, then two distinct triangles can be formed from the given information. See Figure 26.

One Triangle If $a \geq b$, then only one triangle can be formed. See Figure 27.

Figure 26
$b \sin \alpha < a$ and $a < b$

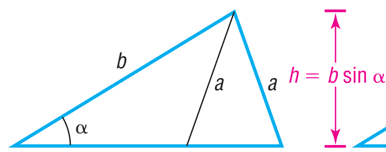
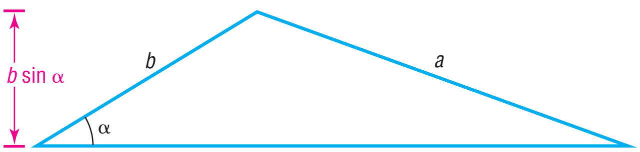


Figure 27
$a \geq b$

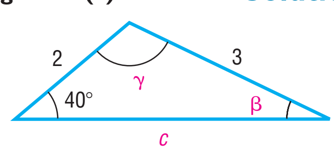


Fortunately, we do not have to rely on an illustration to draw the correct conclusion in the ambiguous case. The Law of Sines will lead us to the correct determination. Let's see how.

EXAMPLE 3 Using the Law of Sines to Solve a SSA Triangle (One Solution)

Solve the triangle: $a = 3, b = 2, \alpha = 40°$

Figure 28(a)



Solution See Figure 28(a). Because $a = 3$, $b = 2$, and $\alpha = 40°$ are known, we use the Law of Sines to find the angle β.

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$$

Then

$$\frac{\sin 40°}{3} = \frac{\sin \beta}{2}$$

$$\sin \beta = \frac{2 \sin 40°}{3} \approx 0.43$$

NOTE
Here we computed β_1 by determining the value of $\sin^{-1}\left(\frac{2 \sin 40°}{3}\right)$. If you use the rounded value and evaluate $\sin^{-1}(0.43)$, you will obtain a slightly different result. ■

There are two angles β, $0° < \beta < 180°$, for which $\sin \beta \approx 0.43$.

$$\beta_1 \approx 25.4° \quad \text{and} \quad \beta_2 \approx 180° - 25.4° = 154.6°$$

The second possibility, $\beta_2 \approx 154.6°$, is ruled out, because $\alpha = 40°$, making $\alpha + \beta_2 \approx 194.6° > 180°$. Now, using $\beta_1 \approx 25.4°$, we find that

$$\gamma = 180° - \alpha - \beta_1 \approx 180° - 40° - 25.4° = 114.6°$$

The third side c may now be determined using the Law of Sines.

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c}$$

$$\frac{\sin 40°}{3} = \frac{\sin 114.6°}{c}$$

$$c = \frac{3 \sin 114.6°}{\sin 40°} \approx 4.24$$

Figure 28(b)

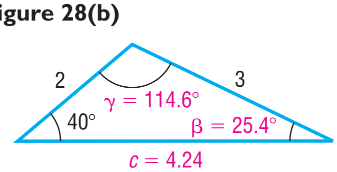
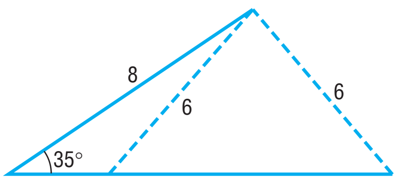


Figure 28(b) illustrates the solved triangle. ◀

EXAMPLE 4 Using the Law of Sines to Solve a SSA Triangle (Two Solutions)

Solve the triangle: $a = 6, b = 8, \alpha = 35°$

Figure 29(a)



Solution See Figure 29(a). Because $a = 6$, $b = 8$, and $\alpha = 35°$ are known, we use the Law of Sines to find the angle β.

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$$

Then

$$\frac{\sin 35°}{6} = \frac{\sin \beta}{8}$$

$$\sin \beta = \frac{8 \sin 35°}{6} \approx 0.76$$

$$\beta_1 \approx 49.9° \quad \text{or} \quad \beta_2 \approx 180° - 49.9° = 130.1°$$

For both choices of β, we have $\alpha + \beta < 180°$. There are two triangles, one containing the angle $\beta_1 \approx 49.9°$ and the other containing the angle $\beta_2 \approx 130.1°$. The third angle γ is either

$$\gamma_1 = 180° - \underbrace{\alpha - \beta_1}_{\substack{\alpha = 35° \\ \beta_1 = 49.9°}} \approx 95.1° \quad \text{or} \quad \gamma_2 = 180° - \underbrace{\alpha - \beta_2}_{\substack{\alpha = 35° \\ \beta_2 = 130.1°}} \approx 14.9°$$

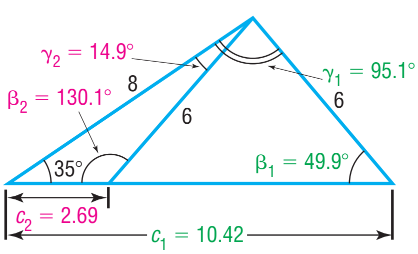
The third side c obeys the Law of Sines, so we have

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma_1}{c_1} \qquad \frac{\sin \alpha}{a} = \frac{\sin \gamma_2}{c_2}$$

$$\frac{\sin 35°}{6} = \frac{\sin 95.1°}{c_1} \qquad \frac{\sin 35°}{6} = \frac{\sin 14.9°}{c_2}$$

$$c_1 = \frac{6 \sin 95.1°}{\sin 35°} \approx 10.42 \qquad c_2 = \frac{6 \sin 14.9°}{\sin 35°} \approx 2.69$$

Figure 29(b)



The two solved triangles are illustrated in Figure 29(b). ◀

EXAMPLE 5 Using the Law of Sines to Solve a SSA Triangle (No Solution)

Solve the triangle: $a = 2, c = 1, \gamma = 50°$

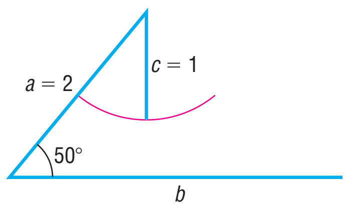
Solution Because $a = 2$, $c = 1$, and $\gamma = 50°$ are known, we use the Law of Sines to find the angle α.

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c}$$

$$\frac{\sin \alpha}{2} = \frac{\sin 50°}{1}$$

$$\sin \alpha = 2 \sin 50° \approx 1.53$$

Figure 30



Since there is no angle α for which $\sin \alpha > 1$, there can be no triangle with the given measurements. Figure 30 illustrates the measurements given. Notice that, no matter how we attempt to position side c, it will never touch side b to form a triangle. ◀

NOW WORK PROBLEMS 25 AND 31.

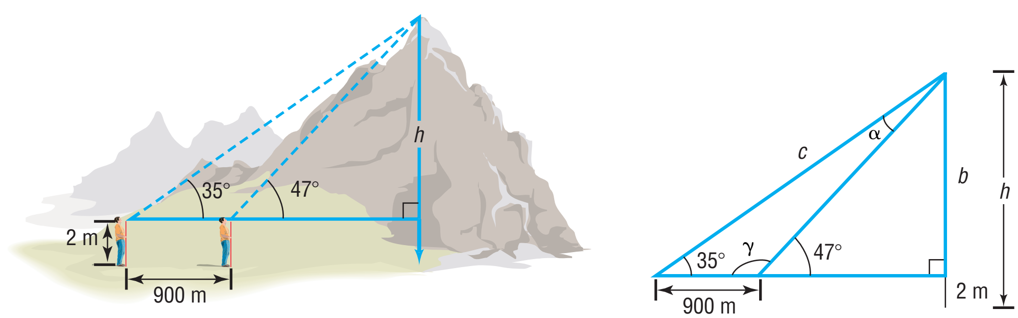
3 Solve Applied Problems

The Law of Sines is particularly useful for solving certain applied problems.

EXAMPLE 6 Finding the Height of a Mountain

To measure the height of a mountain, a surveyor takes two sightings of the peak at a distance 900 meters apart on a direct line to the mountain.* See Figure 31(a). The first observation results in an angle of elevation of 47° and the second results in an angle of elevation of 35°. If the transit is 2 meters high, what is the height h of the mountain?

Figure 31



Solution Figure 31(b) shows the triangles that replicate the illustration in Figure 31(a). Since $\gamma + 47° = 180°$, we find that $\gamma = 133°$. Also, since $\alpha + \gamma + 35° = 180°$, we find that $\alpha = 180° - 35° - \gamma = 145° - 133° = 12°$. We use the Law of Sines to find c.

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c} \qquad \alpha = 12°, \gamma = 133°, a = 900$$

$$c = \frac{900 \sin 133°}{\sin 12°} \approx 3165.86$$

Using the larger right triangle, we have

$$\sin 35° = \frac{b}{c} \qquad c = 3165.86$$

$$b = 3165.86 \sin 35° \approx 1815.86 \approx 1816 \text{ meters}$$

The height of the peak from ground level is approximately 1816 + 2 = 1818 meters. ◀

NOW WORK PROBLEM 39.

*For simplicity, we assume that these sightings are at the same level.

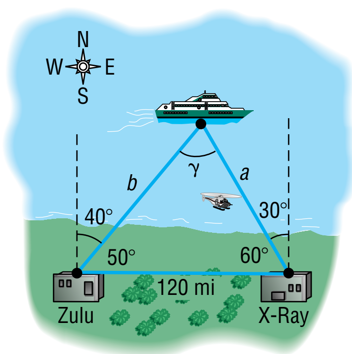
EXAMPLE 7 Rescue at Sea

Coast Guard Station Zulu is located 120 miles due west of Station X-ray. A ship at sea sends an SOS call that is received by each station. The call to Station Zulu indicates that the bearing of the ship from Zulu is N40°E (40° east of north). The call to Station X-ray indicates that the bearing of the ship from X-ray is N30°W (30° west of north).

(a) How far is each station from the ship?

(b) If a helicopter capable of flying 200 miles per hour is dispatched from the nearest station to the ship, how long will it take to reach the ship?

Figure 32



Solution

(a) Figure 32 illustrates the situation. The angle γ is found to be

$$\gamma = 180° - 50° - 60° = 70°$$

The Law of Sines can now be used to find the two distances a and b that we seek.

$$\frac{\sin 50°}{a} = \frac{\sin 70°}{120}$$

$$a = \frac{120 \sin 50°}{\sin 70°} \approx 97.82 \text{ miles}$$

$$\frac{\sin 60°}{b} = \frac{\sin 70°}{120}$$

$$b = \frac{120 \sin 60°}{\sin 70°} \approx 110.59 \text{ miles}$$

Station Zulu is about 111 miles from the ship, and Station X-ray is about 98 miles from the ship.

(b) The time t needed for the helicopter to reach the ship from Station X-ray is found by using the formula

$$(\text{Velocity}, v)(\text{Time}, t) = \text{Distance}, a$$

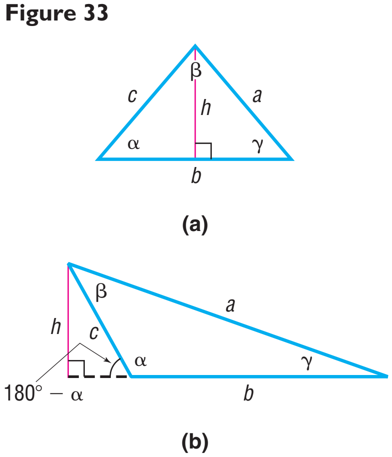
Then

$$t = \frac{a}{v} = \frac{97.82}{200} \approx 0.49 \text{ hour} \approx 29 \text{ minutes}$$

It will take about 29 minutes for the helicopter to reach the ship. ◀

NOW WORK PROBLEM 37.

Figure 33



Proof of the Law of Sines To prove the Law of Sines, we construct an altitude of length h from one of the vertices of a triangle. Figure 33(a) shows h for a triangle with three acute angles, and Figure 33(b) shows h for a triangle with an obtuse angle. In each case, the altitude is drawn from the vertex at β. Using either illustration, we have

$$\sin \gamma = \frac{h}{a}$$

from which

$$h = a \sin \gamma \tag{3}$$

From Figure 33(a), it also follows that

$$\sin \alpha = \frac{h}{c}$$

from which

$$h = c \sin \alpha \tag{4}$$

From Figure 33(b), it follows that

$$\sin(180° - \alpha) = \sin \alpha = \frac{h}{c}$$

$\sin(180° - \alpha) = \sin 180° \cos \alpha - \cos 180° \sin \alpha = \sin \alpha$

which again gives

$$h = c \sin \alpha$$

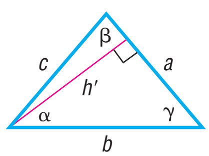
So, whether the triangle has three acute angles or has two acute angles and one obtuse angle, equations (3) and (4) hold. As a result, we may equate the expressions for h in equations (3) and (4) to get

$$a \sin \gamma = c \sin \alpha$$

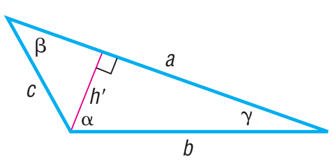
from which

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c} \tag{5}$$

Figure 34



(a)



(b)

In a similar manner, by constructing the altitude h' from the vertex of angle α as shown in Figure 34, we can show that

$$\sin \beta = \frac{h'}{c} \quad \text{and} \quad \sin \gamma = \frac{h'}{b}$$

Equating the expressions for h', we find that

$$h' = c \sin \beta = b \sin \gamma$$

from which

$$\frac{\sin \beta}{b} = \frac{\sin \gamma}{c} \tag{6}$$

When equations (5) and (6) are combined, we have equation (1), the Law of Sines. ■

7.2 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The difference formula for sine is $\sin(\alpha - \beta) =$ _____. (p. 476)
2. If θ is an acute angle, solve the equation $\cos \theta = \dfrac{\sqrt{3}}{2}$. (pp. 497–498)
3. If θ is an acute angle, solve the equation $\sin \theta = 2$. (pp. 496–500)

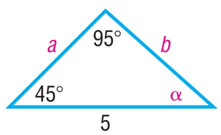
Concepts and Vocabulary

4. If none of the angles of a triangle is a right angle, the triangle is called _____.
5. For a triangle with sides a, b, c and opposite angles α, β, γ, the Law of Sines states that _____.
6. *True or False:* An oblique triangle in which two sides and an angle are given always results in at least one triangle.
7. *True or False:* The sum of the angles of any triangle equals 180°.
8. *True or False:* The ambiguous case refers to the fact that, when two sides and the angle opposite one of them are given, sometimes the Law of Sines cannot be used.

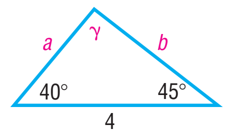
Skill Building

In Problems 9–16, solve each triangle.

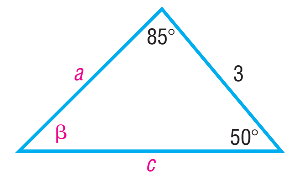
9.



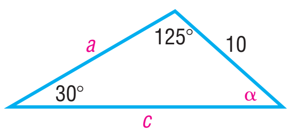
10.



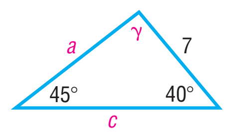
11.



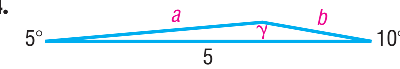
12.



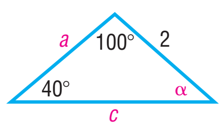
13.



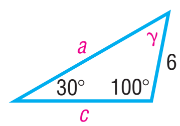
14.



15.



16.



In Problems 17–24, solve each triangle.

17. $\alpha = 40°, \quad \beta = 20°, \quad a = 2$

18. $\alpha = 50°, \quad \gamma = 20°, \quad a = 3$

19. $\beta = 70°, \quad \gamma = 10°, \quad b = 5$

20. $\alpha = 70°, \quad \beta = 60°, \quad c = 4$

21. $\alpha = 110°, \quad \gamma = 30°, \quad c = 3$

22. $\beta = 10°, \quad \gamma = 100°, \quad b = 2$

23. $\alpha = 40°, \quad \beta = 40°, \quad c = 2$

24. $\beta = 20°, \quad \gamma = 70°, \quad a = 1$

In Problems 25–36, two sides and an angle are given. Determine whether the given information results in one triangle, two triangles, or no triangle at all. Solve any triangle(s) that results.

25. $a = 3, \quad b = 2, \quad \alpha = 50°$

26. $b = 4, \quad c = 3, \quad \beta = 40°$

27. $b = 5, \quad c = 3, \quad \beta = 100°$

28. $a = 2, \quad c = 1, \quad \alpha = 120°$

29. $a = 4, \quad b = 5, \quad \alpha = 60°$

30. $b = 2, \quad c = 3, \quad \beta = 40°$

31. $b = 4, \quad c = 6, \quad \beta = 20°$

32. $a = 3, \quad b = 7, \quad \alpha = 70°$

33. $a = 2, \quad c = 1, \quad \gamma = 100°$

34. $b = 4, \quad c = 5, \quad \beta = 95°$

35. $a = 2, \quad c = 1, \quad \gamma = 25°$

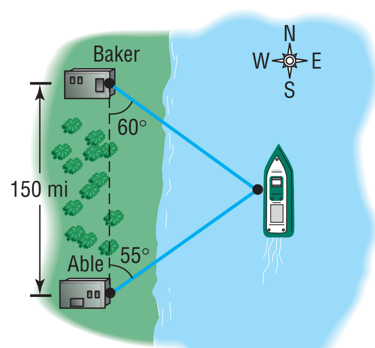
36. $b = 4, \quad c = 5, \quad \beta = 40°$

Applications and Extensions

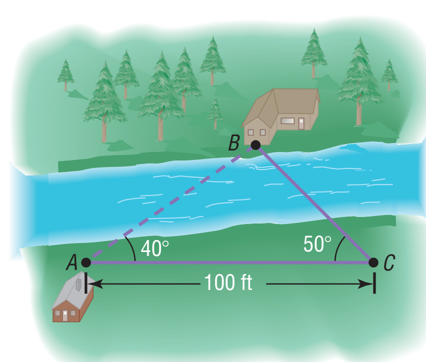
37. Rescue at Sea Coast Guard Station Able is located 150 miles due south of Station Baker. A ship at sea sends an SOS call that is received by each station. The call to Station Able indicates that the ship is located N55°E; the call to Station Baker indicates that the ship is located S60°E.

(a) How far is each station from the ship?

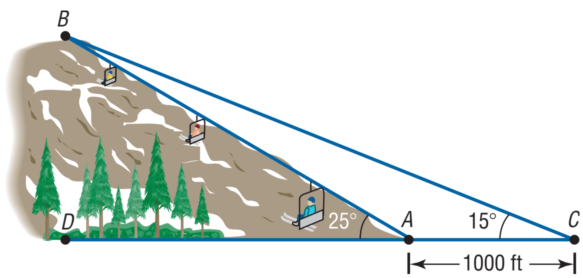
(b) If a helicopter capable of flying 200 miles per hour is dispatched from the station nearest to the ship, how long will it take to reach the ship?



38. Surveying Consult the figure. To find the distance from the house at A to the house at B, a surveyor measures the angle BAC to be 40° and then walks off a distance of 100 feet to C and measures the angle ACB to be 50°. What is the distance from A to B?

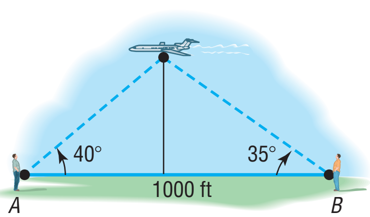


39. Finding the Length of a Ski Lift Consult the figure. To find the length of the span of a proposed ski lift from A to B, a surveyor measures the angle DAB to be 25° and then walks off a distance of 1000 feet to C and measures the angle ACB to be 15°. What is the distance from A to B?



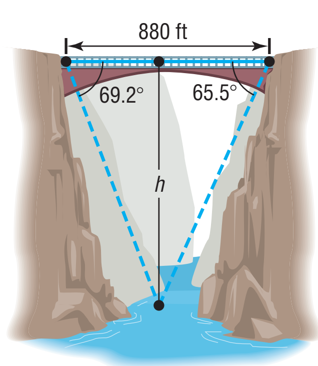
40. Finding the Height of a Mountain Use the illustration in Problem 39 to find the height BD of the mountain at B.

41. Finding the Height of an Airplane An aircraft is spotted by two observers who are 1000 feet apart. As the airplane passes over the line joining them, each observer takes a sighting of the angle of elevation to the plane, as indicated in the figure. How high is the airplane?

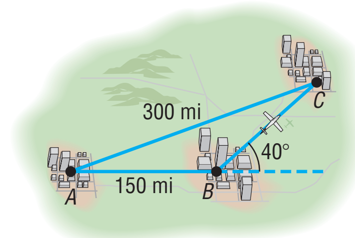


42. Finding the Height of the Bridge over the Royal Gorge The highest bridge in the world is the bridge over the Royal Gorge of the Arkansas River in Colorado. Sightings to the same point at water level directly under the bridge are taken from each side of the 880-foot-long bridge, as indicated in the figure. How high is the bridge?

SOURCE: *Guinness Book of World Records*.



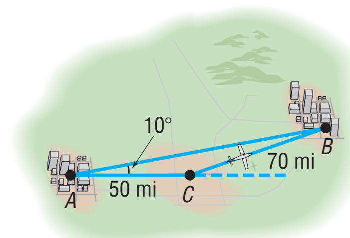
43. Navigation An airplane flies from city A to city B, a distance of 150 miles, and then turns through an angle of 40° and heads toward city C, as shown in the figure.



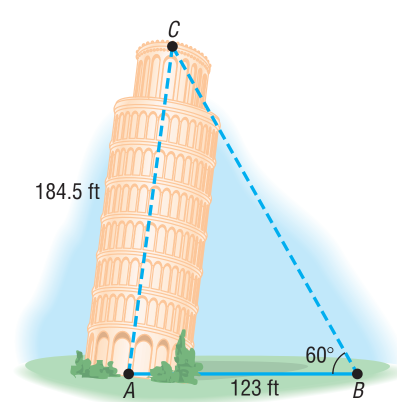
(a) If the distance between cities A and C is 300 miles, how far is it from city B to city C?

(b) Through what angle should the pilot turn at city C to return to city A?

44. Time Lost due to a Navigation Error In attempting to fly from city A to city B, an aircraft followed a course that was 10° in error, as indicated in the figure. After flying a distance of 50 miles, the pilot corrected the course by turning at point C and flying 70 miles farther. If the constant speed of the aircraft was 250 miles per hour, how much time was lost due to the error?



45. Finding the Lean of the Leaning Tower of Pisa The famous Leaning Tower of Pisa was originally 184.5 feet high.* At a distance of 123 feet from the base of the tower, the angle of elevation to the top of the tower is found to be 60°. Find the angle CAB indicated in the figure. Also, find the perpendicular distance from C to AB.

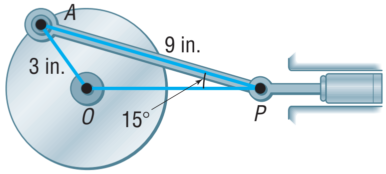


*In their 1986 report on the fragile seven-century-old bell tower, scientists in Pisa, Italy, said that the Leaning Tower of Pisa had increased its famous lean by 1 millimeter, or 0.04 inch. This is about the annual average, although the tilting had slowed to about half that much in the previous 2 years. (*Source:* United Press International, June 29, 1986.)

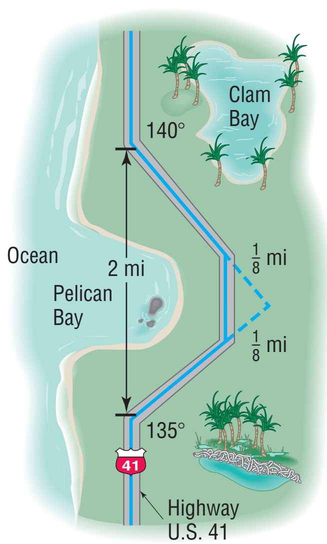
PISA, ITALY. September 1995. The Leaning Tower of Pisa has suddenly shifted, jeopardizing years of preservation work to stabilize it, Italian newspapers said Sunday. The tower, built on shifting subsoil between 1174 and 1350 as a belfry for the nearby cathedral, recently moved 0.07 inch in one night.

Update The tower, which had been closed to tourists since 1990, was reopened in December, 2001, after reinforcement of its base.

46. Crankshafts on Cars On a certain automobile, the crankshaft is 3 inches long and the connecting rod is 9 inches long (see the figure). At the time when the angle OPA is 15°, how far is the piston (P) from the center (O) of the crankshaft?

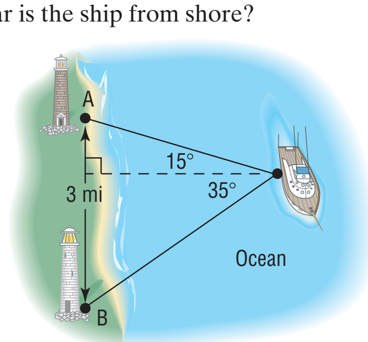


47. Constructing a Highway U.S. 41, a highway whose primary directions are north–south, is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The illustration shows the path that they decide on and the measurements taken. What is the length of highway needed to go around the bay?

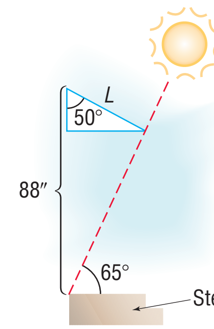


48. Calculating Distances at Sea The navigator of a ship at sea spots two lighthouses that she knows to be 3 miles apart along a straight seashore. She determines that the angles formed between two line-of-sight observations of the lighthouses and the line from the ship directly to shore are 15° and 35°. See the illustration.

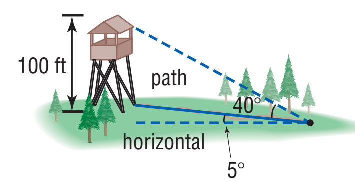
(a) How far is the ship from lighthouse A?
(b) How far is the ship from lighthouse B?
(c) How far is the ship from shore?



49. Designing an Awning An awning that covers a sliding glass door that is 88 inches tall forms an angle of 50° with the wall. The purpose of the awning is to prevent sunlight from entering the house when the angle of elevation of the sun is more than 65°. See the figure. Find the length L of the awning.

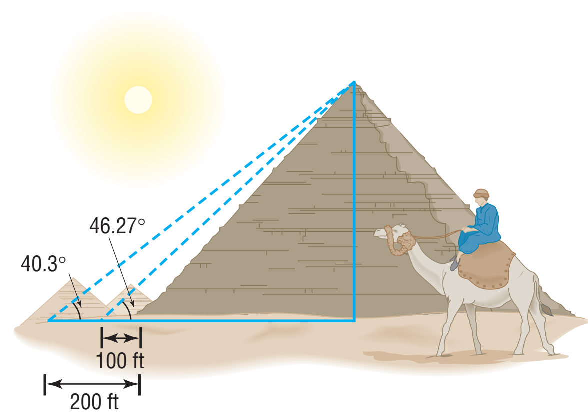


50. Finding Distances A forest ranger is walking on a path inclined at 5° to the horizontal directly toward a 100-foot-tall fire observation tower. The angle of elevation from the path to the top of the tower is 40°. How far is the ranger from the tower at this time?



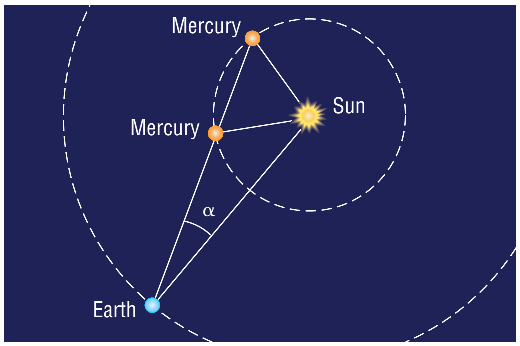
51. Great Pyramid of Cheops One of the original Seven Wonders of the World, the Great Pyramid of Cheops was built about 2580 BC. Its original height was 480 feet 11 inches, but due to the loss of its topmost stones, it is now shorter. Find the current height of the Great Pyramid, using the information given in the illustration.

SOURCE: *Guinness Book of World Records*.



52. Determining the Height of an Aircraft Two sensors are spaced 700 feet apart along the approach to a small airport. When an aircraft is nearing the airport, the angle of elevation from the first sensor to the aircraft is 20°, and from the second sensor to the aircraft it is 15°. Determine how high the aircraft is at this time.

53. Mercury The distance from the Sun to Earth is approximately 149,600,000 kilometers (km). The distance from the Sun to Mercury is approximately 57,910,000 km. The **elongation angle** α is the angle formed between the line of sight from Earth to the Sun and the line of sight from Earth to Mercury. See the figure. Suppose that the elongation angle for Mercury is 15°. Use this information to find the possible distances between Earth and Mercury.

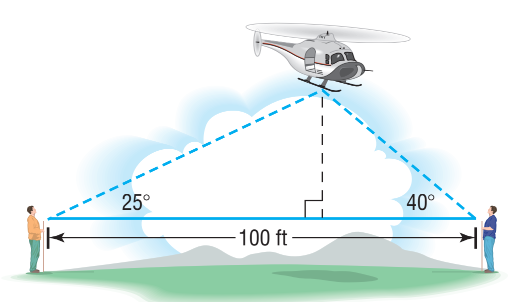


54. Venus The distance from the Sun to Earth is approximately 149,600,000 km. The distance from the Sun to Venus is approximately 108,200,000 km. The elongation angle α is the angle formed between the line of sight from Earth to the Sun and the line of sight from Earth to Venus. Suppose that the elongation angle for Venus is 10°. Use this information to find the possible distances between Earth and Venus.

55. Landscaping Pat needs to determine the height of a tree before cutting it down to be sure that it will not fall on a nearby fence. The angle of elevation of the tree from one position on a flat path from the tree is 30°, and from a second position 40 feet farther along this path it is 20°. What is the height of the tree?

56. Construction A loading ramp 10 feet long that makes an angle of 18° with the horizontal is to be replaced by one that makes an angle of 12° with the horizontal. How long is the new ramp?

57. Finding the Height of a Helicopter Two observers simultaneously measure the angle of elevation of a helicopter. One angle is measured as 25°, the other as 40° (see the figure). If the observers are 100 feet apart and the helicopter lies over the line joining them, how high is the helicopter?



58. Mollweide's Formula For any triangle, Mollweide's Formula (named after Karl Mollweide, 1774–1825) states that

$$\frac{a+b}{c} = \frac{\cos\left[\frac{1}{2}(\alpha - \beta)\right]}{\sin\left(\frac{1}{2}\gamma\right)}$$

Derive it.

[**Hint:** Use the Law of Sines and then a Sum-to-Product Formula. Notice that this formula involves all six parts of a triangle. As a result, it is sometimes used to check the solution of a triangle.]

59. Mollweide's Formula Another form of Mollweide's Formula is

$$\frac{a-b}{c} = \frac{\sin\left[\frac{1}{2}(\alpha - \beta)\right]}{\cos\left(\frac{1}{2}\gamma\right)}$$

Derive it.

60. For any triangle, derive the formula

$$a = b\cos\gamma + c\cos\beta$$

[**Hint:** Use the fact that $\sin\alpha = \sin(180° - \beta - \gamma)$.]

61. Law of Tangents For any triangle, derive the Law of Tangents.

$$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(\alpha - \beta)\right]}{\tan\left[\frac{1}{2}(\alpha + \beta)\right]}$$

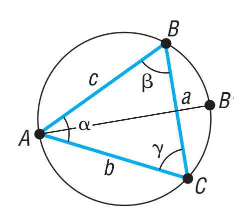
[**Hint:** Use Mollweide's Formula.]

62. Circumscribing a Triangle Show that

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{1}{2r}$$

where r is the radius of the circle circumscribing the triangle ABC whose sides are a, b, and c, as shown in the figure.

[**Hint:** Draw the diameter AB'. Then $\beta = \text{angle } ABC = \text{angle } AB'C$, and angle $ACB' = 90°$.]



Discussion and Writing

63. Make up three problems involving oblique triangles. One should result in one triangle, the second in two triangles, and the third in no triangle.

64. What do you do first if you are asked to solve a triangle and are given one side and two angles?

65. What do you do first if you are asked to solve a triangle and are given two sides and the angle opposite one of them?

'Are You Prepared?' Answers

1. $\sin\alpha\cos\beta - \cos\alpha\sin\beta$ **2.** 30° **3.** No solution

7.3 The Law of Cosines

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Trigonometric Equations (I) (Section 6.7, pp. 496–500)
- Distance Formula (Section 1.1, p. 5)

Now work the 'Are You Prepared?' problems on page 546.

OBJECTIVES
1. Solve SAS Triangles
2. Solve SSS Triangles
3. Solve Applied Problems

In the previous section, we used the Law of Sines to solve Case 1 (SAA or ASA) and Case 2 (SSA) of an oblique triangle. In this section, we derive the Law of Cosines and use it to solve the remaining cases, 3 and 4.

Case 3: Two sides and the included angle are known (SAS).

Case 4: Three sides are known (SSS).

Theorem

Law of Cosines

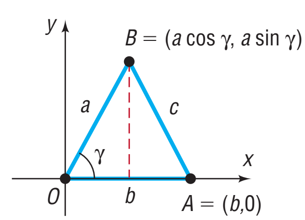
For a triangle with sides a, b, c and opposite angles α, β, γ, respectively,

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \qquad (1)$$

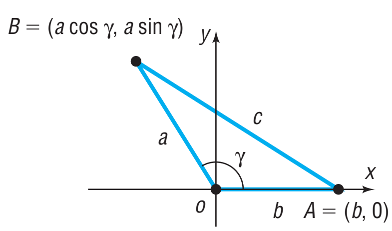
$$b^2 = a^2 + c^2 - 2ac\cos\beta \qquad (2)$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \qquad (3)$$

Figure 35



(a) Angle γ is acute



(b) Angle γ is obtuse

Proof We will prove only formula (1) here. Formulas (2) and (3) may be proved using the same argument.

We begin by strategically placing a triangle on a rectangular coordinate system so that the vertex of angle γ is at the origin and side b lies along the positive x-axis. Regardless of whether γ is acute, as in Figure 35(a), or obtuse, as in Figure 35(b), the vertex B has coordinates $(a\cos\gamma, a\sin\gamma)$. Vertex A has coordinates $(b, 0)$.

We can now use the distance formula to compute c^2.

$$\begin{aligned} c^2 &= (b - a\cos\gamma)^2 + (0 - a\sin\gamma)^2 \\ &= b^2 - 2ab\cos\gamma + a^2\cos^2\gamma + a^2\sin^2\gamma \\ &= b^2 - 2ab\cos\gamma + a^2(\cos^2\gamma + \sin^2\gamma) \\ &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}$$

■

Each of formulas (1), (2), and (3) may be stated in words as follows:

Theorem

Law of Cosines

The square of one side of a triangle equals the sum of the squares of the other two sides minus twice their product times the cosine of their included angle.

Observe that if the triangle is a right triangle (so that, say, $\gamma = 90°$), then formula (1) becomes the familiar Pythagorean Theorem: $c^2 = a^2 + b^2$. The Pythagorean Theorem is a special case of the Law of Cosines!

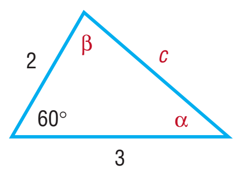
1 Solve SAS Triangles

Let's see how to use the Law of Cosines to solve Case 3 (SAS), which applies to triangles for which two sides and the included angle are known.

EXAMPLE 1 Using the Law of Cosines to Solve a SAS Triangle

Solve the triangle: $a = 2, \quad b = 3, \quad \gamma = 60°$

Figure 36



Solution See Figure 36. The Law of Cosines makes it easy to find the third side, c.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ &= 4 + 9 - 2\cdot 2\cdot 3\cdot \cos 60° \\ &= 13 - \left(12\cdot\frac{1}{2}\right) = 7 \\ c &= \sqrt{7} \end{aligned}$$

Side c is of length $\sqrt{7}$. To find the angles α and β, we may use either the Law of Sines or the Law of Cosines. It is preferable to use the Law of Cosines, since it will lead to an equation with one solution. Using the Law of Sines would lead to an equation with two solutions that would need to be checked to determine which solution fits the given data. We choose to use formulas (2) and (3) of the Law of Cosines to find α and β.

For α:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ 2bc\cos\alpha &= b^2 + c^2 - a^2 \\ \cos\alpha &= \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 7 - 4}{2\cdot 3\sqrt{7}} = \frac{12}{6\sqrt{7}} = \frac{2\sqrt{7}}{7} \\ \alpha &\approx 40.9° \end{aligned}$$

For β:

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\beta \\ \cos\beta &= \frac{a^2 + c^2 - b^2}{2ac} = \frac{4 + 7 - 9}{4\sqrt{7}} = \frac{1}{2\sqrt{7}} = \frac{\sqrt{7}}{14} \\ \beta &\approx 79.1° \end{aligned}$$

Notice that $\alpha + \beta + \gamma = 40.9° + 79.1° + 60° = 180°$, as required. ◀

NOW WORK PROBLEM 9.

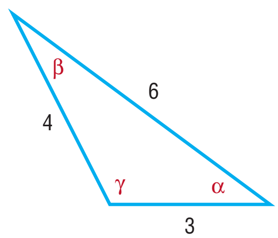
2 Solve SSS Triangles

The next example illustrates how the Law of Cosines is used when three sides of a triangle are known, Case 4 (SSS).

EXAMPLE 2 Using the Law of Cosines to Solve a SSS Triangle

Solve the triangle: $a = 4, b = 3, c = 6$

Figure 37



Solution See Figure 37. To find the angles α, β, and γ, we proceed as we did in the latter part of the solution to Example 1.

For α:

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 36 - 16}{2 \cdot 3 \cdot 6} = \frac{29}{36}$$
$$\alpha \approx 36.3°$$

For β:

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{16 + 36 - 9}{2 \cdot 4 \cdot 6} = \frac{43}{48}$$
$$\beta \approx 26.4°$$

Since we know α and β,

$$\gamma = 180° - \alpha - \beta \approx 180° - 36.3° - 26.4° = 117.3°$$

◀

NOW WORK PROBLEM 15.

3 Solve Applied Problems

EXAMPLE 3 **Correcting a Navigational Error**

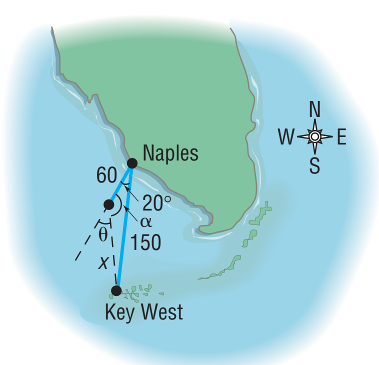
A motorized sailboat leaves Naples, Florida, bound for Key West, 150 miles away. Maintaining a constant speed of 15 miles per hour, but encountering heavy crosswinds and strong currents, the crew finds, after 4 hours, that the sailboat is off course by 20°.

(a) How far is the sailboat from Key West at this time?

(b) Through what angle should the sailboat turn to correct its course?

(c) How much time has been added to the trip because of this? (Assume that the speed remains at 15 miles per hour.)

Figure 38



Solution See Figure 38. With a speed of 15 miles per hour, the sailboat has gone 60 miles after 4 hours. We seek the distance x of the sailboat from Key West. We also seek the angle θ that the sailboat should turn through to correct its course.

(a) To find x, we use the Law of Cosines, since we know two sides and the included angle.

$$x^2 = 150^2 + 60^2 - 2(150)(60)\cos 20° \approx 9186$$
$$x \approx 95.8$$

The sailboat is about 96 miles from Key West.

(b) We now know three sides of the triangle, so we can use the Law of Cosines again to find the angle α opposite the side of length 150 miles.

$$150^2 = 96^2 + 60^2 - 2(96)(60)\cos \alpha$$
$$9684 = -11{,}520 \cos \alpha$$
$$\cos \alpha \approx -0.8406$$
$$\alpha \approx 147.2°$$

The sailboat should turn through an angle of

$$\theta = 180° - \alpha \approx 180° - 147.2° = 32.8°$$

The sailboat should turn through an angle of about 33° to correct its course.

(c) The total length of the trip is now $60 + 96 = 156$ miles. The extra 6 miles will only require about 0.4 hour or 24 minutes more if the speed of 15 miles per hour is maintained. ◀

NOW WORK PROBLEM 35.

Historical Feature

The Law of Sines was known vaguely long before it was explicitly stated by Nasir Eddin (about AD 1250). Ptolemy (about AD 150) was aware of it in a form using a chord function instead of the sine function. But it was first clearly stated in Europe by Regiomontanus, writing in 1464.

The Law of Cosines appears first in Euclid's *Elements* (Book II), but in a well-disguised form in which squares built on the sides of triangles are added and a rectangle representing the cosine term is subtracted. It was thus known to all mathematicians because of their familiarity with Euclid's work. An early modern form of the Law of Cosines, that for finding the angle when the sides are known, was stated by François Viète (in 1593).

The Law of Tangents (see Problem 61 of Exercise 7.2) has become obsolete. In the past it was used in place of the Law of Cosines, because the Law of Cosines was very inconvenient for calculation with logarithms or slide rules. Mixing of addition and multiplication is now very easy on a calculator, however, and the Law of Tangents has been shelved along with the slide rule.

7.3 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Write the formula for the distance d from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$. (p. 5)

2. If θ is an acute angle, solve the equation $\cos\theta = \frac{\sqrt{2}}{2}$. (pp. 497–498)

Concepts and Vocabulary

3. If three sides of a triangle are given, the Law of _______ is used to solve the triangle.

4. If one side and two angles of a triangle are given, the Law of _______ is used to solve the triangle.

5. If two sides and the included angle of a triangle are given, the Law of _______ is used to solve the triangle.

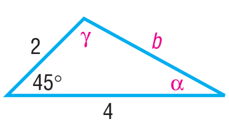
6. *True or False:* Given only the three sides of a triangle, there is insufficient information to solve the triangle.

7. *True or False:* Given two sides and the included angle, the first thing to do to solve the triangle is to use the Law of Sines.

8. *True or False:* A special case of the Law of Cosines is the Pythagorean Theorem.

Skill Building

In Problems 9–16, solve each triangle.

9.


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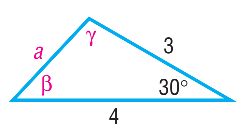
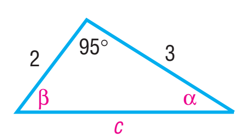


11.


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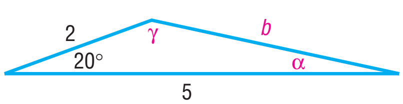
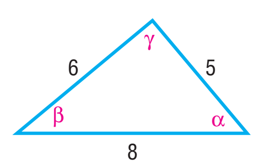
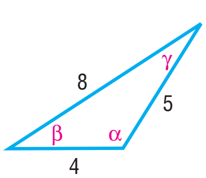
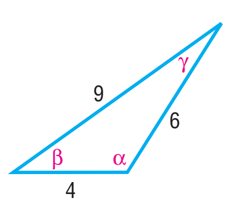


13.


14.


15.


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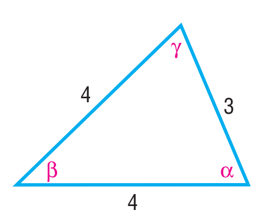


In Problems 17–32, solve each triangle.

17. $a = 3, \quad b = 4, \quad \gamma = 40°$

18. $a = 2, \quad c = 1, \quad \beta = 10°$

19. $b = 1, \quad c = 3, \quad \alpha = 80°$

20. $a = 6, \quad b = 4, \quad \gamma = 60°$

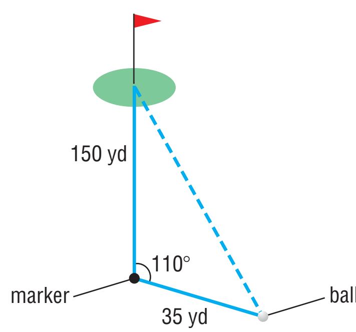
21. $a = 3, \quad c = 2, \quad \beta = 110°$

22. $b = 4, \quad c = 1, \quad \alpha = 120°$

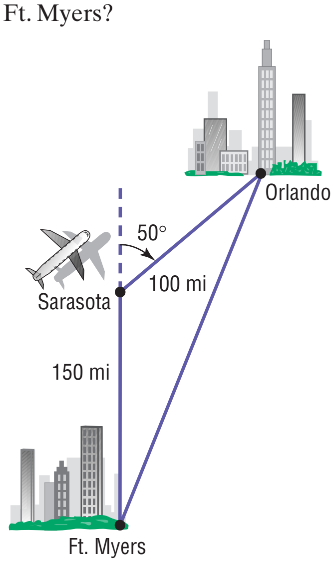
23. $a = 2, \quad b = 2, \quad \gamma = 50°$
24. $a = 3, \quad c = 2, \quad \beta = 90°$
25. $a = 12, \quad b = 13, \quad c = 5$
26. $a = 4, \quad b = 5, \quad c = 3$
27. $a = 2, \quad b = 2, \quad c = 2$
28. $a = 3, \quad b = 3, \quad c = 2$
29. $a = 5, \quad b = 8, \quad c = 9$
30. $a = 4, \quad b = 3, \quad c = 6$
31. $a = 10, \quad b = 8, \quad c = 5$
32. $a = 9, \quad b = 7, \quad c = 10$

Applications and Extensions

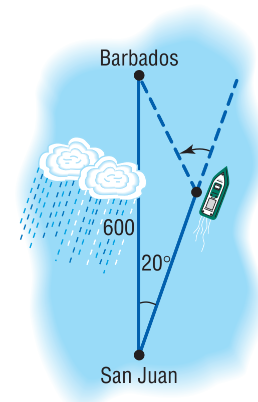
33. Distance to the Green A golfer hits an errant tee shot that lands in the rough. A marker in the center of the fairway is 150 yards from the center of the green. While standing on the marker and facing the green, the golfer turns 110° toward his ball. He then paces off 35 yards to his ball. See the figure. How far is the ball from the center of the green?



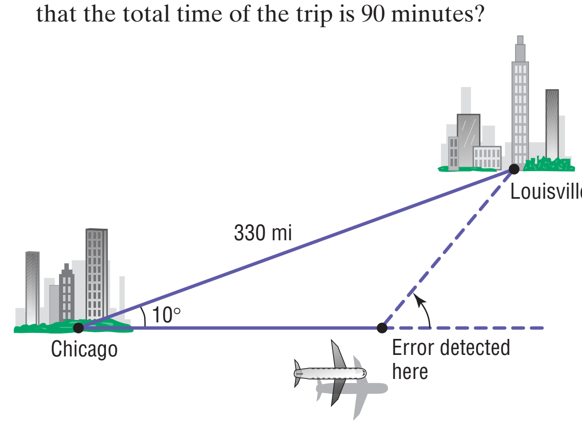
34. Navigation An airplane flies from Ft. Myers to Sarasota, a distance of 150 miles, and then turns through an angle of 50° and flies to Orlando, a distance of 100 miles (see the figure).
(a) How far is it from Ft. Myers to Orlando?
(b) Through what angle should the pilot turn at Orlando to return to Ft. Myers?



35. Avoiding a Tropical Storm A cruise ship maintains an average speed of 15 knots in going from San Juan, Puerto Rico, to Barbados, West Indies, a distance of 600 nautical miles. To avoid a tropical storm, the captain heads out of San Juan in a direction of 20° off a direct heading to Barbados. The captain maintains the 15-knot speed for 10 hours, after which time the path to Barbados becomes clear of storms.
(a) Through what angle should the captain turn to head directly to Barbados?
(b) Once the turn is made, how long will it be before the ship reaches Barbados if the same 15-knot speed is maintained?



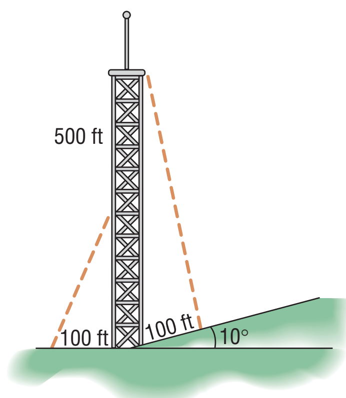
36. Revising a Flight Plan In attempting to fly from Chicago to Louisville, a distance of 330 miles, a pilot inadvertently took a course that was 10° in error, as indicated in the figure.
(a) If the aircraft maintains an average speed of 220 miles per hour and if the error in direction is discovered after 15 minutes, through what angle should the pilot turn to head toward Louisville?
(b) What new average speed should the pilot maintain so that the total time of the trip is 90 minutes?



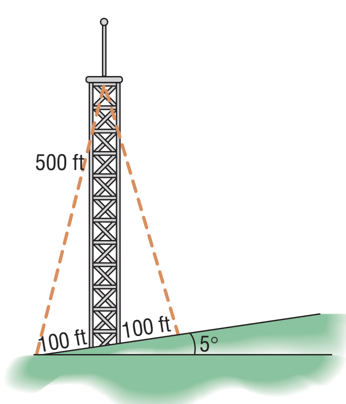
37. Major League Baseball Field A Major League baseball diamond is actually a square 90 feet on a side. The pitching rubber is located 60.5 feet from home plate on a line joining home plate and second base.
(a) How far is it from the pitching rubber to first base?
(b) How far is it from the pitching rubber to second base?
(c) If a pitcher faces home plate, through what angle does he need to turn to face first base?

38. Little League Baseball Field According to Little League baseball official regulations, the diamond is a square 60 feet on a side. The pitching rubber is located 46 feet from home plate on a line joining home plate and second base.
(a) How far is it from the pitching rubber to first base?
(b) How far is it from the pitching rubber to second base?
(c) If a pitcher faces home plate, through what angle does he need to turn to face first base?

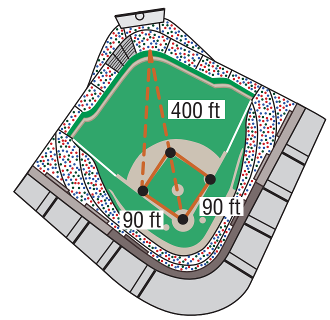
39. Finding the Length of a Guy Wire The height of a radio tower is 500 feet, and the ground on one side of the tower slopes upward at an angle of 10° (see the figure).
(a) How long should a guy wire be if it is to connect to the top of the tower and be secured at a point on the sloped side 100 feet from the base of the tower?
(b) How long should a second guy wire be if it is to connect to the middle of the tower and be secured at a point 100 feet from the base on the flat side?



40. Finding the Length of a Guy Wire A radio tower 500 feet high is located on the side of a hill with an inclination to the horizontal of 5° (see the figure). How long should two guy wires be if they are to connect to the top of the tower and be secured at two points 100 feet directly above and directly below the base of the tower?



41. Wrigley Field, Home of the Chicago Cubs The distance from home plate to the fence in dead center in Wrigley Field is 400 feet (see the figure). How far is it from the fence in dead center to third base?



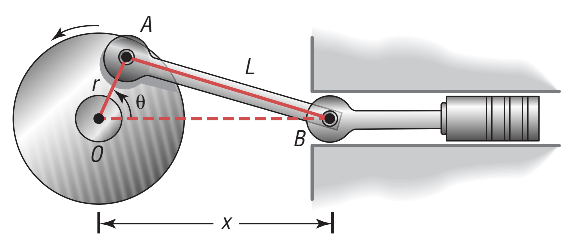
42. Little League Baseball The distance from home plate to the fence in dead center at the Oak Lawn Little League field is 280 feet. How far is it from the fence in dead center to third base?

[**Hint:** The distance between the bases in Little League is 60 feet.]

43. Rods and Pistons Rod OA (see the figure) rotates about the fixed point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston. Show that the distance x between point O and point B is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$

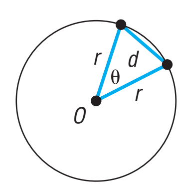
where θ is the angle of rotation of rod OA.



44. Geometry Show that the length d of a chord of a circle of radius r is given by the formula

$$d = 2r\sin\frac{\theta}{2}$$

where θ is the central angle formed by the radii to the ends of the chord (see the figure). Use this result to derive the fact that $\sin\theta < \theta$, where $\theta > 0$ is measured in radians.



45. For any triangle, show that

$$\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

where $s = \frac{1}{2}(a + b + c)$.

[**Hint:** Use a Half-angle Formula and the Law of Cosines.]

46. For any triangle show that

$$\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

where $s = \frac{1}{2}(a+b+c)$.

47. Use the Law of Cosines to prove the identity

$$\frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c} = \frac{a^2+b^2+c^2}{2abc}$$

Discussion and Writing

48. What do you do first if you are asked to solve a triangle and are given two sides and the included angle?

49. What do you do first if you are asked to solve a triangle and are given three sides?

50. Make up an applied problem that requires using the Law of Cosines.

51. Write down your strategy for solving an oblique triangle.

'Are You Prepared?' Answers

1. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **2.** $\theta = 45°$

7.4 Area of a Triangle

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Geometry Review (Appendix, Section A.2, pp. 963–964)

Now work the 'Are You Prepared?' problem on page 552.

OBJECTIVES 1 Find the Area of SAS Triangles
2 Find the Area of SSS Triangles

In this section, we will derive several formulas for calculating the area A of a triangle. The most familiar of these is the following:

Theorem

The area A of a triangle is

$$A = \frac{1}{2}bh \qquad \textbf{(1)}$$

where b is the base and h is an altitude drawn to that base.

Proof The derivation of this formula is rather easy once a rectangle of base b and height h is constructed around the triangle. See Figures 39 and 40.

Triangles 1 and 2 in Figure 40 are equal in area, as are triangles 3 and 4. Consequently, the area of the triangle with base b and altitude h is exactly half the area of the rectangle, which is bh.

Figure 39

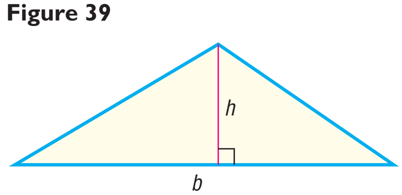
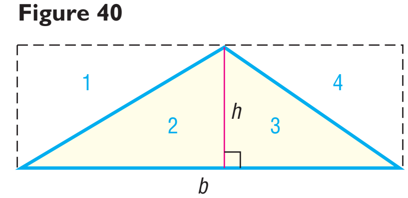


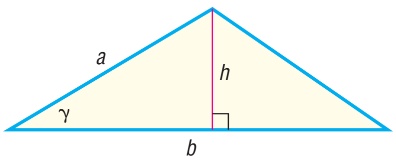
Figure 40



■

1 Find the Area of SAS Triangles

Figure 41



If the base b and altitude h to that base are known, then we can find the area of such a triangle using formula (1). Usually, though, the information required to use formula (1) is not given. Suppose, for example, that we know two sides a and b and the included angle γ (see Figure 41). Then the altitude h can be found by noting that

$$\frac{h}{a} = \sin \gamma$$

so that

$$h = a \sin \gamma$$

Using this fact in formula (1) produces

$$A = \frac{1}{2}bh = \frac{1}{2}b(a \sin \gamma) = \frac{1}{2}ab \sin \gamma$$

We now have the formula

$$A = \frac{1}{2}ab \sin \gamma \qquad \textbf{(2)}$$

By dropping altitudes from the other two vertices of the triangle, we obtain the following corresponding formulas:

$$A = \frac{1}{2}bc \sin \alpha \qquad \textbf{(3)}$$

$$A = \frac{1}{2}ac \sin \beta \qquad \textbf{(4)}$$

It is easiest to remember these formulas using the following wording:

Theorem

The area A of a triangle equals one-half the product of two of its sides times the sine of their included angle.

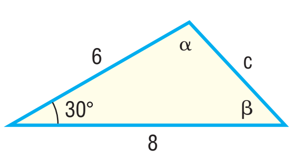
EXAMPLE 1 **Finding the Area of a SAS Triangle**

Find the area A of the triangle for which $a = 8$, $b = 6$, and $\gamma = 30°$.

Solution See Figure 42. We use formula (2) to get

$$A = \frac{1}{2}ab \sin \gamma = \frac{1}{2} \cdot 8 \cdot 6 \cdot \sin 30° = 12$$

Figure 42



NOW WORK PROBLEM 5.

2 Find the Area of SSS Triangles

If the three sides of a triangle are known, another formula, called **Heron's Formula** (named after Heron of Alexandria), can be used to find the area of a triangle.

Theorem

Heron's Formula

The area A of a triangle with sides a, b, and c is

$$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}} \qquad \textbf{(5)}$$

where $s = \frac{1}{2}(a + b + c)$.

A proof of Heron's Formula is given at the end of this section.

EXAMPLE 2 Finding the Area of a SSS Triangle

Find the area of a triangle whose sides are 4, 5, and 7.

Solution We let $a = 4$, $b = 5$, and $c = 7$. Then

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(4 + 5 + 7) = 8$$

Heron's Formula then gives the area A as

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{8 \cdot 4 \cdot 3 \cdot 1} = \sqrt{96} = 4\sqrt{6}$$ ◀

NOW WORK PROBLEM 11.

Proof of Heron's Formula The proof that we shall give uses the Law of Cosines and is quite different from the proof given by Heron.

From the Law of Cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

and the Half-angle Formula

$$\cos^2\frac{\gamma}{2} = \frac{1 + \cos\gamma}{2}$$

we find that

$$\begin{aligned}
\cos^2\frac{\gamma}{2} &= \frac{1+\cos\gamma}{2} = \frac{1 + \dfrac{a^2+b^2-c^2}{2ab}}{2} \\
&= \frac{a^2 + 2ab + b^2 - c^2}{4ab} = \frac{(a+b)^2 - c^2}{4ab} \\
&\underset{\substack{\uparrow \\ \text{Factor}}}{=} \frac{(a+b-c)(a+b+c)}{4ab} \underset{\substack{\uparrow \\ a+b-c = a+b+c-2c \\ = 2s - 2c = 2(s-c)}}{=} \frac{2(s-c)\cdot 2s}{4ab} = \frac{s(s-c)}{ab}
\end{aligned} \qquad \textbf{(6)}$$

Similarly,

$$\sin^2\frac{\gamma}{2} = \frac{(s-a)(s-b)}{ab} \qquad \textbf{(7)}$$

Now we use formula (2) for the area.

$$\begin{aligned} A &= \frac{1}{2}ab\sin\gamma \\ &= \frac{1}{2}ab\cdot 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2} && \sin\gamma = \sin\left[2\left(\frac{\gamma}{2}\right)\right] = 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2} \\ &= ab\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{s(s-c)}{ab}} && \text{Use equations (6) and (7).} \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

■

Historical Feature

Heron's Formula (also known as *Hero's Formula*) is due to Heron of Alexandria (first century AD), who had, besides his mathematical talents, a good deal of engineering skills. In various temples his mechanical devices produced effects that seemed supernatural, and visitors presumably were thus influenced to generosity. Heron's book *Metrica*, on making such devices, has survived and was discovered in 1896 in the city of Constantinople.

Heron's Formulas for the area of a triangle caused some mild discomfort in Greek mathematics, because a product with two factors was an area, while one with three factors was a volume, but four factors seemed contradictory in Heron's time.

7.4 Assess Your Understanding

'Are You Prepared?'

Answer given at the end of these exercises. If you get the wrong answer, read the page listed in red.

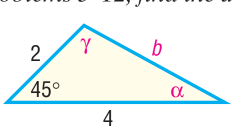
1. The area A of a triangle whose base is b and whose height is h is _____ (p. 963)

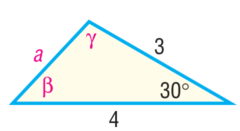
Concepts and Vocabulary

2. If three sides of a triangle are given, _______ Formula is used to find the area of the triangle.
3. *True or False:* No formula exists for finding the area of a triangle when only three sides are given.
4. *True or False:* Given two sides and the included angle, there is a formula that can be used to find the area of the triangle.

Skill Building

In Problems 5–12, find the area of each triangle. Round answers to two decimal places.

5.


6.


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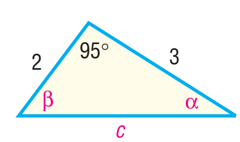

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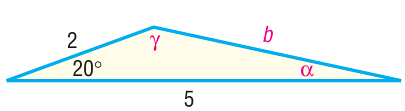

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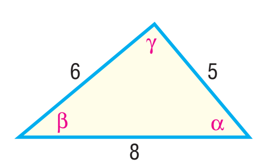

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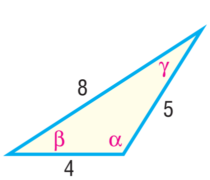

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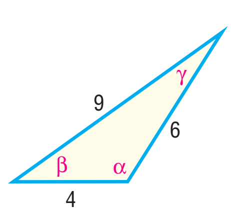

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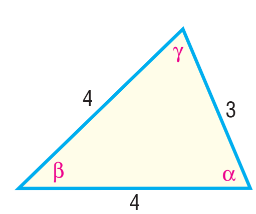


In Problems 13–24, find the area of each triangle. Round answers to two decimal places.

13. $a = 3, \quad b = 4, \quad \gamma = 40°$
14. $a = 2, \quad c = 1, \quad \beta = 10°$
15. $b = 1, \quad c = 3, \quad \alpha = 80°$
16. $a = 6, \quad b = 4, \quad \gamma = 60°$
17. $a = 3, \quad c = 2, \quad \beta = 110°$
18. $b = 4, \quad c = 1, \quad \alpha = 120°$
19. $a = 12, \quad b = 13, \quad c = 5$
20. $a = 4, \quad b = 5, \quad c = 3$
21. $a = 2, \quad b = 2, \quad c = 2$
22. $a = 3, \quad b = 3, \quad c = 2$
23. $a = 5, \quad b = 8, \quad c = 9$
24. $a = 4, \quad b = 3, \quad c = 6$

Applications and Extensions

25. Area of a Triangle Prove that the area A of a triangle is given by the formula

$$A = \frac{a^2 \sin \beta \sin \gamma}{2 \sin \alpha}$$

26. Area of a Triangle Prove the two other forms of the formula given in Problem 25.

$$A = \frac{b^2 \sin \alpha \sin \gamma}{2 \sin \beta} \quad \text{and} \quad A = \frac{c^2 \sin \alpha \sin \beta}{2 \sin \gamma}$$

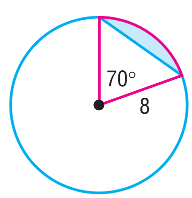
In Problems 27–32, use the results of Problem 25 or 26 to find the area of each triangle. Round answers to two decimal places.

27. $\alpha = 40°, \quad \beta = 20°, \quad a = 2$
28. $\alpha = 50°, \quad \gamma = 20°, \quad a = 3$
29. $\beta = 70°, \quad \gamma = 10°, \quad b = 5$
30. $\alpha = 70°, \quad \beta = 60°, \quad c = 4$
31. $\alpha = 110°, \quad \gamma = 30°, \quad c = 3$
32. $\beta = 10°, \quad \gamma = 100°, \quad b = 2$

---

33. Area of a Segment Find the area of the segment (shaded in blue in the figure) of a circle whose radius is 8 feet, formed by a central angle of 70°.

[**Hint:** Subtract the area of the triangle from the area of the sector to obtain the area of the segment.]



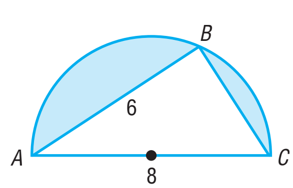
34. Area of a Segment Find the area of the segment of a circle whose radius is 5 inches, formed by a central angle of 40°.

35. Cost of a Triangular Lot The dimensions of a triangular lot are 100 feet by 50 feet by 75 feet. If the price of such land is \$3 per square foot, how much does the lot cost?

36. Amount of Materials to Make a Tent A cone-shaped tent is made from a circular piece of canvas 24 feet in diameter by removing a sector with central angle 100° and connecting the ends. What is the surface area of the tent?

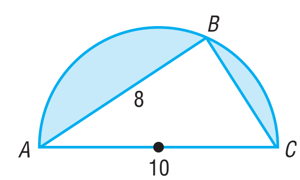
37. Computing Areas Find the area of the shaded region enclosed in a semicircle of diameter 8 centimeters. The length of the chord AB is 6 centimeters.

[**Hint:** Triangle ABC is a right triangle.]

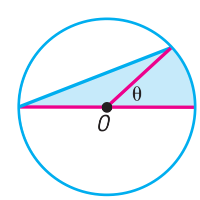


38. Computing Areas Find the area of the shaded region enclosed in a semicircle of diameter 10 inches. The length of the chord AB is 8 inches.

[**Hint:** Triangle ABC is a right triangle.]

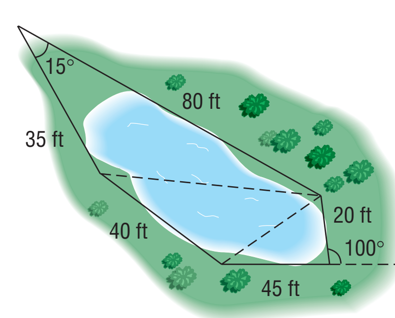


39. Geometry Consult the figure, which shows a circle of radius r with center at O. Find the area A of the shaded region as a function of the central angle θ.



40. Approximating the Area of a Lake To approximate the area of a lake, a surveyor walks around the perimeter of the lake, taking the measurements shown in the illustration. Using this technique, what is the approximate area of the lake?

[**Hint:** Use the Law of Cosines on the three triangles shown and then find the sum of their areas.]



41. Refer to the figure. If $|OA| = 1$, show that:

(a) Area $\Delta OAC = \frac{1}{2}\sin\alpha\cos\alpha$

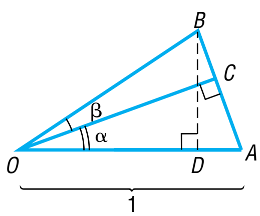
(b) Area $\Delta OCB = \frac{1}{2}|OB|^2\sin\beta\cos\beta$

(c) Area $\Delta OAB = \frac{1}{2}|OB|\sin(\alpha+\beta)$

(d) $|OB| = \dfrac{\cos\alpha}{\cos\beta}$

(e) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

[**Hint:** Area ΔOAB = Area ΔOAC + Area ΔOCB.]



42. Refer to the figure, in which a unit circle is drawn. The line DB is tangent to the circle.

(a) Express the area of ΔOBC in terms of $\sin\theta$ and $\cos\theta$.

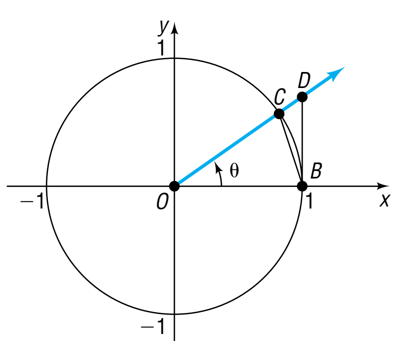
(b) Express the area of ΔOBD in terms of $\sin\theta$ and $\cos\theta$.

(c) The area of the sector $\widehat{OBC}$ of the circle is $\frac{1}{2}\theta$, where θ is measured in radians. Use the results of parts (a) and (b) and the fact that

$$\text{Area } \Delta OBC < \text{Area } \widehat{OBC} < \text{Area } \Delta OBD$$

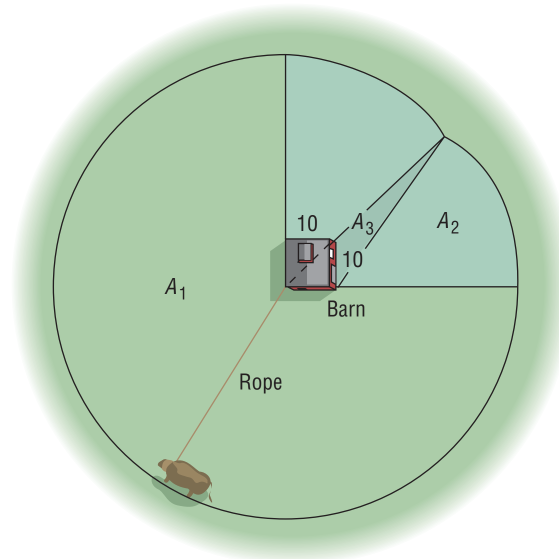
to show that

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$



43. The Cow Problem* A cow is tethered to one corner of a square barn, 10 feet by 10 feet, with a rope 100 feet long. What is the maximum grazing area for the cow?

[**Hint:** See the illustration.]



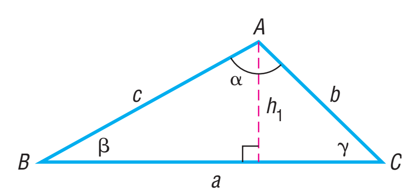
44. Another Cow Problem If the barn in Problem 43 is rectangular, 10 feet by 20 feet, what is the maximum grazing area for the cow?

45. If h_1, h_2, and h_3 are the altitudes dropped from A, B, and C, respectively, in a triangle (see the figure), show that

$$\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{s}{K}$$

where K is the area of the triangle and $s = \frac{1}{2}(a+b+c)$.

[**Hint:** $h_1 = \dfrac{2K}{a}$.]



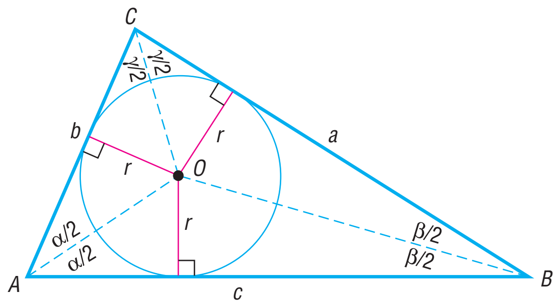
46. Show that a formula for the altitude h from a vertex to the opposite side a of a triangle is

$$h = \frac{a\sin\beta\sin\gamma}{\sin\alpha}$$

Inscribed Circle *For Problems 47–50, the lines that bisect each angle of a triangle meet in a single point O, and the perpendicular distance r from O to each side of the triangle is the same. The circle*

*Suggested by Professor Teddy Koukounas of Suffolk Community College, who learned of it from an old farmer in Virginia. Solution provided by Professor Kathleen Miranda of SUNY at Old Westbury.

with center at O and radius r is called the ***inscribed circle*** *of the triangle (see the figure).*



47. Apply Problem 46 to triangle OAB to show that

$$r = \frac{c \sin \dfrac{\alpha}{2} \sin \dfrac{\beta}{2}}{\cos \dfrac{\gamma}{2}}$$

48. Use the results of Problems 46 and Problem 47 in Section 7.3 to show that

$$\cot \frac{\gamma}{2} = \frac{s - c}{r}$$

49. Show that

$$\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = \frac{s}{r}$$

50. Show that the area K of triangle ABC is $K = rs$. Then show that

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

where $s = \frac{1}{2}(a + b + c)$.

Discussion and Writing

51. What do you do first if you are asked to find the area of a triangle and are given two sides and the included angle?

52. What do you do first if you are asked to find the area of a triangle and are given three sides?

'Are You Prepared?' Answer

1. $A = \frac{1}{2}bh$

7.5 Simple Harmonic Motion; Damped Motion; Combining Waves

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Sinusoidal Graphs (Section 5.4, pp. 408–414)

Now work the 'Are You Prepared?' problem on page 562.

OBJECTIVES
1. Find an Equation for an Object in Simple Harmonic Motion
2. Analyze Simple Harmonic Motion
3. Analyze an Object in Damped Motion
4. Graph the Sum of Two Functions

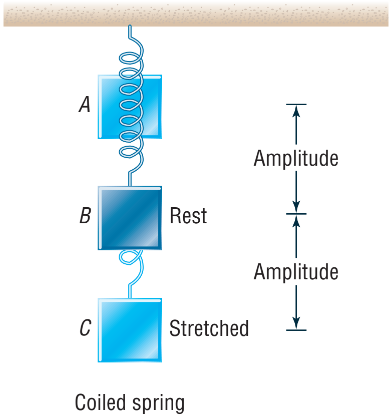
Vibrating tuning fork

1 Find an Equation for an Object in Simple Harmonic Motion

Many physical phenomena can be described as simple harmonic motion. Radio and television waves, light waves, sound waves, and water waves exhibit motion that is simple harmonic.

The swinging of a pendulum, the vibrations of a tuning fork, and the bobbing of a weight attached to a coiled spring are examples of vibrational motion. In this type of motion, an object swings back and forth over the same path. In Figure 43, the point B is the **equilibrium (rest) position** of the vibrating object. The **amplitude** is the distance from the object's rest position to its point of greatest displacement

Figure 43



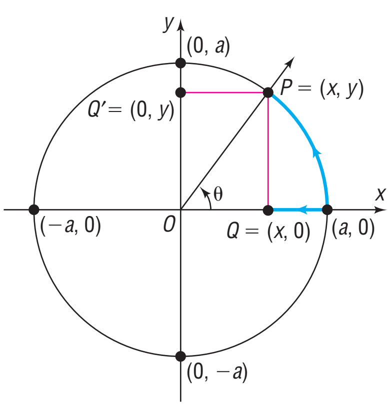
(either point A or point C in Figure 43). The **period** is the time required to complete one vibration, that is, the time it takes to go from, say, point A through B to C and back to A.

Simple harmonic motion is a special kind of vibrational motion in which the acceleration a of the object is directly proportional to the negative of its displacement d from its rest position. That is, $a = -kd$, $k > 0$.

For example, when the mass hanging from the spring in Figure 43 is pulled down from its rest position B to the point C, the force of the spring tries to restore the mass to its rest position. Assuming that there is no frictional force* to retard the motion, the amplitude will remain constant. The force increases in direct proportion to the distance that the mass is pulled from its rest position. Since the force increases directly, the acceleration of the mass of the object must do likewise, because (by Newton's Second Law of Motion) force is directly proportional to acceleration. As a result, the acceleration of the object varies directly with its displacement, and the motion is an example of simple harmonic motion.

Simple harmonic motion is related to circular motion. To see this relationship, consider a circle of radius a, with center at $(0, 0)$. See Figure 44. Suppose that an object initially placed at $(a, 0)$ moves counterclockwise around the circle at a constant angular speed ω. Suppose further that after time t has elapsed the object is at the point $P = (x, y)$ on the circle. The angle θ, in radians, swept out by the ray $\overrightarrow{OP}$ in this time t is

Figure 44



$$\theta = \omega t$$

The coordinates of the point P at time t are

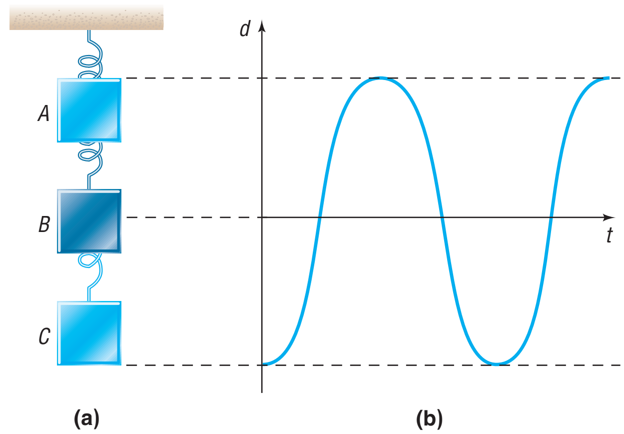
$$x = a\cos\theta = a\cos(\omega t)$$
$$y = a\sin\theta = a\sin(\omega t)$$

Corresponding to each position $P = (x, y)$ of the object moving about the circle, there is the point $Q = (x, 0)$, called the **projection of *P* on the *x*-axis**. As P moves around the circle at a constant rate, the point Q moves back and forth between the points $(a, 0)$ and $(-a, 0)$ along the x-axis with a motion that is simple harmonic. Similarly, for each point P there is a point $Q' = (0, y)$, called the **projection of *P* on the *y*-axis**. As P moves around the circle, the point Q' moves back and forth between the points $(0, a)$ and $(0, -a)$ on the y-axis with a motion that is simple harmonic. Simple harmonic motion can be described as the projection of constant circular motion on a coordinate axis.

To put it another way, again consider a mass hanging from a spring where the mass is pulled down from its rest position to the point C and then released. See Figure 45(a). The graph shown in Figure 45(b) describes the displacement d of the object from its rest position as a function of time t, assuming that no frictional force is present.

*If friction is present, the amplitude will decrease with time to 0. This type of motion is an example of **damped motion**, which is discussed later in this section.

Figure 45



Theorem

Simple Harmonic Motion

An object that moves on a coordinate axis so that the distance d from its rest position at time t is given by either

$$d = a\cos(\omega t) \quad \text{or} \quad d = a\sin(\omega t)$$

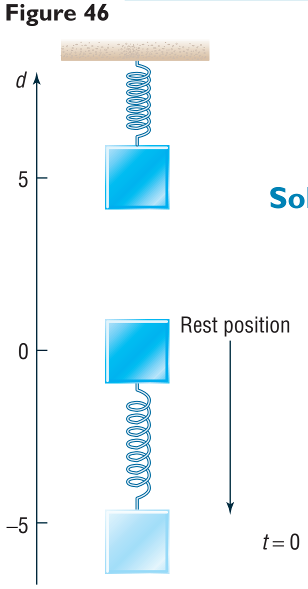
where a and $\omega > 0$ are constants, moves with simple harmonic motion. The motion has amplitude $|a|$ and period $\dfrac{2\pi}{\omega}$.

The **frequency** f of an object in simple harmonic motion is the number of oscillations per unit time. Since the period is the time required for one oscillation, it follows that the frequency is the reciprocal of the period; that is,

$$f = \frac{\omega}{2\pi} \qquad \omega > 0$$

EXAMPLE 1 Finding an Equation for an Object in Harmonic Motion

Figure 46



Suppose that an object attached to a coiled spring is pulled down a distance of 5 inches from its rest position and then released. If the time for one oscillation is 3 seconds, write an equation that relates the displacement d of the object from its rest position after time t (in seconds). Assume no friction.

Solution The motion of the object is simple harmonic. See Figure 46. When the object is released ($t = 0$), the displacement of the object from the rest position is -5 units (since the object was pulled down). Because $d = -5$ when $t = 0$, it is easier to use the cosine function*

$$d = a\cos(\omega t)$$

to describe the motion. Now the amplitude is $|-5| = 5$ and the period is 3, so

$$a = -5 \quad \text{and} \quad \frac{2\pi}{\omega} = \text{period} = 3, \quad \omega = \frac{2\pi}{3}$$

*No phase shift is required if a cosine function is used.

An equation of the motion of the object is

$$d = -5\cos\left[\frac{2\pi}{3}t\right]$$

NOTE In the solution to Example 1, we let $a = -5$, since the initial motion is down. If the initial direction were up, we would let $a = 5$.

NOW WORK PROBLEM 5.

2 Analyze Simple Harmonic Motion

EXAMPLE 2 Analyzing the Motion of an Object

Suppose that the displacement d (in meters) of an object at time t (in seconds) satisfies the equation

$$d = 10\sin(5t)$$

(a) Describe the motion of the object.
(b) What is the maximum displacement from its resting position?
(c) What is the time required for one oscillation?
(d) What is the frequency?

Solution We observe that the given equation is of the form

$$d = a\sin(\omega t) \qquad d = 10\sin(5t)$$

where $a = 10$ and $\omega = 5$.

(a) The motion is simple harmonic.
(b) The maximum displacement of the object from its resting position is the amplitude: $|a| = 10$ meters.
(c) The time required for one oscillation is the period:

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{5}\text{ seconds}$$

(d) The frequency is the reciprocal of the period. Thus,

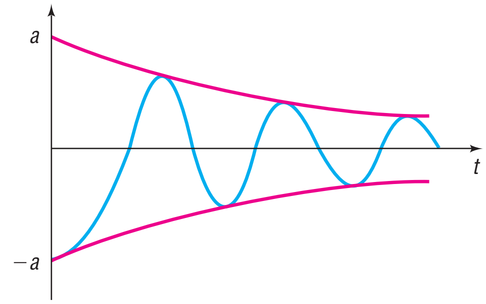
$$\text{Frequency} = f = \frac{5}{2\pi}\text{ oscillations per second}$$

NOW WORK PROBLEM 13.

3 Analyze an Object in Damped Motion

Most physical phenomena are affected by friction or other resistive forces. These forces remove energy from a moving system and thereby damp its motion. For example, when a mass hanging from a spring is pulled down a distance a and released, the friction in the spring causes the distance that the mass moves from its at-rest position to decrease over time. As a result, the amplitude of any real oscillating spring or swinging pendulum decreases with time due to air resistance, friction, and so forth. See Figure 47.

Figure 47



A function that describes this phenomenon maintains a sinusoidal component, but the amplitude of this component will decrease with time to account for the damping effect. In addition, the period of the oscillating component will be affected by the damping. The next result, from physics, describes damped motion.

Theorem

Damped Motion

The displacement d of an oscillating object from its at-rest position at time t is given by

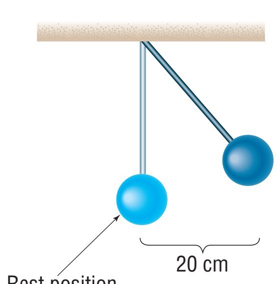
$$\boxed{d(t) = ae^{-bt/2m}\cos\left(\sqrt{\omega^2 - \frac{b^2}{4m^2}}\,t\right)}$$

where b is a **damping factor** (most physics texts call this a **damping coefficient**) and m is the mass of the oscillating object. Here $|a|$ is the displacement at $t = 0$ and $\dfrac{2\pi}{\omega}$ is the period under simple harmonic motion (no damping).

Notice for $b = 0$ (zero damping) that we have the formula for simple harmonic motion with amplitude $|a|$ and period $\dfrac{2\pi}{\omega}$.

EXAMPLE 3 Analyzing the Motion of a Pendulum with Damped Motion

Figure 48



Suppose that a simple pendulum with a bob of mass 10 grams and a damping factor of 0.8 gram/second is pulled 20 centimeters from its at-rest position and released. See Figure 48. The period of the pendulum without the damping effect is 4 seconds.

(a) Find an equation that describes the position of the pendulum bob.

(b) Using a graphing utility, graph the function found in part (a).

(c) What is the displacement of the bob at the start of the second oscillation?

(d) What happens to the displacement of the bob as time increases without bound?

Solution

(a) We have $m = 10$, $a = 20$, and $b = 0.8$. Since the period of the pendulum under simple harmonic motion is 4 seconds, we have

$$4 = \frac{2\pi}{\omega}$$

$$\omega = \frac{2\pi}{4} = \frac{\pi}{2}$$

Figure 49

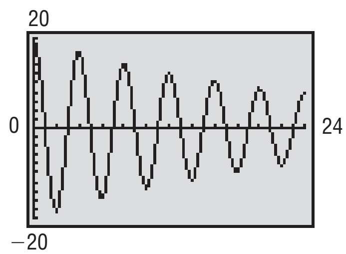
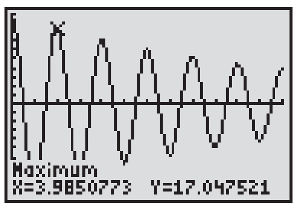


Figure 50



Substituting these values into the equation for damped motion, we obtain

$$d = 20e^{-0.8t/2(10)}\cos\left(\sqrt{\left(\frac{\pi}{2}\right)^2 - \frac{0.8^2}{4(10)^2}}\,t\right)$$

$$d = d(t) = 20e^{-0.8t/20}\cos\left(\sqrt{\frac{\pi^2}{4} - \frac{0.64}{400}}\,t\right)$$

(b) See Figure 49 for the graph of $d = d(t)$.

(c) See Figure 50. At the start of the second oscillation the displacement is approximately 17.05 centimeters.

(d) As t increases without bound $e^{-0.8t/20} \to 0$, so the displacement of the bob approaches zero. As a result, the pendulum will eventually come to rest. ◀

NOW WORK PROBLEM 21.

4 Graph the Sum of Two Functions

Many physical and biological applications require the graph of the sum of two functions, such as

$$f(x) = x + \sin x \quad \text{or} \quad g(x) = \sin x + \cos(2x)$$

For example, if two tones are emitted, the sound produced is the sum of the waves produced by the two tones. See Problem 43 for an explanation of Touch-Tone phones.

To graph the sum of two (or more) functions, we can use the method of adding y-coordinates described next.

EXAMPLE 4 Graphing the Sum of Two Functions

Use the method of adding y-coordinates to graph $f(x) = x + \sin x$.

Solution First, we graph the component functions,

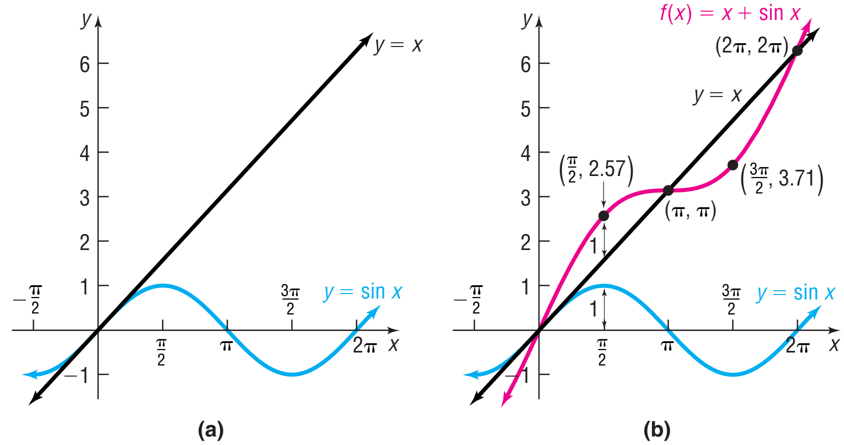
$$y = f_1(x) = x \qquad y = f_2(x) = \sin x$$

in the same coordinate system. See Figure 51(a). Now, select several values of x, say, $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$, and $x = 2\pi$, at which we compute $f(x) = f_1(x) + f_2(x)$. Table 1 shows the computation. We plot these points and connect them to get the graph, as shown in Figure 51(b).

Table 1

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = f_1(x) = x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = f_2(x) = \sin x$	0	1	0	-1	0
$f(x) = x + \sin x$	0	$\frac{\pi}{2} + 1 \approx 2.57$	π	$\frac{3\pi}{2} - 1 \approx 3.71$	2π
Point on graph of f	$(0, 0)$	$\left(\frac{\pi}{2}, 2.57\right)$	(π, π)	$\left(\frac{3\pi}{2}, 3.71\right)$	$(2\pi, 2\pi)$

Figure 51



In Figure 51(b), notice that the graph of $f(x) = x + \sin x$ intersects the line $y = x$ whenever $\sin x = 0$. Also, notice that the graph of f is not periodic.

✔ **CHECK:** Graph $Y_1 = x + \sin x$ and compare the result with Figure 51(b). Use INTERSECT to verify that the graphs intersect when $\sin x = 0$. ◀

The next example shows a periodic graph of the sum of two functions.

EXAMPLE 5 Graphing the Sum of Two Sinusoidal Functions

Use the method of adding y-coordinates to graph

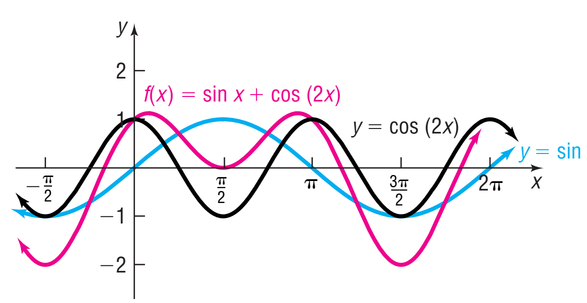
$$f(x) = \sin x + \cos(2x)$$

Solution Table 2 shows the steps for computing several points on the graph of f. Figure 52 illustrates the graphs of the component functions, $y = f_1(x) = \sin x$ and $y = f_2(x) = \cos(2x)$, and the graph of $f(x) = \sin x + \cos(2x)$, which is shown in red.

Table 2

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = f_1(x) = \sin x$	-1	0	1	0	-1	0
$y = f_2(x) = \cos(2x)$	-1	1	-1	1	-1	1
$f(x) = \sin x + \cos(2x)$	-2	1	0	1	-2	1
Point on graph of f	$\left(-\frac{\pi}{2}, -2\right)$	$(0, 1)$	$\left(\frac{\pi}{2}, 0\right)$	$(\pi, 1)$	$\left(\frac{3\pi}{2}, -2\right)$	$(2\pi, 1)$

Figure 52



✔ **CHECK:** Graph $Y_1 = \sin x + \cos(2x)$ and compare the result with Figure 52. ◀

NOW WORK PROBLEM 33.

7.5 Assess Your Understanding

'Are You Prepared?'

Answer given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The amplitude A and period T of $f(x) = 5\sin(4x)$ are _____ and _____. (pp. 408–410)

Concepts and Vocabulary

2. The motion of an object obeys the equation $d = 4\cos(6t)$. Such motion is described as _____ _____. The number 4 is called the _____.
3. When a mass hanging from a spring is pulled down and then released, the motion is called _____ _____ _____ if there is no frictional force to retard the motion, and the motion is called _____ _____ if there is friction.
4. *True or False:* If the distance d of an object from its rest position at time t is given by a sinusoidal graph, the motion of the object is simple harmonic motion.

Skill Building

In Problems 5–8, an object attached to a coiled spring is pulled down a distance a from its rest position and then released. Assuming that the motion is simple harmonic with period T, write an equation that relates the displacement d of the object from its rest position after t seconds. Also assume that the positive direction of the motion is up.

5. $a = 5$; $T = 2$ seconds
6. $a = 10$; $T = 3$ seconds
7. $a = 6$; $T = \pi$ seconds
8. $a = 4$; $T = \frac{\pi}{2}$ seconds
9. Rework Problem 5 under the same conditions except that, at time $t = 0$, the object is at its resting position and moving down.
10. Rework Problem 6 under the same conditions except that, at time $t = 0$, the object is at its resting position and moving down.
11. Rework Problem 7 under the same conditions except that, at time $t = 0$, the object is at its resting position and moving down.
12. Rework Problem 8 under the same conditions except that, at time $t = 0$, the object is at its resting position and moving down.

In Problems 13–20, the displacement d (in meters) of an object at time t (in seconds) is given.

(a) *Describe the motion of the object.*
(b) *What is the maximum displacement from its resting position?*
(c) *What is the time required for one oscillation?*
(d) *What is the frequency?*

13. $d = 5\sin(3t)$
14. $d = 4\sin(2t)$
15. $d = 6\cos(\pi t)$
16. $d = 5\cos\left(\frac{\pi}{2}t\right)$
17. $d = -3\sin\left(\frac{1}{2}t\right)$
18. $d = -2\cos(2t)$
19. $d = 6 + 2\cos(2\pi t)$
20. $d = 4 + 3\sin(\pi t)$

In Problems 21–26, an object of mass m attached to a coiled spring with damping factor b is pulled down a distance a from its rest position and then released. Assume that the positive direction of the motion is up and the period is T under simple harmonic motion.

(a) *Write an equation that relates the distance d of the object from its rest position after t seconds.*
(b) *Graph the equation found in part (a) for 5 oscillations using a graphing utility.*

21. $m = 25$ grams; $a = 10$ centimeters; $b = 0.7$ gram/second; $T = 5$ seconds
22. $m = 20$ grams; $a = 15$ centimeters; $b = 0.75$ gram/second; $T = 6$ seconds
23. $m = 30$ grams; $a = 18$ centimeters; $b = 0.6$ gram/second; $T = 4$ seconds
24. $m = 15$ grams; $a = 16$ centimeters; $b = 0.65$ gram/second; $T = 5$ seconds
25. $m = 10$ grams; $a = 5$ centimeters; $b = 0.8$ gram/second; $T = 3$ seconds
26. $m = 10$ grams; $a = 5$ centimeters; $b = 0.7$ gram/second; $T = 3$ seconds

In Problems 27–32, the distance d (in meters) of the bob of a pendulum of mass m (in kilograms) from its rest position at time t (in seconds) is given. The bob is released from the left of its rest position and represents a negative direction.
(a) *Describe the motion of the object. Be sure to give the mass and damping factor.*
(b) *What is the initial displacement of the bob? That is, what is the displacement at $t = 0$?*
(c) *Graph the motion using a graphing utility.*
(d) *What is the displacement of the bob at the start of the second oscillation?*
(e) *What happens to the displacement of the bob as time increases without bound?*

27. $d = -20e^{-0.7t/40}\cos\left(\sqrt{\left(\frac{2\pi}{5}\right)^2 - \frac{0.49}{1600}}\,t\right)$

28. $d = -20e^{-0.8t/40}\cos\left(\sqrt{\left(\frac{2\pi}{5}\right)^2 - \frac{0.64}{1600}}\,t\right)$

29. $d = -30e^{-0.6t/80}\cos\left(\sqrt{\left(\frac{2\pi}{7}\right)^2 - \frac{0.36}{6400}}\,t\right)$

30. $d = -30e^{-0.5t/70}\cos\left(\sqrt{\left(\frac{\pi}{2}\right)^2 - \frac{0.25}{4900}}\,t\right)$

31. $d = -15e^{-0.9t/30}\cos\left(\sqrt{\left(\frac{\pi}{3}\right)^2 - \frac{0.81}{900}}\,t\right)$

32. $d = -10e^{-0.8t/50}\cos\left(\sqrt{\left(\frac{2\pi}{3}\right)^2 - \frac{0.64}{2500}}\,t\right)$

In Problem 33–40, use the method of adding y-coordinates to graph each function. Verify your result using a graphing utility.

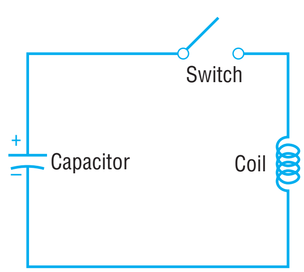
33. $f(x) = x + \cos x$
34. $f(x) = x + \cos(2x)$
35. $f(x) = x - \sin x$
36. $f(x) = x - \cos x$
37. $f(x) = \sin x + \cos x$
38. $f(x) = \sin(2x) + \cos x$
39. $g(x) = \sin x + \sin(2x)$
40. $g(x) = \cos(2x) + \cos x$

Applications and Extensions

41. Charging a Capacitor If a charged capacitor is connected to a coil by closing a switch (see the figure), energy is transferred to the coil and then back to the capacitor in an oscillatory motion. The voltage V (in volts) across the capacitor will gradually diminish to 0 with time t (in seconds).
(a) By hand, graph the equation relating V and t:

$$V(t) = e^{-t/3}\cos(\pi t), \qquad 0 \le t \le 3$$

(b) At what times t will the graph of V touch the graph of $y = e^{-t/3}$? When does V touch the graph of $y = -e^{-t/3}$?
(c) When will the voltage V be between -0.4 and 0.4 volt?



42. The Sawtooth Curve An oscilloscope often displays a *sawtooth curve*. This curve can be approximated by sinusoidal curves of varying periods and amplitudes.
(a) Use a graphing utility to graph the following function, which can be used to approximate the sawtooth curve.

$$f(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x), \qquad 0 \le x \le 2$$

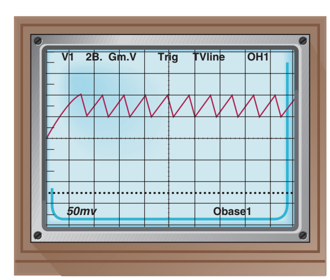
(b) A better approximation to the sawtooth curve is given by

$$f(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x) + \frac{1}{8}\sin(8\pi x)$$

Use a graphing utility to graph this function for $0 \le x \le 4$ and compare the result to the graph obtained in part (a).
(c) A third and even better approximation to the sawtooth curve is given by

$$f(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x) + \frac{1}{8}\sin(8\pi x) + \frac{1}{16}\sin(16\pi x)$$

Use a graphing utility to graph this function for $0 \le x \le 4$ and compare the result to the graphs obtained in parts (a) and (b).
(d) What do you think the next approximation to the sawtooth curve is?



43. Touch-Tone Phones On a Touch-Tone phone, each button produces a unique sound. The sound produced is the sum of two tones, given by

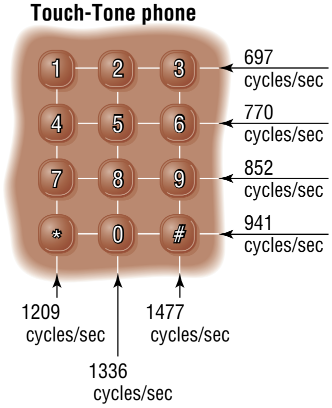
$$y = \sin(2\pi lt) \quad \text{and} \quad y = \sin(2\pi ht)$$

where l and h are the low and high frequencies (cycles per second) shown on the illustration on the next page. For example, if you touch 7, the low frequency is $l = 852$ cycles per second and the high frequency is $h = 1209$ cycles per second. The sound emitted by touching 7 is

$$y = \sin[2\pi(852)t] + \sin[2\pi(1209)t]$$

Use a graphing utility to graph the sound emitted by touching 7.

Touch-Tone phone



44. Use a graphing utility to graph the sound emitted by the * key on a Touch-Tone phone. See Problem 43.

45. CBL Experiment Pendulum motion is analyzed to estimate simple harmonic motion. A plot is generated with the position of the pendulum over time. The graph is used to find a sinusoidal curve of the form $y = A\cos[B(x - C)] + D$. Determine the amplitude, period, and frequency. (Activity 16, Real-World Math with the CBL System.)

46. CBL Experiment The sound from a tuning fork is collected over time. Determine the amplitude, frequency, and period of the graph. A model of the form $y = A\cos[B(x - C)]$ is fitted to the data. (Activity 23, Real-World Math with the CBL System.)

Discussion and Writing

47. Use a graphing utility to graph the function $f(x) = \dfrac{\sin x}{x}$, $x > 0$. Based on the graph, what do you conjecture about the value of $\dfrac{\sin x}{x}$ for x close to 0?

48. Use a graphing utility to graph $y = x\sin x$, $y = x^2\sin x$, and $y = x^3\sin x$ for $x > 0$. What patterns do you observe?

49. Use a graphing utility to graph $y = \dfrac{1}{x}\sin x$, $y = \dfrac{1}{x^2}\sin x$, and $y = \dfrac{1}{x^3}\sin x$ for $x > 0$. What patterns do you observe?

50. How would you explain to a friend what simple harmonic motion is? How would you explain damped motion?

'Are You Prepared?' Answer

1. $5; \dfrac{\pi}{2}$

Chapter Review

Things to Know

Formulas

Law of Sines (p. 532) $\qquad \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$

Law of Cosines (p. 543) $\qquad c^2 = a^2 + b^2 - 2ab\cos\gamma$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

Area of a triangle (pp. 549–551) $\qquad A = \dfrac{1}{2}bh \quad A = \dfrac{1}{2}ab\sin\gamma \quad A = \dfrac{1}{2}bc\sin\alpha \quad A = \dfrac{1}{2}ac\sin\beta$

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where} \quad s = \frac{1}{2}(a + b + c)$$

Objectives

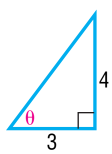
Section		You should be able to . . .	Review Exercises
7.1	1	Find the value of trigonometric functions of acute angles (p. 518)	1–4
	2	Use the Complementary Angle Theorem (p. 520)	5–10
	3	Solve right triangles (p. 521)	11–14
	4	Solve applied problems using right triangle trigonometry (p. 522)	45, 46, 56–58
7.2	1	Solve SAA or ASA triangles (p. 532)	15–16, 32
	2	Solve SSA triangles (p. 533)	17–20, 22, 27–28, 31

	3	Solve applied problems using the Law of Sines (p. 536)	47, 49, 50
7.3	1	Solve SAS triangles (p. 544)	21, 25–26, 33–34
	2	Solve SSS triangles (p. 544)	23–24, 29–30
	3	Solve applied problems using the Law of Cosines (p. 545)	48, 51–53
7.4	1	Find the area of SAS triangles (p. 550)	35–38, 53–55
	2	Find the area of SSS triangles (p. 550)	39–42
7.5	1	Find an equation for an object in simple harmonic motion (p. 555)	59–60
	2	Analyze simple harmonic motion (p. 558)	61–64
	3	Analyze an object in damped motion (p. 558)	65–68
	4	Graph the sum of two functions (p. 560)	69, 70

Review Exercises

In Problems 1–4, find the exact value of the six trigonometric functions of the angle θ in each figure.

1.



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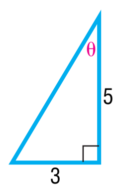

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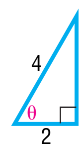

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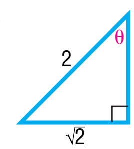


In Problems 5–10, find the exact value of each expression. Do not use a calculator.

5. $\cos 62° - \sin 28°$

6. $\tan 15° - \cot 75°$

7. $\dfrac{\sec 55°}{\csc 35°}$

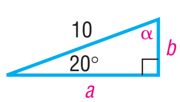
8. $\dfrac{\tan 40°}{\cot 50°}$

9. $\cos^2 40° + \cos^2 50°$

10. $\tan^2 40° - \csc^2 50°$

In Problems 11–14, solve each triangle.

11.



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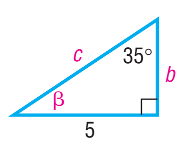

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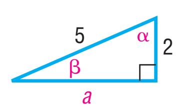

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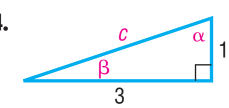


In Problems 15–34, find the remaining angle(s) and side(s) of each triangle, if it (they) exists. If no triangle exists, say "No triangle."

15. $\alpha = 50°, \quad \beta = 30°, \quad a = 1$

16. $\alpha = 10°, \quad \gamma = 40°, \quad c = 2$

17. $\alpha = 100°, \quad a = 5, \quad c = 2$

18. $a = 2, \quad c = 5, \quad \alpha = 60°$

19. $a = 3, \quad c = 1, \quad \gamma = 110°$

20. $a = 3, \quad c = 1, \quad \gamma = 20°$

21. $a = 3, \quad c = 1, \quad \beta = 100°$

22. $a = 3, \quad b = 5, \quad \beta = 80°$

23. $a = 2, \quad b = 3, \quad c = 1$

24. $a = 10, \quad b = 7, \quad c = 8$

25. $a = 1, \quad b = 3, \quad \gamma = 40°$

26. $a = 4, \quad b = 1, \quad \gamma = 100°$

27. $a = 5, \quad b = 3, \quad \alpha = 80°$

28. $a = 2, \quad b = 3, \quad \alpha = 20°$

29. $a = 1, \quad b = \dfrac{1}{2}, \quad c = \dfrac{4}{3}$

30. $a = 3, \quad b = 2, \quad c = 2$

31. $a = 3, \quad \alpha = 10°, \quad b = 4$

32. $a = 4, \quad \alpha = 20°, \quad \beta = 100°$

33. $c = 5, \quad b = 4, \quad \alpha = 70°$

34. $a = 1, \quad b = 2, \quad \gamma = 60°$

In Problems 35–44, find the area of each triangle.

35. $a = 2, \quad b = 3, \quad \gamma = 40°$

36. $b = 5, \quad c = 5, \quad \alpha = 20°$

37. $b = 4, \quad c = 10, \quad \alpha = 70°$

38. $a = 2, \quad b = 1, \quad \gamma = 100°$

39. $a = 4, \quad b = 3, \quad c = 5$

40. $a = 10, \quad b = 7, \quad c = 8$

41. $a = 4, \quad b = 2, \quad c = 5$

42. $a = 3, \quad b = 2, \quad c = 2$

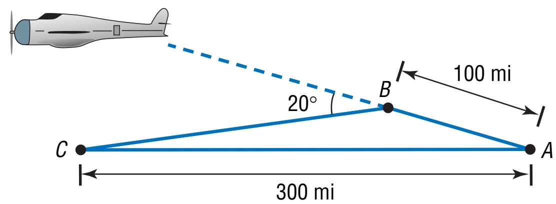
43. $\alpha = 50°, \quad \beta = 30°, \quad a = 1$

44. $\alpha = 10°, \quad \gamma = 40°, \quad c = 3$

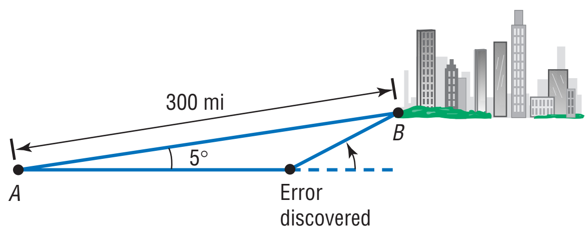
45. Finding the Grade of a Mountain Trail A straight trail with a uniform inclination leads from a hotel, elevation 5000 feet, to a lake in a valley, elevation 4100 feet. The length of the trail is 4100 feet. What is the inclination (grade) of the trail?

46. Geometry The hypotenuse of a right triangle is 12 feet. If one leg is 8 feet, find the degree measure of each angle.

47. Navigation An airplane flies from city A to city B, a distance of 100 miles, and then turns through an angle of 20° and heads toward city C, as indicated in the figure. If the distance from A to C is 300 miles, how far is it from city B to city C?

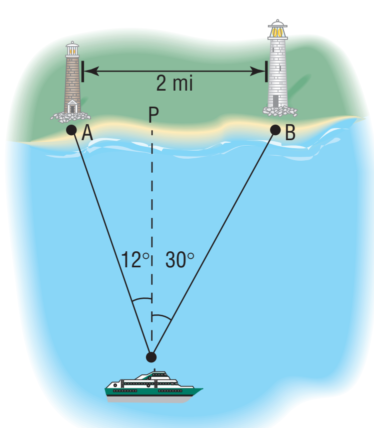


48. Correcting a Navigation Error Two cities A and B are 300 miles apart. In flying from city A to city B, a pilot inadvertently took a course that was 5° in error.

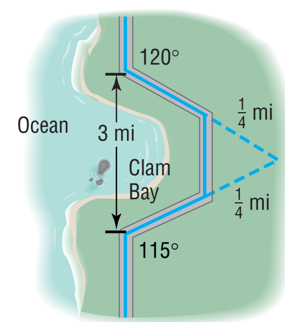


(a) If the error was discovered after flying 10 minutes at a constant speed of 420 miles per hour, through what angle should the pilot turn to correct the course? (Consult the figure.)
(b) What new constant speed should be maintained so that no time is lost due to the error? (Assume that the speed would have been a constant 420 miles per hour if no error had occurred.)

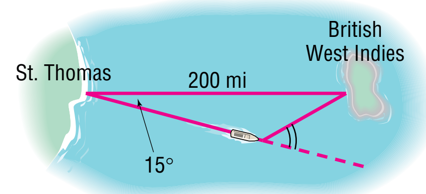
49. Determining Distances at Sea Rebecca, the navigator of a ship at sea, spots two lighthouses that she knows to be 2 miles apart along a straight shoreline. She determines that the angles formed between two line-of-sight observations of the lighthouses and the line from the ship directly to shore are 12° and 30°. See the illustration.
(a) How far is the ship from lighthouse A?
(b) How far is the ship from lighthouse B?
(c) How far is the ship from shore?



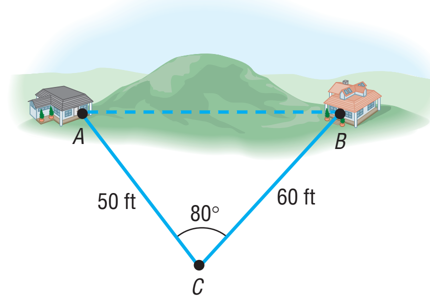
50. Constructing a Highway A highway whose primary directions are north–south is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The illustration shows the path that they decide on and the measurements taken. What is the length of highway needed to go around the bay?



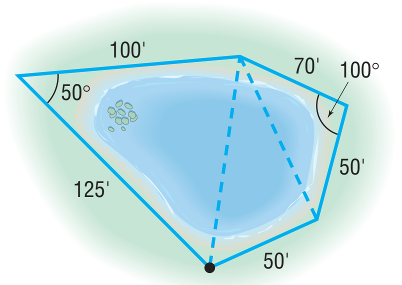
51. Correcting a Navigational Error A sailboat leaves St. Thomas bound for an island in the British West Indies, 200 miles away. Maintaining a constant speed of 18 miles per hour, but encountering heavy crosswinds and strong currents, the crew finds after 4 hours that the sailboat is off course by 15°.
(a) How far is the sailboat from the island at this time?
(b) Through what angle should the sailboat turn to correct its course?
(c) How much time has been added to the trip because of this? (Assume that the speed remains at 18 miles per hour.)



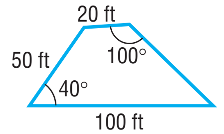
52. Surveying Two homes are located on opposite sides of a small hill. See the illustration. To measure the distance between them, a surveyor walks a distance of 50 feet from house A to point C, uses a transit to measure the angle ACB, which is found to be 80°, and then walks to house B, a distance of 60 feet. How far apart are the houses?



53. Approximating the Area of a Lake To approximate the area of a lake, Cindy walks around the perimeter of the lake, taking the measurements shown in the illustration. Using this technique, what is the approximate area of the lake? [**Hint:** Use the Law of Cosines on the three triangles shown and then find the sum of their areas.]



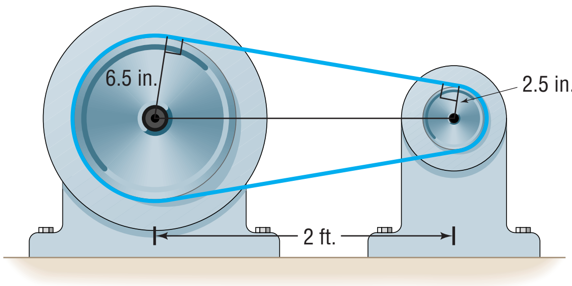
54. Calculating the Cost of Land The irregular parcel of land shown in the figure is being sold for \$100 per square foot. What is the cost of this parcel?



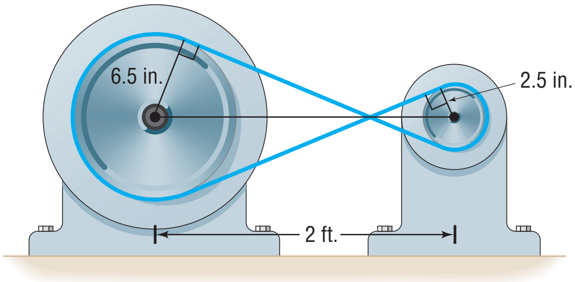
55. Area of a Segment Find the area of the segment of a circle whose radius is 6 inches formed by a central angle of 50°.

56. Finding the Bearing of a Ship The *Majesty* leaves the Port at Boston for Bermuda with a bearing of S80°E at an average speed of 10 knots. After 1 hour, the ship turns 90° toward the southwest. After 2 hours at an average speed of 20 knots, what is the bearing of the ship from Boston?

57. Pulleys The drive wheel of an engine is 13 inches in diameter, and the pulley on the rotary pump is 5 inches in diameter. If the shafts of the drive wheel and the pulley are 2 feet apart, what length of belt is required to join them as shown in the figure?



58. Rework Problem 57 if the belt is crossed, as shown in the figure.



In Problems 59 and 60, an object attached to a coiled spring is pulled down a distance a from its rest position and then released. Assuming that the motion is simple harmonic with period T, write an equation that relates the displacement d of the object from its rest position after t seconds. Also assume that the positive direction of the motion is up.

59. $a = 3$; $T = 4$ seconds

60. $a = 5$; $T = 6$ seconds

In Problems 61–64, the distance d (in feet) that an object travels in time t (in seconds) is given.
(a) *Describe the motion of the object.*
(b) *What is the maximum displacement from its rest position?*
(c) *What is the time required for one oscillation?*
(d) *What is the frequency?*

61. $d = 6\sin(2t)$

62. $d = 2\cos(4t)$

63. $d = -2\cos(\pi t)$

64. $d = -3\sin\left[\frac{\pi}{2}t\right]$.

In Problems 65 and 66, an object of mass m attached to a coiled spring with damping factor b is pulled down a distance a from its rest position and then released. Assume that the positive direction of the motion is up and the period is T under simple harmonic motion.
(a) *Write an equation that relates the distance d of the object from its rest position after t seconds.*
(b) *Graph the equation found in part (a) for 5 oscillations.*

65. $m = 40$ grams; $a = 15$ centimeters; $b = 0.75$ gram/second; $T = 5$ seconds

66. $m = 25$ grams; $a = 13$ centimeters; $b = 0.65$ gram/second; $T = 4$ seconds

In Problems 67 and 68, the distance d (in meters) of the bob of a pendulum of mass m (in kilograms) from its rest position at time t (in seconds) is given.
(a) *Describe the motion of the object.*
(b) *What is the initial displacement of the bob? That is, what is the displacement at $t = 0$?*
(c) *Graph the motion using a graphing utility.*
(d) *What is the displacement of the bob at the start of the second oscillation?*
(e) *What happens to the displacement of the bob as time increases without bound?*

67. $d = -15e^{-0.6t/40}\cos\left(\sqrt{\left(\frac{2\pi}{5}\right)^2 - \frac{0.36}{1600}}\,t\right)$

68. $d = -20e^{-0.5t/60}\cos\left(\sqrt{\left(\frac{2\pi}{3}\right)^2 - \frac{0.25}{3600}}\,t\right)$

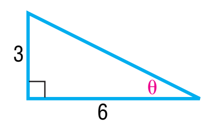
In Problems 69 and 70, use the method of adding y-coordinates to graph each function. Verify your result using a graphing utility.

69. $y = 2\sin x + \cos(2x)$

70. $y = 2\cos(2x) + \sin\dfrac{x}{2}$

Chapter Test

1. Find the exact value of the six trigonometric functions of the angle θ in the figure.



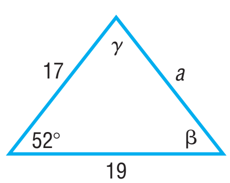
2. Find the exact value of $\sin 40° - \cos 50°$.

3. A 12 foot ladder leans against a building. The top of the ladder leans against the wall 10.5 feet from the ground. What is the angle formed by the ground and the ladder?

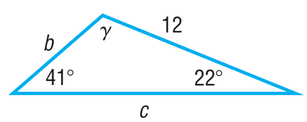
4. A hot air balloon is flying at a height of 600 feet and is directly above the Marshall Space Flight Center in Huntsville, AL. The pilot of the balloon looks down at the airport that is known to be 5 miles from the Marshall Space Flight Center. What is the angle of depression from the balloon to the airport?

In Problems 5–7, use the given information to determine the three remaining parts of each triangle.

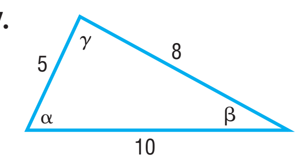
5.



6.



7.



In Problems 8–10, solve each triangle.

8. $\alpha = 55°, \quad \gamma = 20°, \quad a = 4$

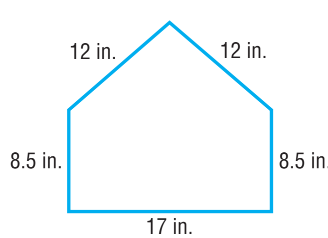
9. $a = 3, \quad b = 7, \quad \alpha = 40°$

10. $a = 8, \quad b = 4, \quad \gamma = 70°$

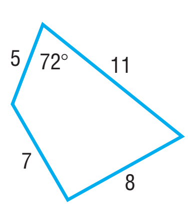
11. Find the area of the triangle described in Problem 10.

12. Find the area of the triangle described in Problem 7.

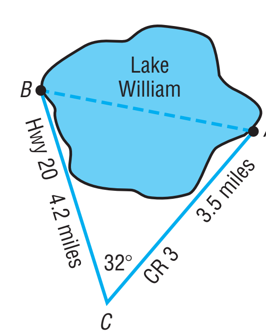
13. The dimensions of home plate at any major league baseball stadium are shown. Find the area of home plate.



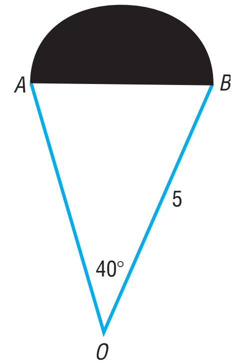
14. Find the area of the quadrilateral shown below.



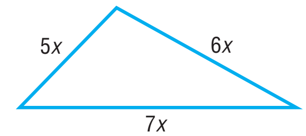
15. Madison wants to swim across Lake William from the fishing lodge (A) to the boat ramp (B) but she wants to know the distance first. Highway 20 goes right past the boat ramp and County Road 3 goes to the lodge. The two roads intersect at point (C), 4.2 miles from the ramp and 3.5 miles from the lodge. Madison uses a transit to measure the angle of intersection of the two roads to be 32°. How far will she need to swim?



16. Given that $\triangle OAB$ is an isosceles triangle and the shaded sector is a semicircle, find the area of the entire region. Express your answer as a decimal rounded to two places.



17. The area of the triangle shown below is $54\sqrt{6}$ square units; find the lengths of the sides.

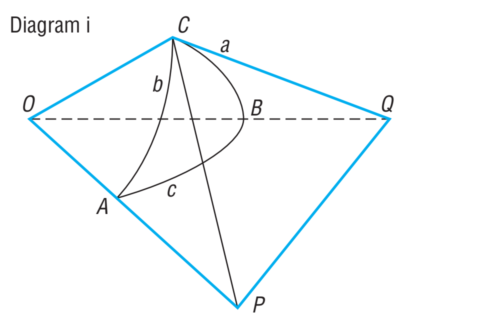


18. Logan is playing on her swing. One full swing (front to back to front) takes 6 seconds and at the peak of her swing she is at an angle of 42° with the vertical. If her swing is 5 feet long, and we ignore all resistive forces, write an equation that relates her horizontal displacement (from the rest position) after time t.

Chapter Projects

1. A. Spherical Trigonometry When the distance between two locations on the surface of Earth is small, we can compute the distance in statutory miles. Using this assumption, we can use the Law of Sines and the Law of Cosines to approximate distances and angles. However, if you look at a globe, you notice that Earth is a sphere, so, as the distance between two points on its surface increases, the linear distance is less accurate because of curvature. Under this circumstance, we need to take into account the curvature of Earth when using the Law of Sines and the Law of Cosines.

(a) Draw a spherical triangle and label each vertex by A, B, and C. Then connect each vertex by a radius to the center O of the sphere. Now, draw tangent lines to the sides a and b of the triangle that go through C. Extend the lines OA and OB to intersect the tangent lines at P and Q, respectively. See the diagram. List the plane right triangles. Determine the measures of the central angles.



(b) Apply the Law of Cosines to triangles OPQ and CPQ to find two expressions for the length of PQ.

(c) Subtract the expressions in part (b) from each other. Solve for the term containing $\cos c$.

(d) Use the Pythagorean Theorem to find another value for $OQ^2 - CQ^2$ and $OP^2 - CP^2$. Now solve for $\cos c$.

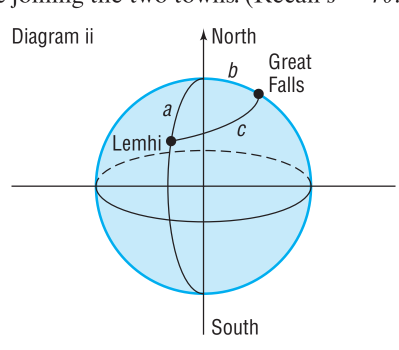
(e) Replacing the ratios in part (d) by the cosines of the sides of the spherical triangle, you should now have the Law of Cosines for spherical triangles:

$$\cos c = \cos a \cos b + \sin a \sin b \cos c$$

SOURCE: For spherical Law of Cosines: *Mathematics from the Birth of Numbers* by Jan Gullberg. W. W. Norton & Co., Publishers. 1996, pp. 491–494.

B. The Lewis and Clark Expedition Lewis and Clark followed several rivers in their trek from what is now Great Falls, Montana, to the Pacific coast. First, they went down the Missouri and Jefferson rivers from Great Falls to Lemhi, Idaho. Because the two cities are on different longitudes and different latitudes, we must account for the curvature of Earth when computing the distance that they traveled. Assume that the radius of Earth is 3960 miles.

(a) Great Falls is at approximately 47.5°N and 111.3°W. Lemhi is at approximately 45.0°N and 113.5°W. (We will assume that the rivers flow straight from Great Falls to Lemhi on the surface of Earth.) This line is called a geodesic line. Apply the Law of Cosines for a spherical triangle to find the angle between Great Falls and Lemhi. (The central angles are found by using the differences in the latitudes and longitudes of the towns. See the diagram.) Then find the length of the arc joining the two towns. (Recall $s = r\theta$.)



(b) From Lemhi, they went up the Bitteroot River and the Snake River to what is now Lewiston and Clarkston on the border of Idaho and Washington. Although this is not really a side to a triangle, we will make a side that goes from Lemhi to Lewiston and Clarkston. If Lewiston and Clarkston are at about 46.5°N 117.0°W, find the distance from Lemhi using the Law of Cosines for a spherical triangle and the arc length.

(c) How far did the explorers travel just to get that far?

(d) Draw a plane triangle connecting the three towns. If the distance from Lewiston to Great Falls is 282 miles and the angle at Great Falls is 42° and the angle at Lewiston is 48.5°, find the distance from Great Falls to Lemhi and from Lemhi to Lewiston. How do these distances compare with the ones computed in parts (a) and (b)?

SOURCE: For Lewis and Clark Expedition: *American Journey: The Quest for Liberty to 1877, Texas Edition*. Prentice Hall, 1992, p. 345.

SOURCE: For map coordinates: *National Geographic Atlas of the World*, published by National Geographic Society, 1981, pp. 74–75.

The following projects are available at the Instructor's Resource Center (IRC):

2. **Project at Motorola** ***How Can You Build or Analyze a Vibration Profile?***
3. **Leaning Tower of Pisa**
4. **Locating Lost Treasure**

Cumulative Review

1. Find the real solutions, if any, of the equation $3x^2 + 1 = 4x$.

2. Find an equation for the circle with center at the point $(-5, 1)$ and radius 3. Graph this circle.

3. What is the domain of the function

$$f(x) = \sqrt{x^2 - 3x - 4}?$$

4. Graph the function $y = 3\sin(\pi x)$.

5. Graph the function $y = -2\cos(2x - \pi)$.

6. If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find the exact value of:

(a) $\sin\theta$ (b) $\cos\theta$ (c) $\sin(2\theta)$
(d) $\cos(2\theta)$ (e) $\sin\left(\frac{1}{2}\theta\right)$ (f) $\cos\left(\frac{1}{2}\theta\right)$

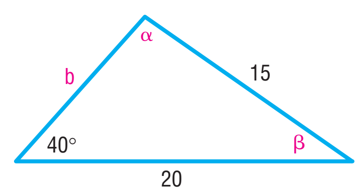
7. Graph each of the following functions on the interval $[0, 4]$:

(a) $y = e^x$ (b) $y = \sin x$
(c) $y = e^x \sin x$ (d) $y = 2x + \sin x$

8. Sketch the graph of each of the following functions:

(a) $y = x$ (b) $y = x^2$ (c) $y = \sqrt{x}$
(d) $y = x^3$ (e) $y = e^x$ (f) $y = \ln x$
(g) $y = \sin x$ (h) $y = \cos x$ (i) $y = \tan x$

9. Solve the triangle:



10. In the complex number system, solve the equation

$$3x^5 - 10x^4 + 21x^3 - 42x^2 + 36x - 8 = 0$$

11. Analyze the graph of the rational function

$$R(x) = \frac{2x^2 - 7x - 4}{x^2 + 2x - 15}$$

12. Solve $3^x = 12$. Round your answer to two decimal places.

13. Solve $\log_3(x + 8) + \log_3 x = 2$.

14. Suppose that $f(x) = 4x + 5$ and $g(x) = x^2 + 5x - 24$.

(a) Solve $f(x) = 0$.
(b) Solve $f(x) = 13$.
(c) Solve $f(x) = g(x)$.
(d) Solve $f(x) > 0$.
(e) Solve $g(x) \le 0$.
(f) Graph $y = f(x)$.
(g) Graph $y = g(x)$.