Polar Coordinates; Vectors

8

Earth Scientists Use Fractals to Measure and Predict Natural Disasters

Predicting the size, location, and timing of natural hazards is virtually impossible, but now earth scientists are able to forecast hurricanes, floods, earthquakes, volcanic eruptions, wildfires, and landslides using fractals. A fractal is a mathematical formula of a pattern that repeats over a wide range of size and time scales. These patterns are hidden within more complex systems. A good example of a fractal is the branching system of a river. Small tributaries join to form larger and larger "branches" in the system, but each small piece of the system closely resembles the branching pattern as a whole.

At the American Geophysical Union meeting held last month, Benoit Mandelbrot, a professor of mathematical sciences at Yale University who is considered to be the father of fractals, described how he has been using fractals to find order within complex systems in nature, such as the natural shape of a coastline. As a result of his research, earth scientists are taking Mandelbrot's fractal approach one step further and are measuring past events and making probability forecasts about the size, location, and timing of future natural disasters.

"By understanding the fractal order and scale embedded in patterns of chaos, researchers found a deeper level of understanding that can be used to predict natural hazards," says Christopher Barton, a research geologist at the United States Geological Survey. "They can measure past events like a hurricane and then apply fractal mathematics to predict future hurricane events."

Thanks to Dr. Mandelbrot, earth scientists like Dr. Barton have a powerful, new tool to predict future chaotic events of nature.

SOURCE: American Institute of Physics, January 31, 2002.

—*See Chapter Project 1.*

A LOOK BACK, A LOOK AHEAD This chapter is in two parts: Polar Coordinates, Sections 8.1–8.3, and Vectors, Sections 8.4–8.7. They are independent of each other and may be covered in any order.

Sections 8.1–8.3: In Chapter 1 we introduced rectangular coordinates (x, y) and discussed the graph of an equation in two variables involving x and y. In Sections 8.1 and 8.2, we introduce an alternative to rectangular coordinates, polar coordinates, and discuss graphing equations that involve polar coordinates. In Section 4.3, we discussed raising a real number to a real power. In Section 8.3 we extend this idea by raising a complex number to a real power. As it turns out, polar coordinates are useful for the discussion.

Sections 8.4–8.7: We have seen in many chapters that often we are required to solve an equation to obtain a solution to applied problems. In the last four sections of this chapter, we develop the notion of a vector, and show how they can be used to solve certain types of applied problems, particularly in physics and engineering.

OUTLINE

8.1 Polar Coordinates
8.2 Polar Equations and Graphs
8.3 The Complex Plane; De Moivre's Theorem
8.4 Vectors
8.5 The Dot Product
8.6 Vectors in Space
8.7 The Cross Product
Chapter Review Chapter Test Chapter Projects
Cumulative Review

8.1 Polar Coordinates

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Rectangular Coordinates (Section 1.1, pp. 2–5)
- Definitions of the Sine and Cosine Functions (Section 5.2, pp. 371–372)
- Inverse Tangent Function (Section 6.1, pp. 455–457)
- Completing the Square (Appendix, Section A.5, pp. 991–992)

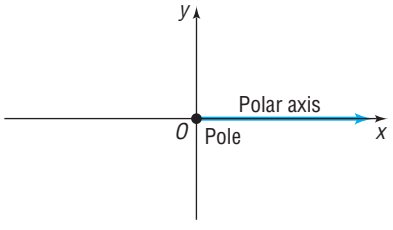
Now work the 'Are You Prepared?' problems on page 579.

OBJECTIVES
1. Plot Points Using Polar Coordinates
2. Convert from Polar Coordinates to Rectangular Coordinates
3. Convert from Rectangular Coordinates to Polar Coordinates

So far, we have always used a system of rectangular coordinates to plot points in the plane. Now we are ready to describe another system called *polar coordinates*. As we shall soon see, in many instances polar coordinates offer certain advantages over rectangular coordinates.

In a rectangular coordinate system, you will recall, a point in the plane is represented by an ordered pair of numbers (x, y), where x and y equal the signed distance of the point from the y-axis and x-axis, respectively. In a polar coordinate system, we select a point, called the **pole**, and then a ray with vertex at the pole, called the **polar axis**. See Figure 1. Comparing the rectangular and polar coordinate systems, we see that the origin in rectangular coordinates coincides with the pole in polar coordinates, and the positive x-axis in rectangular coordinates coincides with the polar axis in polar coordinates.

Figure 1



1 Plot Points Using Polar Coordinates

A point P in a polar coordinate system is represented by an ordered pair of numbers (r, θ). If $r > 0$, then r is the distance of the point from the pole; θ is an angle (in degrees or radians) formed by the polar axis and a ray from the pole through the point. We call the ordered pair (r, θ) the **polar coordinates** of the point. See Figure 2.

As an example, suppose that the polar coordinates of a point P are $\left(2, \frac{\pi}{4}\right)$. We locate P by first drawing an angle of $\frac{\pi}{4}$ radian, placing its vertex at the pole and its initial side along the polar axis. Then we go out a distance of 2 units along the terminal side of the angle to reach the point P. See Figure 3.

Figure 2

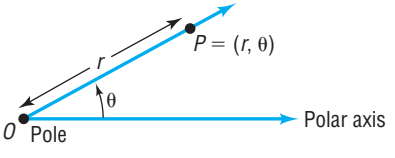
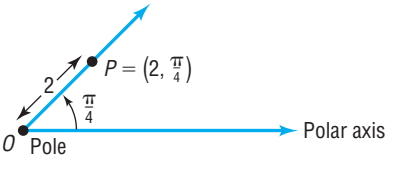


Figure 3



NOW WORK PROBLEM 19.

In using polar coordinates (r, θ), it is possible for the first entry r to be negative. When this happens, instead of the point being on the terminal side of θ, it is on the ray from the pole extending in the direction *opposite* the terminal side of θ at a distance $|r|$ units from the pole. See Figure 4 for an illustration.

For example, to plot the point $\left(-3, \frac{2\pi}{3}\right)$, we use the ray in the opposite direction of $\frac{2\pi}{3}$ and go out $|-3| = 3$ units along that ray. See Figure 5.

Figure 4

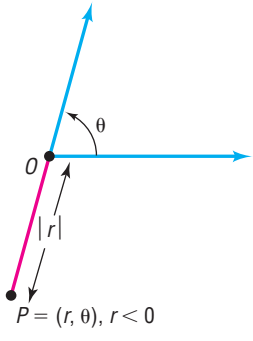
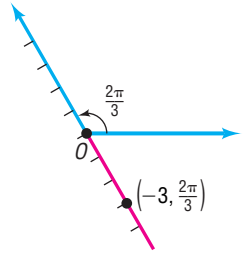


Figure 5



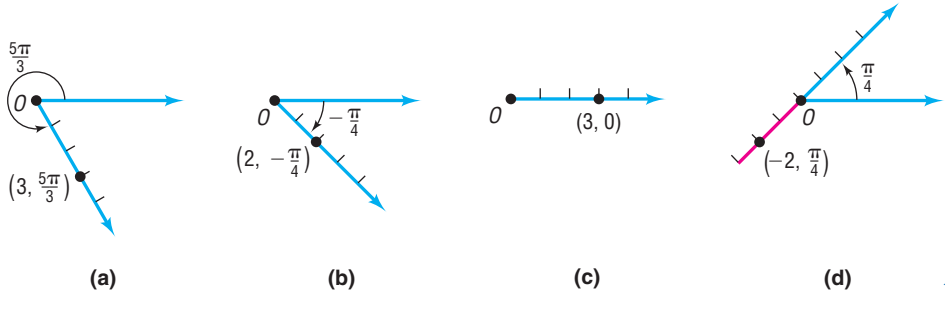
EXAMPLE 1 Plotting Points Using Polar Coordinates

Plot the points with the following polar coordinates:

(a) $\left(3, \frac{5\pi}{3}\right)$ (b) $\left(2, -\frac{\pi}{4}\right)$ (c) $(3, 0)$ (d) $\left(-2, \frac{\pi}{4}\right)$

Solution Figure 6 shows the points.

Figure 6



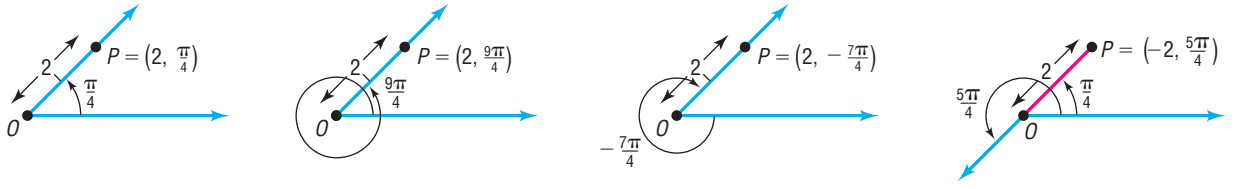
NOW WORK PROBLEMS 11 AND 27.

Recall that an angle measured counterclockwise is positive and an angle measured clockwise is negative. This convention has some interesting consequences relating to polar coordinates. Let's see what these consequences are.

EXAMPLE 2 Finding Several Polar Coordinates of a Single Point

Consider again the point P with polar coordinates $\left(2, \frac{\pi}{4}\right)$, as shown in Figure 7(a). Because $\frac{\pi}{4}$, $\frac{9\pi}{4}$, and $-\frac{7\pi}{4}$ all have the same terminal side, we also could have located this point P by using the polar coordinates $\left(2, \frac{9\pi}{4}\right)$ or $\left(2, -\frac{7\pi}{4}\right)$, as shown in Figures 7(b) and (c). The point $\left(2, \frac{\pi}{4}\right)$ can also be represented by the polar coordinates $\left(-2, \frac{5\pi}{4}\right)$. See Figure 7(d).

Figure 7



(a) (b) (c) (d)

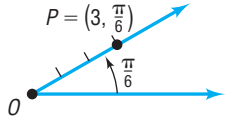
EXAMPLE 3 Finding Other Polar Coordinates of a Given Point

Plot the point P with polar coordinates $\left(3, \frac{\pi}{6}\right)$, and find other polar coordinates (r, θ) of this same point for which:

(a) $r > 0, \quad 2\pi \le \theta < 4\pi$ (b) $r < 0, \quad 0 \le \theta < 2\pi$

(c) $r > 0, \quad -2\pi \le \theta < 0$

Figure 8



Solution The point $\left(3, \frac{\pi}{6}\right)$ is plotted in Figure 8.

(a) We add 1 revolution (2π radians) to the angle $\frac{\pi}{6}$ to get $P = \left(3, \frac{\pi}{6} + 2\pi\right) = \left(3, \frac{13\pi}{6}\right)$. See Figure 9.

(b) We add $\frac{1}{2}$ revolution (π radians) to the angle $\frac{\pi}{6}$ and replace 3 by -3 to get $P = \left(-3, \frac{\pi}{6} + \pi\right) = \left(-3, \frac{7\pi}{6}\right)$. See Figure 10.

(c) We subtract 2π from the angle $\frac{\pi}{6}$ to get $P = \left(3, \frac{\pi}{6} - 2\pi\right) = \left(3, -\frac{11\pi}{6}\right)$. See Figure 11.

Figure 9

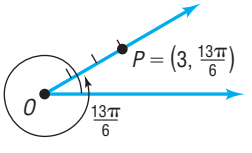


Figure 10

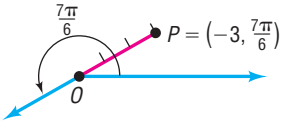
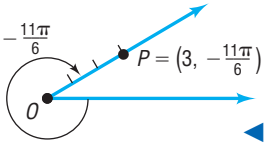


Figure 11



NOW WORK PROBLEM 31.

These examples show a major difference between rectangular coordinates and polar coordinates. In the former, each point has exactly one pair of rectangular coordinates; in the latter, a point can have infinitely many pairs of polar coordinates.

Summary

A point with polar coordinates (r, θ) also can be represented by either of the following:

$$(r, \theta + 2k\pi) \quad \text{or} \quad (-r, \theta + \pi + 2k\pi), \qquad k \text{ any integer}$$

The polar coordinates of the pole are $(0, \theta)$, where θ can be any angle.

2 Convert from Polar Coordinates to Rectangular Coordinates

It is sometimes convenient and, indeed, necessary to be able to convert coordinates or equations in rectangular form to polar form, and vice versa. To do this, we recall that the origin in rectangular coordinates is the pole in polar coordinates and that the positive x-axis in rectangular coordinates is the polar axis in polar coordinates.

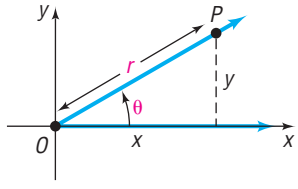
Theorem

Conversion from Polar Coordinates to Rectangular Coordinates

If P is a point with polar coordinates (r, θ), the rectangular coordinates (x, y) of P are given by

$$x = r\cos\theta \qquad y = r\sin\theta \tag{1}$$

Figure 12



Proof Suppose that P has the polar coordinates (r, θ). We seek the rectangular coordinates (x, y) of P. Refer to Figure 12.

If $r = 0$, then, regardless of θ, the point P is the pole, for which the rectangular coordinates are $(0, 0)$. Formula (1) is valid for $r = 0$.

If $r > 0$, the point P is on the terminal side of θ, and $r = d(O, P) = \sqrt{x^2 + y^2}$. Since

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

we have

$$x = r\cos\theta \qquad y = r\sin\theta$$

If $r < 0$, then the point $P = (r, \theta)$ can be represented as $(-r, \pi + \theta)$, where $-r > 0$. Since

$$\cos(\pi + \theta) = -\cos\theta = \frac{x}{-r} \qquad \sin(\pi + \theta) = -\sin\theta = \frac{y}{-r}$$

we have

$$x = r\cos\theta \qquad y = r\sin\theta$$

■

EXAMPLE 4 Converting from Polar Coordinates to Rectangular Coordinates

Find the rectangular coordinates of the points with the following polar coordinates:

(a) $\left(6, \frac{\pi}{6}\right)$ (b) $\left(-4, -\frac{\pi}{4}\right)$

Solution We use formula (1): $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Figure 13(a) shows $\left(6, \frac{\pi}{6}\right)$ plotted. Notice that $\left(6, \frac{\pi}{6}\right)$ lies in quadrant I of the rectangular coordinate system. So, we expect both the x-coordinate and the y-coordinate to be positive. With $r = 6$ and $\theta = \frac{\pi}{6}$, we have

$$x = r\cos\theta = 6\cos\frac{\pi}{6} = 6\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$$

$$y = r\sin\theta = 6\sin\frac{\pi}{6} = 6\cdot\frac{1}{2} = 3$$

The rectangular coordinates of the point $\left(6, \frac{\pi}{6}\right)$ are $(3\sqrt{3}, 3)$, which lies in quadrant I, as expected.

(b) Figure 13(b) shows $\left(-4, -\frac{\pi}{4}\right)$ plotted. Notice that $\left(-4, -\frac{\pi}{4}\right)$ lies in quadrant II of the rectangular coordinate system. With $r = -4$ and $\theta = -\frac{\pi}{4}$, we have

$$x = r\cos\theta = -4\cos\left(-\frac{\pi}{4}\right) = -4\cdot\frac{\sqrt{2}}{2} = -2\sqrt{2}$$

$$y = r\sin\theta = -4\sin\left(-\frac{\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

The rectangular coordinates of the point $\left(-4, -\frac{\pi}{4}\right)$ are $(-2\sqrt{2}, 2\sqrt{2})$, which lies in quadrant II, as expected. ◀

Figure 13

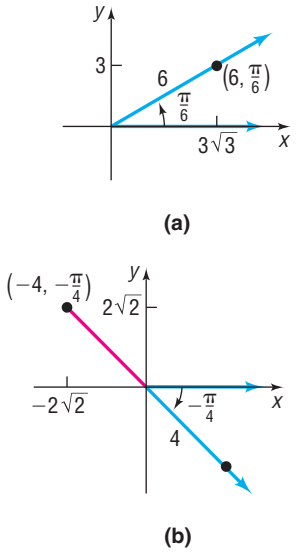
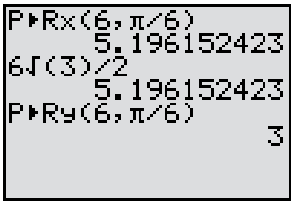


Figure 14



Most calculators have the capability of converting from polar coordinates to rectangular coordinates. Consult your owner's manual for the proper key strokes. Since in most cases this procedure is tedious, you will find that using formula (1) is faster.

Figure 14 verifies the result obtained in Example 4(a) using a TI-84 Plus. Note that the calculator is in radian mode.

NOW WORK PROBLEMS 39 AND 51.

3 Convert from Rectangular Coordinates to Polar Coordinates

Converting from rectangular coordinates (x, y) to polar coordinates (r, θ) is a little more complicated. Notice that we begin each example by plotting the given rectangular coordinates.

EXAMPLE 5 Converting from Rectangular Coordinates to Polar Coordinates

Find polar coordinates of a point whose rectangular coordinates are $(0, 3)$.

Solution See Figure 15. The point $(0, 3)$ lies on the y-axis a distance of 3 units from the origin (pole), so $r = 3$. A ray with vertex at the pole through $(0, 3)$ forms an angle $\theta = \frac{\pi}{2}$ with the polar axis. Polar coordinates for this point can be given by $\left(3, \frac{\pi}{2}\right)$. ◀

Figure 15

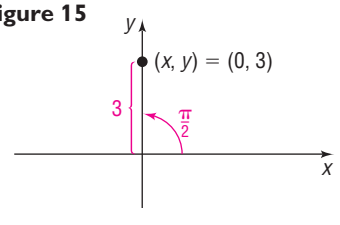
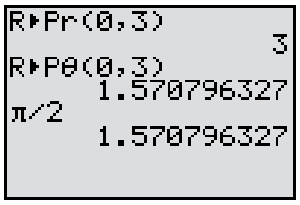


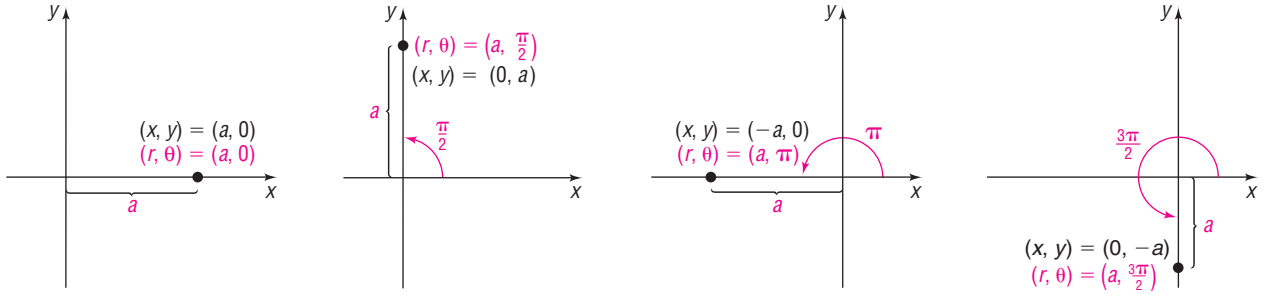
Figure 16



Most graphing calculators have the capability of converting from rectangular coordinates to polar coordinates. Consult you owner's manual for the proper keystrokes. Figure 16 verifies the results obtained in Example 5 using a TI-84 Plus. Note that the calculator is in radian mode.

Figure 17 shows polar coordinates of points that lie on either the x-axis or the y-axis. In each illustration, $a > 0$.

Figure 17



(a) $(x, y) = (a, 0)$, $a > 0$ **(b)** $(x, y) = (0, a)$, $a > 0$ **(c)** $(x, y) = (-a, 0)$, $a > 0$ **(d)** $(x, y) = (0, -a)$, $a > 0$

NOW WORK PROBLEM 55.

EXAMPLE 6 Converting from Rectangular Coordinates to Polar Coordinates

Find polar coordinates of a point whose rectangular coordinates are:

(a) $(2, -2)$ (b) $\left(-1, -\sqrt{3}\right)$

Solution (a) See Figure 18(a). The distance r from the origin to the point $(2, -2)$ is

$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

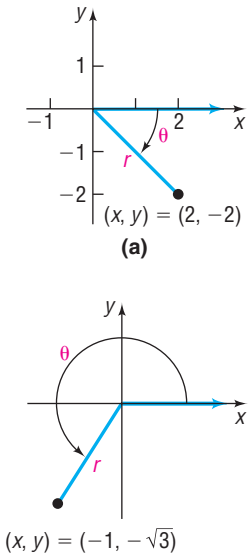
We find θ by recalling that $\tan\theta = \dfrac{y}{x}$, so $\theta = \tan^{-1}\dfrac{y}{x}, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

Since $(2, -2)$ lies in quadrant IV, we know that $-\dfrac{\pi}{2} < \theta < 0$. As a result,

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

A set of polar coordinates for this point is $\left(2\sqrt{2}, -\dfrac{\pi}{4}\right)$. Other possible representations include $\left(2\sqrt{2}, \dfrac{7\pi}{4}\right)$ and $\left(-2\sqrt{2}, \dfrac{3\pi}{4}\right)$.

Figure 18



(b) See Figure 18(b). The distance r from the origin to the point $\left(-1, -\sqrt{3}\right)$ is

$$r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

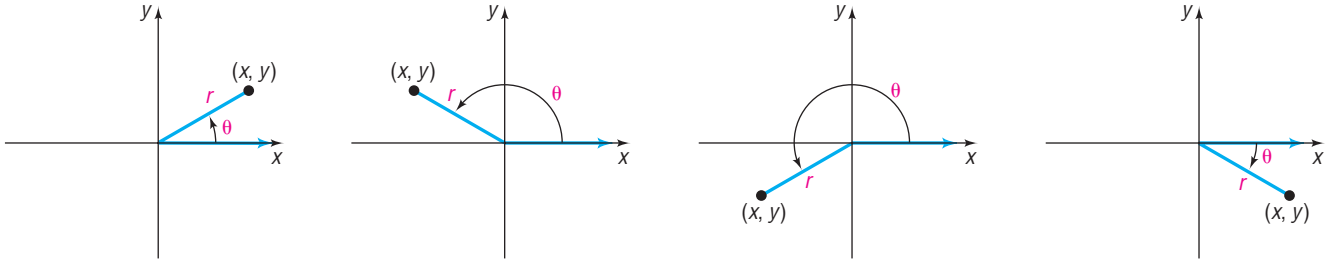
To find θ, we use $\theta = \tan^{-1}\dfrac{y}{x}, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Since the point $\left(-1, -\sqrt{3}\right)$ lies in quadrant III and the inverse tangent function gives an angle in quadrant I, we add π to the result to obtain an angle in quadrant III.

$$\theta = \pi + \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \pi + \tan^{-1}\sqrt{3} = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

A set of polar coordinates for this point is $\left(2, \dfrac{4\pi}{3}\right)$. Other possible representations include $\left(-2, \dfrac{\pi}{3}\right)$ and $\left(2, -\dfrac{2\pi}{3}\right)$. ◀

Figure 19 shows how to find polar coordinates of a point that lies in a quadrant when its rectangular coordinates (x, y) are given.

Figure 19



(a) $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$

(b) $r = \sqrt{x^2 + y^2}$, $\theta = \pi + \tan^{-1}\frac{y}{x}$

(c) $r = \sqrt{x^2 + y^2}$, $\theta = \pi + \tan^{-1}\frac{y}{x}$

(d) $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$

Based on the preceding discussion, we have the formulas

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad \text{if } x \neq 0 \tag{2}$$

To use formula (2) effectively, follow these steps:

Steps for Converting from Rectangular to Polar Coordinates

STEP 1: Always plot the point (x, y) first, as we did in Examples 5 and 6.

STEP 2: If $x = 0$ or $y = 0$, use your illustration to find (r, θ). See Figure 7.

STEP 3: If $x \neq 0$ and $y \neq 0$, then $r = \sqrt{x^2 + y^2}$.

STEP 4: To find θ, first determine the quadrant that the point lies in.

Quadrant I: $\theta = \tan^{-1}\frac{y}{x}$ Quadrant II: $\theta = \pi + \tan^{-1}\frac{y}{x}$

Quadrant III: $\theta = \pi + \tan^{-1}\frac{y}{x}$ Quadrant IV: $\theta = \tan^{-1}\frac{y}{x}$

See Figure 19.

NOW WORK PROBLEM 59.

Formulas (1) and (2) may also be used to transform equations from polar form to rectangular form and vice-versa. Two common techniques for transforming an equation from polar form to rectangular form are (1) multiplying both sides of the equation by r and (2) squaring both sides of the equation.

EXAMPLE 7 Transforming an Equation from Polar to Rectangular Form

Transform the equation $r = 4\sin\theta$ from polar coordinates to rectangular coordinates, and identify the graph.

Solution If we multiply each side by r, it will be easier to apply formulas (1) and (2).

$$r = 4\sin\theta$$

$$r^2 = 4r\sin\theta \qquad \text{Multiply each side by } r.$$

$$x^2 + y^2 = 4y \qquad r^2 = x^2 + y^2;\ y = r\sin\theta.$$

This is the equation of a circle; we proceed to complete the square to obtain the standard form of the equation.

$$x^2 + y^2 = 4y$$

$$x^2 + (y^2 - 4y) = 0 \qquad \text{General form}$$

$$x^2 + (y^2 - 4y + 4) = 4 \qquad \text{Complete the square in } y.$$

$$x^2 + (y - 2)^2 = 4 \qquad \text{Factor}$$

This is the standard form of the equation of a circle with center $(0, 2)$ and radius 2. ◀

NOW WORK PROBLEM 75.

EXAMPLE 8 Transforming an Equation from Rectangular to Polar Form

Transform the equation $4xy = 9$ from rectangular coordinates to polar coordinates.

Solution We use formula (1): $x = r\cos\theta$ and $y = r\cos\theta$.

$$4xy = 9$$

$$4(r\cos\theta)(r\sin\theta) = 9 \qquad x = r\cos\theta,\ y = r\sin\theta$$

$$4r^2\cos\theta\sin\theta = 9$$

$$2r^2(2\sin\theta\cos\theta) = 9 \qquad \text{Factor out } 2r^2.$$

$$2r^2\sin(2\theta) = 9 \qquad \text{Double-angle Formula}$$ ◀

8.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Plot the point whose rectangular coordinates are $(3, -1)$. (pp. 2–5)
2. To complete the square of $x^2 + 6x$, add _____. (p. 991)
3. If $P = (x, y)$ is a point on a unit circle and on the terminal side of the angle θ, then $\sin\theta =$ _____. (p. 372)
4. $\tan^{-1}(-1) =$ _____ (pp. 455–457)

Concepts and Vocabulary

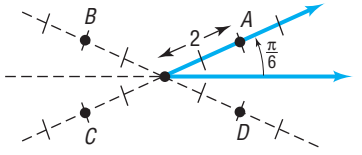
5. In polar coordinates, the origin is called the _____ and the positive x-axis is referred to as the _____ _____.
6. Another representation in polar coordinates for the point $\left(2, \frac{\pi}{3}\right)$ is $\left(____, \frac{4\pi}{3}\right)$.
7. The polar coordinates $\left(-2, \frac{\pi}{6}\right)$ are represented in rectangular coordinates by (_____, _____).
8. *True or False:* The polar coordinates of a point are unique.
9. *True or False:* The rectangular coordinates of a point are unique.
10. *True or False:* In (r, θ), the number r can be negative.

Skill Building

In Problems 11–18, match each point in polar coordinates with either A, B, C, or D on the graph.

11. $\left(2, -\frac{11\pi}{6}\right)$ **12.** $\left(-2, -\frac{\pi}{6}\right)$ **13.** $\left(-2, \frac{\pi}{6}\right)$ **14.** $\left(2, \frac{7\pi}{6}\right)$

15. $\left(2, \frac{5\pi}{6}\right)$ **16.** $\left(-2, \frac{5\pi}{6}\right)$ **17.** $\left(-2, \frac{7\pi}{6}\right)$ **18.** $\left(2, \frac{11\pi}{6}\right)$



In Problems 19–30, plot each point given in polar coordinates.

19. $(3, 90°)$ **20.** $(4, 270°)$ **21.** $(-2, 0)$ **22.** $(-3, \pi)$

23. $\left(6, \frac{\pi}{6}\right)$ **24.** $\left(5, \frac{5\pi}{3}\right)$ **25.** $(-2, 135°)$ **26.** $(-3, 120°)$

27. $\left(-1, -\frac{\pi}{3}\right)$ **28.** $\left(-3, -\frac{3\pi}{4}\right)$ **29.** $(-2, -\pi)$ **30.** $\left(-3, -\frac{\pi}{2}\right)$

In Problems 31–38, plot each point given in polar coordinates, and find other polar coordinates (r, θ) of the point for which:
(a) $r > 0, \quad -2\pi \le \theta < 0$ (b) $r < 0, \quad 0 \le \theta < 2\pi$ (c) $r > 0, \quad 2\pi \le \theta < 4\pi$

31. $\left(5, \frac{2\pi}{3}\right)$ **32.** $\left(4, \frac{3\pi}{4}\right)$ **33.** $(-2, 3\pi)$ **34.** $(-3, 4\pi)$

35. $\left(1, \frac{\pi}{2}\right)$ **36.** $(2, \pi)$ **37.** $\left(-3, -\frac{\pi}{4}\right)$ **38.** $\left(-2, -\frac{2\pi}{3}\right)$

In Problems 39–54, the polar coordinates of a point are given. Find the rectangular coordinates of each point. Verify your results using a graphing utility.

39. $\left(3, \frac{\pi}{2}\right)$ **40.** $\left(4, \frac{3\pi}{2}\right)$ **41.** $(-2, 0)$ **42.** $(-3, \pi)$

43. $(6, 150°)$ **44.** $(5, 300°)$ **45.** $\left(-2, \frac{3\pi}{4}\right)$ **46.** $\left(-2, \frac{2\pi}{3}\right)$

47. $\left(-1, -\frac{\pi}{3}\right)$ **48.** $\left(-3, -\frac{3\pi}{4}\right)$ **49.** $(-2, -180°)$ **50.** $(-3, -90°)$

51. $(7.5, 110°)$ **52.** $(-3.1, 182°)$ **53.** $(6.3, 3.8)$ **54.** $(8.1, 5.2)$

In Problems 55–66, the rectangular coordinates of a point are given. Find polar coordinates for each point. Verify your results using a graphing utility.

55. $(3, 0)$ **56.** $(0, 2)$ **57.** $(-1, 0)$ **58.** $(0, -2)$

59. $(1, -1)$ **60.** $(-3, 3)$ **61.** $(\sqrt{3}, 1)$ **62.** $(-2, -2\sqrt{3})$

63. $(1.3, -2.1)$ **64.** $(-0.8, -2.1)$ **65.** $(8.3, 4.2)$ **66.** $(-2.3, 0.2)$

In Problems 67–74, the letters x and y represent rectangular coordinates. Write each equation using polar coordinates (r, θ).

67. $2x^2 + 2y^2 = 3$ **68.** $x^2 + y^2 = x$ **69.** $x^2 = 4y$ **70.** $y^2 = 2x$

71. $2xy = 1$ **72.** $4x^2y = 1$ **73.** $x = 4$ **74.** $y = -3$

In Problems 75–82, the letters r and θ represent polar coordinates. Write each equation using rectangular coordinates (x, y).

75. $r = \cos\theta$ **76.** $r = \sin\theta + 1$ **77.** $r^2 = \cos\theta$ **78.** $r = \sin\theta - \cos\theta$

79. $r = 2$ **80.** $r = 4$ **81.** $r = \frac{4}{1 - \cos\theta}$ **82.** $r = \frac{3}{3 - \cos\theta}$

Applications and Extensions

83. Show that the formula for the distance d between two points $P_1 = (r_1, \theta_1)$ and $P_2 = (r_2, \theta_2)$ is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

Discussion and Writing

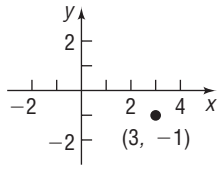
84. In converting from polar coordinates to rectangular coordinates, what formulas will you use?

85. Explain how you proceed to convert from rectangular coordinates to polar coordinates.

86. Is the street system in your town based on a rectangular coordinate system, a polar coordinate system, or some other system? Explain.

'Are You Prepared?' Answers

1.



2. 9 **3.** y **4.** $-\frac{\pi}{4}$

8.2 Polar Equations and Graphs

PREPARING FOR THIS SECTION *Before getting started, review the following:*

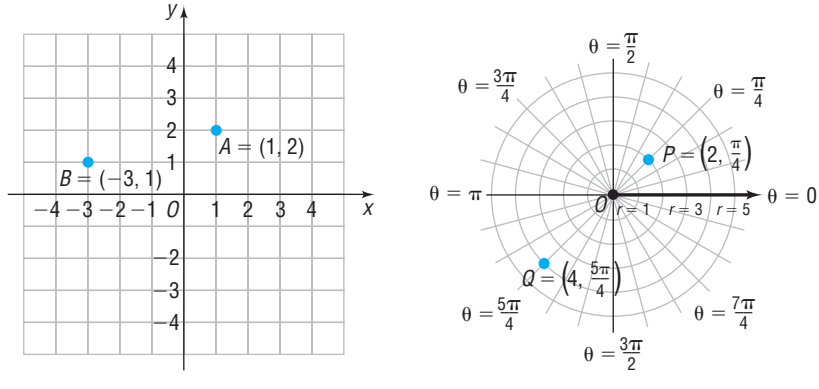
- Symmetry; (Section 1.2, pp. 17–19)
- Circles (Section 1.5, pp. 44–49)
- Even–Odd Properties of Trigonometric Functions (Section 5.3, pp. 398–399)
- Difference Formulas for Sine and Cosine (Section 6.4, pp. 473 and 476)
- Value of the Sine and Cosine Functions at Certain Angles (Section 5.2, pp. 374–381)

Now work the 'Are You Prepared?' problems on page 597.

OBJECTIVES
1. Graph and Identify Polar Equations by Converting to Rectangular Equations
2. Graph Polar Equations Using a Graphing Utility
3. Test Polar Equations for Symmetry
4. Graph Polar Equations by Plotting Points

Just as a rectangular grid may be used to plot points given by rectangular coordinates, as in Figure 20(a), we can use a grid consisting of concentric circles (with centers at the pole) and rays (with vertices at the pole) to plot points given by polar coordinates, as shown in Figure 20(b). We shall use such **polar grids** to graph *polar equations*.

Figure 20



(a) Rectangular grid **(b)** Polar grid

An equation whose variables are polar coordinates is called a **polar equation**. The **graph of a polar equation** consists of all points whose polar coordinates satisfy the equation.

1 Graph and Identify Polar Equations by Converting to Rectangular Equations

One method that we can use to graph a polar equation is to convert the equation to rectangular coordinates. In the discussion that follows, (x, y) represent the rectangular coordinates of a point P, and (r, θ) represent polar coordinates of the point P.

EXAMPLE 1 Identifying and Graphing a Polar Equation By Hand (Circle)

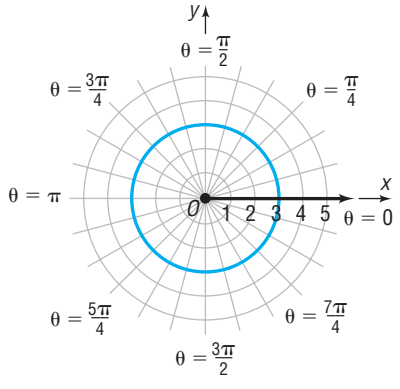
Identify and graph the equation: $r = 3$

Solution We convert the polar equation to a rectangular equation.

$$r = 3$$
$$r^2 = 9 \qquad \text{Square both sides.}$$
$$x^2 + y^2 = 9 \qquad r^2 = x^2 + y^2$$

The graph of $r = 3$ is a circle, with center at the pole and radius 3. See Figure 21.

Figure 21
$r = 3$ or $x^2 + y^2 = 9$



NOW WORK PROBLEM 13.

EXAMPLE 2 Identifying and Graphing a Polar Equation By Hand (Line)

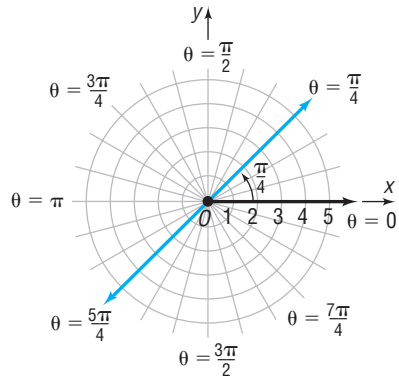
Identify and graph the equation: $\theta = \dfrac{\pi}{4}$

Solution We convert the polar equation to a rectangular equation.

$$\theta = \frac{\pi}{4}$$
$$\tan\theta = \tan\frac{\pi}{4} = 1$$
$$\frac{y}{x} = 1 \qquad \tan\theta = \frac{y}{x}$$
$$y = x$$

The graph of $\theta = \frac{\pi}{4}$ is a line passing through the pole making an angle of $\frac{\pi}{4}$ with the polar axis. See Figure 22.

Figure 22
$\theta = \frac{\pi}{4}$ or $y = x$



NOW WORK PROBLEM 15.

EXAMPLE 3 Identifying and Graphing a Polar Equation By Hand (Horizontal Line)

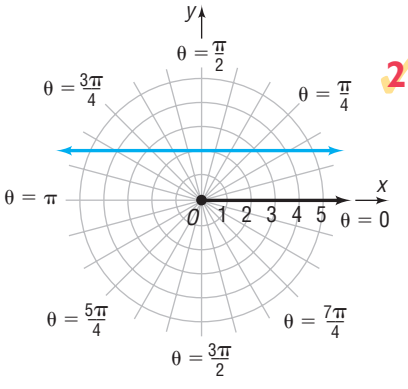
Identify and graph the equation: $r \sin \theta = 2$

Solution Since $y = r \sin \theta$, we can write the equation as

$$y = 2$$

We conclude that the graph of $r \sin \theta = 2$ is a horizontal line 2 units above the pole. See Figure 23.

Figure 23
$r \sin \theta = 2$ or $y = 2$



2 Graph Polar Equations Using a Graphing Utility

A second method we can use to graph a polar equation is to graph the equation using a graphing utility.

Most graphing utilities require the following steps to obtain the graph of an equation:

Graphing a Polar Equation Using a Graphing Utility

STEP 1: Solve the equation for r in terms of θ.

STEP 2: Select the viewing window in POLar mode. In addition to setting *X*min, *X*max, *X*scl, and so forth, the viewing window in polar mode requires setting minimum and maximum values for θ and an increment setting for θ (θstep). Finally, a square screen and radian measure should be used.

STEP 3: Enter the expression involving θ that you found in Step 1. (Consult your manual for the correct way to enter the expression.)

STEP 4: Press graph.

EXAMPLE 4 Graphing a Polar Equation Using a Graphing Utility

Use a graphing utility to graph the polar equation $r \sin \theta = 2$.

Solution **STEP 1:** We solve the equation for r in terms of θ.

$$r \sin \theta = 2$$

$$r = \frac{2}{\sin \theta}$$

STEP 2: From the polar mode, select a square viewing window. We will use the one given next.

$$\theta\text{min} = 0 \qquad X\text{min} = -9 \qquad Y\text{min} = -6$$
$$\theta\text{max} = 2\pi \qquad X\text{max} = 9 \qquad Y\text{max} = 6$$
$$\theta\text{step} = \frac{\pi}{24} \qquad X\text{scl} = 1 \qquad Y\text{scl} = 1$$

θstep determines the number of points the graphing utility will plot. For example, if θstep is $\frac{\pi}{24}$, then the graphing utility will evaluate r at $\theta = 0$ (θmin), $\frac{\pi}{24}, \frac{2\pi}{24}, \frac{3\pi}{24}$, and so forth, up to 2π(θmax).

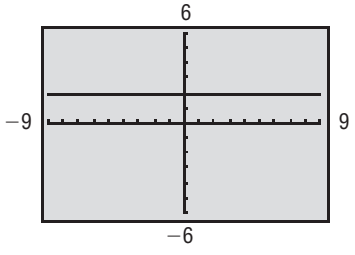
The smaller θstep, the more points the graphing utility will plot. The student is encouraged to experiment with different values for θmin, θmax, and θstep to see how the graph is affected.

STEP 3: Enter the expression $\frac{2}{\sin\theta}$ after the prompt $r =$.

STEP 4: Graph.

The graph is shown in Figure 24. ◀

Figure 24



EXAMPLE 5 Identifying and Graphing a Polar Equation (Vertical Line)

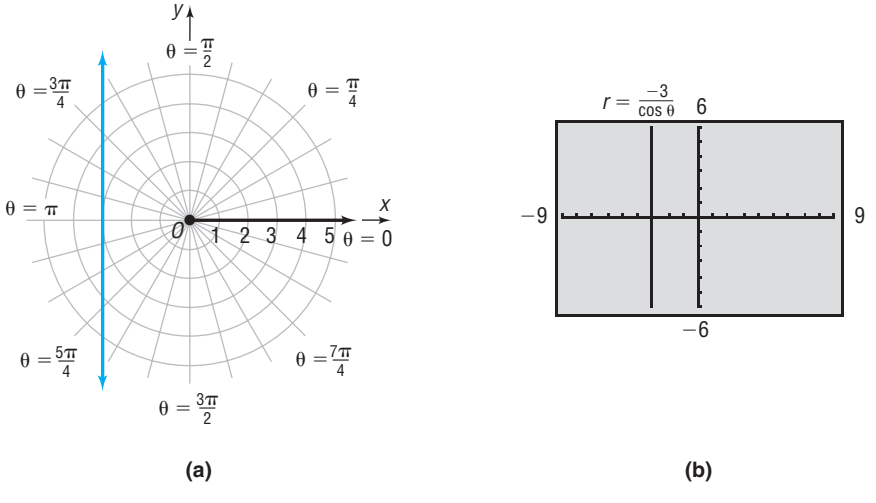
Identify and graph the equation: $r\cos\theta = -3$

Solution Since $x = r\cos\theta$, we can write the equation as

$$x = -3$$

We conclude that the graph of $r\cos\theta = -3$ is a vertical line 3 units to the left of the pole. Figure 25(a) shows the graph drawn by hand. Figure 25(b) shows the graph using a graphing utility with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \frac{\pi}{24}$.

Figure 25
$r\cos\theta = -3$ or $x = -3$



◀

Based on Examples 3, 4, and 5, we are led to the following results. (The proofs are left as exercises.)

Theorem

Let a be a nonzero real number. Then the graph of the equation

$$r \sin \theta = a$$

is a horizontal line a units above the pole if $a > 0$ and $|a|$ units below the pole if $a < 0$.

The graph of the equation

$$r \cos \theta = a$$

is a vertical line a units to the right of the pole if $a > 0$ and $|a|$ units to the left of the pole if $a < 0$.

NOW WORK PROBLEM 19.

EXAMPLE 6 Identifying and Graphing a Polar Equation (Circle)

Identify and graph the equation: $r = 4 \sin \theta$

Solution To transform the equation to rectangular coordinates, we multiply each side by r.

$$r^2 = 4r \sin \theta$$

Now we use the facts that $r^2 = x^2 + y^2$ and $y = r \sin \theta$. Then

$$x^2 + y^2 = 4y$$

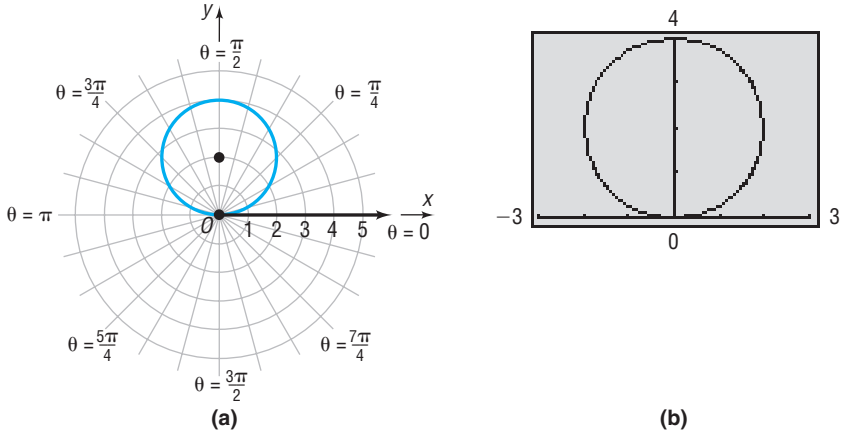
$$x^2 + (y^2 - 4y) = 0$$

$$x^2 + (y^2 - 4y + 4) = 4 \qquad \text{Complete the square in } y.$$

$$x^2 + (y - 2)^2 = 4 \qquad \text{Factor.}$$

This is the standard equation of a circle with center at $(0, 2)$ in rectangular coordinates and radius 2. Figure 26(a) shows the graph drawn by hand. Figure 26(b) shows the graph using a graphing utility with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \dfrac{\pi}{24}$.

Figure 26
$r = 4 \sin \theta$ or $x^2 + (y - 2)^2 = 4$



EXAMPLE 7 Identifying and Graphing a Polar Equation (Circle)

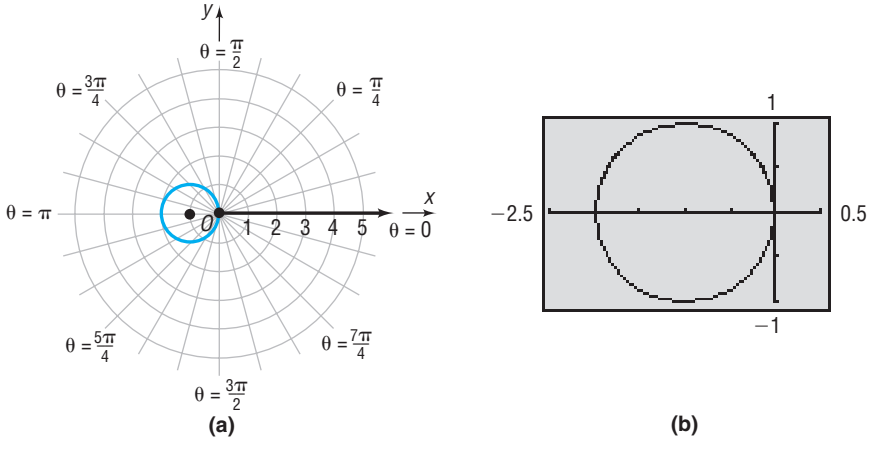
Identify and graph the equation: $r = -2\cos\theta$

Solution We proceed as in Example 6.

$$r^2 = -2r\cos\theta \qquad \text{Multiply both sides by } r.$$
$$x^2 + y^2 = -2x \qquad r^2 = x^2 + y^2;\quad x = r\cos\theta$$
$$x^2 + 2x + y^2 = 0$$
$$(x^2 + 2x + 1) + y^2 = 1 \qquad \text{Complete the square in } x.$$
$$(x + 1)^2 + y^2 = 1 \qquad \text{Factor.}$$

This is the standard equation of a circle with center at $(-1, 0)$ in rectangular coordinates and radius 1. Figure 27(a) shows the graph drawn by hand. Figure 27(b) shows the graph using a graphing utility with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \dfrac{\pi}{24}$.

Figure 27
$r = -2\cos\theta$ or $(x + 1)^2 + y^2 = 1$



Exploration

Using a square screen, graph $r_1 = \sin\theta$, $r_2 = 2\sin\theta$, and $r_3 = 3\sin\theta$. Do you see the pattern? Clear the screen and graph $r_1 = -\sin\theta$, $r_2 = -2\sin\theta$, and $r_3 = -3\sin\theta$. Do you see the pattern? Clear the screen and graph $r_1 = \cos\theta$, $r_2 = 2\cos\theta$, and $r_3 = 3\cos\theta$. Do you see the pattern? Clear the screen and graph $r_1 = -\cos\theta$, $r_2 = -2\cos\theta$, and $r_3 = -3\cos\theta$. Do you see the pattern?

Based on Examples 6 and 7 and the preceding Exploration, we are led to the following results. (The proofs are left as exercises.)

Theorem Let a be a positive real number. Then,

Equation	Description
(a) $r = 2a\sin\theta$	Circle: radius a; center at $(0, a)$ in rectangular coordinates
(b) $r = -2a\sin\theta$	Circle: radius a; center at $(0, -a)$ in rectangular coordinates
(c) $r = 2a\cos\theta$	Circle: radius a; center at $(a, 0)$ in rectangular coordinates
(d) $r = -2a\cos\theta$	Circle: radius a; center at $(-a, 0)$ in rectangular coordinates

Each circle passes through the pole.

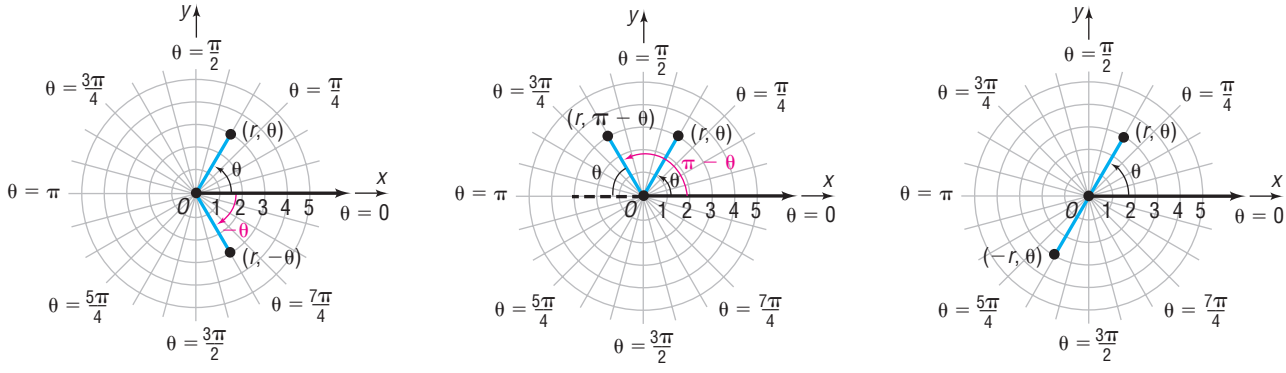
NOW WORK PROBLEM 21.

The method of converting a polar equation to an identifiable rectangular equation to obtain the graph is not always helpful, nor is it always necessary. Usually, we set up a table that lists several points on the graph. By checking for symmetry, it may be possible to reduce the number of points needed to draw the graph.

3 Test Polar Equations for Symmetry

In polar coordinates, the points (r, θ) and $(r, -\theta)$ are symmetric with respect to the polar axis (and to the x-axis). See Figure 28(a). The points (r, θ) and $(r, \pi - \theta)$ are symmetric with respect to the line $\theta = \dfrac{\pi}{2}$ (the y-axis). See Figure 28(b). The points (r, θ) and $(-r, \theta)$ are symmetric with respect to the pole (the origin). See Figure 28(c).

Figure 28



(a) Points symmetric with respect to the polar axis

(b) Points symmetric with respect to the line $\theta = \frac{\pi}{2}$

(c) Points symmetric with respect to the pole

The following tests are a consequence of these observations.

Theorem

Tests for Symmetry

Symmetry with Respect to the Polar Axis (x-Axis)

In a polar equation, replace θ by $-\theta$. If an equivalent equation results, the graph is symmetric with respect to the polar axis.

Symmetry with Respect to the Line $\theta = \dfrac{\pi}{2}$ (y-Axis)

In a polar equation, replace θ by $\pi - \theta$. If an equivalent equation results, the graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

Symmetry with Respect to the Pole (Origin)

In a polar equation, replace r by $-r$. If an equivalent equation results, the graph is symmetric with respect to the pole.

The three tests for symmetry given here are *sufficient* conditions for symmetry, but they are not *necessary* conditions. That is, an equation may fail these tests and still have a graph that is symmetric with respect to the polar axis, the line $\theta = \dfrac{\pi}{2}$, or the pole. For example, the graph of $r = \sin(2\theta)$ turns out to be symmetric with respect to the polar axis, the line $\theta = \dfrac{\pi}{2}$, and the pole, but all three tests given here fail. See also Problems 87, 88, and 89.

4 Graph Polar Equations by Plotting Points

EXAMPLE 8 Graphing a Polar Equation (Cardioid)

Graph the equation: $r = 1 - \sin\theta$

Solution We check for symmetry first.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 1 - \sin(-\theta) = 1 + \sin\theta$$

The test fails, so the graph may or may not be symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace θ by $\pi - \theta$. The result is

$$\begin{aligned} r = 1 - \sin(\pi - \theta) &= 1 - (\sin\pi\cos\theta - \cos\pi\sin\theta) \\ &= 1 - [0\cdot\cos\theta - (-1)\sin\theta] = 1 - \sin\theta \end{aligned}$$

The test is satisfied, so the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Replace r by $-r$. Then the result is $-r = 1 - \sin\theta$, so $r = -1 + \sin\theta$. The test fails, so the graph may or may not be symmetric with respect to the pole.

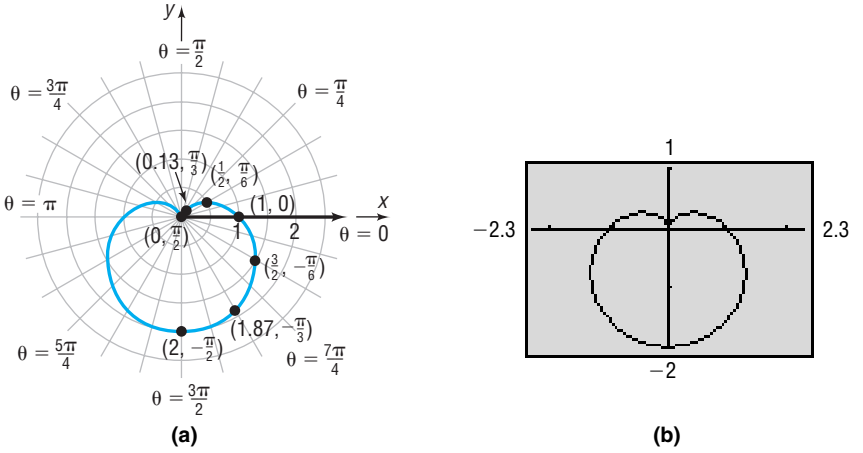
Next, we identify points on the graph by assigning values to the angle θ and calculating the corresponding values of r. Due to the symmetry with respect to the line $\theta = \frac{\pi}{2}$, we only need to assign values to θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, as given in Table 1.

Now we plot the points (r, θ) from Table 1 and trace out the graph, beginning at the point $\left(2, -\frac{\pi}{2}\right)$ and ending at the point $\left(0, \frac{\pi}{2}\right)$. Then we reflect this portion of the graph about the line $\theta = \frac{\pi}{2}$ (the y-axis) to obtain the complete graph. Figure 29(a) shows the graph drawn by hand. Figure 29(b) shows the graph using a graphing utility with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \frac{\pi}{24}$.

Table 1

θ	$r = 1 - \sin\theta$
$-\frac{\pi}{2}$	$1 - (-1) = 2$
$-\frac{\pi}{3}$	$1 - \left(-\frac{\sqrt{3}}{2}\right) \approx 1.87$
$-\frac{\pi}{6}$	$1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$
0	$1 - 0 = 1$
$\frac{\pi}{6}$	$1 - \frac{1}{2} = \frac{1}{2}$
$\frac{\pi}{3}$	$1 - \frac{\sqrt{3}}{2} \approx 0.13$
$\frac{\pi}{2}$	$1 - 1 = 0$

Figure 29



(a) (b)

The curve in Figure 29 is an example of a *cardioid* (a heart-shaped curve).

Exploration

Graph $r_1 = 1 + \sin\theta$. Clear the screen and graph $r_1 = 1 - \cos\theta$. Clear the screen and graph $r_1 = 1 + \cos\theta$. Do you see a pattern?

Cardioids are characterized by equations of the form

$$r = a(1 + \cos\theta) \qquad r = a(1 + \sin\theta)$$
$$r = a(1 - \cos\theta) \qquad r = a(1 - \sin\theta)$$

where $a > 0$. The graph of a cardioid passes through the pole.

NOW WORK PROBLEM 43.

EXAMPLE 9 Graphing a Polar Equation (Limaçon without Inner Loop)

Graph the equation: $r = 3 + 2\cos\theta$

Solution We check for symmetry first.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 3 + 2\cos(-\theta) = 3 + 2\cos\theta$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace θ by $\pi - \theta$. The result is

$$r = 3 + 2\cos(\pi - \theta) = 3 + 2(\cos\pi\cos\theta + \sin\pi\sin\theta)$$
$$= 3 - 2\cos\theta$$

The test fails, so the graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Replace r by $-r$. The test fails, so the graph may or may not be symmetric with respect to the pole.

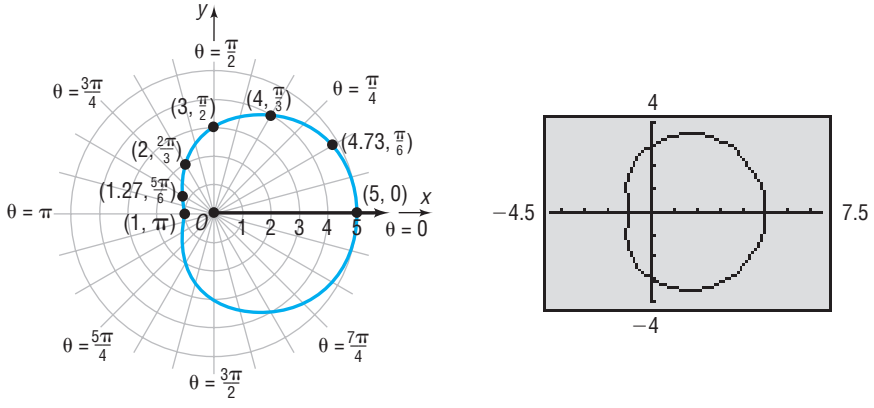
Next, we identify points on the graph by assigning values to the angle θ and calculating the corresponding values of r. Due to the symmetry with respect to the polar axis, we only need to assign values to θ from 0 to π, as given in Table 2.

Now we plot the points (r, θ) from Table 2 and trace out the graph, beginning at the point $(5, 0)$ and ending at the point $(1, \pi)$. Then we reflect this portion of the graph about the polar axis (the x-axis) to obtain the complete graph. Figure 30(a) shows the graph drawn by hand. Figure 30(b) shows the graph using a graphing utility with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \frac{\pi}{24}$.

Table 2

θ	$r = 3 + 2\cos\theta$
0	$3 + 2(1) = 5$
$\frac{\pi}{6}$	$3 + 2\left(\frac{\sqrt{3}}{2}\right) \approx 4.73$
$\frac{\pi}{3}$	$3 + 2\left(\frac{1}{2}\right) = 4$
$\frac{\pi}{2}$	$3 + 2(0) = 3$
$\frac{2\pi}{3}$	$3 + 2\left(-\frac{1}{2}\right) = 2$
$\frac{5\pi}{6}$	$3 + 2\left(-\frac{\sqrt{3}}{2}\right) \approx 1.27$
π	$3 + 2(-1) = 1$

Figure 30



(a) (b)

Exploration

Graph $r_1 = 3 - 2\cos\theta$. Clear the screen and graph $r_1 = 3 + 2\sin\theta$. Clear the screen and graph $r_1 = 3 - 2\sin\theta$. Do you see a pattern?

The curve in Figure 30 is an example of a *limaçon* (the French word for *snail*) *without an inner loop*.

Limaçons without an inner loop are characterized by equations of the form

$$r = a + b\cos\theta \qquad r = a + b\sin\theta$$
$$r = a - b\cos\theta \qquad r = a - b\sin\theta$$

where $a > 0$, $b > 0$, and $a > b$. The graph of a limaçon without an inner loop does not pass through the pole.

NOW WORK PROBLEM 49.

EXAMPLE 10 Graphing a Polar Equation (Limaçon with Inner Loop)

Graph the equation: $r = 1 + 2\cos\theta$

Solution First, we check for symmetry.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 1 + 2\cos(-\theta) = 1 + 2\cos\theta$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace θ by $\pi - \theta$. The result is

$$r = 1 + 2\cos(\pi - \theta) = 1 + 2(\cos\pi\cos\theta + \sin\pi\sin\theta)$$
$$= 1 - 2\cos\theta$$

The test fails, so the graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Replace r by $-r$. The test fails, so the graph may or may not be symmetric with respect to the pole.

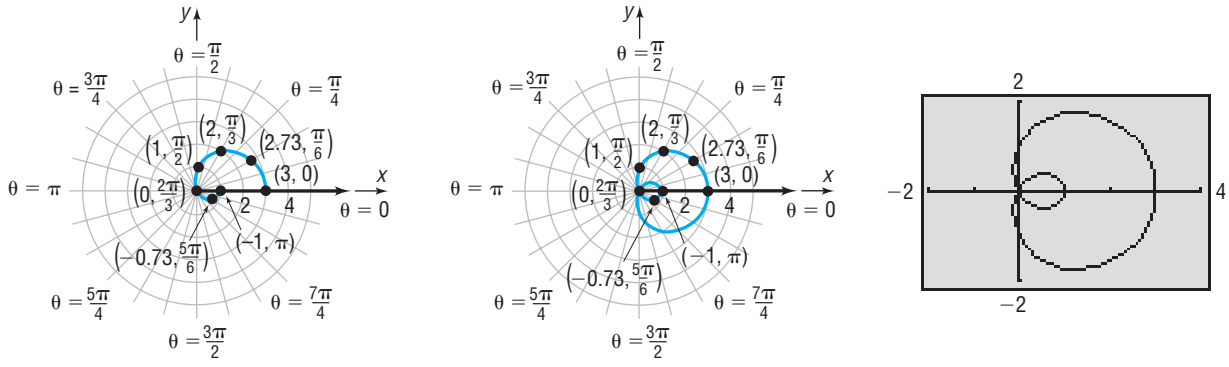
Next, we identify points on the graph of $r = 1 + 2\cos\theta$ by assigning values to the angle θ and calculating the corresponding values of r. Due to the symmetry with respect to the polar axis, we only need to assign values to θ from 0 to π, as given in Table 3.

Now we plot the points (r, θ) from Table 3, beginning at $(3, 0)$ and ending at $(-1, \pi)$. See Figure 31(a). Finally, we reflect this portion of the graph about the polar axis (the x-axis) to obtain the complete graph. See Figure 31(b). Figure 31(c) shows the graph using a graphing utility with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \frac{\pi}{24}$.

Table 3

θ	$r = 1 + 2\cos\theta$
0	$1 + 2(1) = 3$
$\frac{\pi}{6}$	$1 + 2\left(\frac{\sqrt{3}}{2}\right) \approx 2.73$
$\frac{\pi}{3}$	$1 + 2\left(\frac{1}{2}\right) = 2$
$\frac{\pi}{2}$	$1 + 2(0) = 1$
$\frac{2\pi}{3}$	$1 + 2\left(-\frac{1}{2}\right) = 0$
$\frac{5\pi}{6}$	$1 + 2\left(-\frac{\sqrt{3}}{2}\right) \approx -0.73$
π	$1 + 2(-1) = -1$

Figure 31



(a) (b) $r = 1 + 2\cos\theta$ (c)

— Exploration —

Graph $r_1 = 1 - 2\cos\theta$. Clear the screen and graph $r_1 = 1 + 2\sin\theta$. Clear the screen and graph $r_1 = 1 - 2\sin\theta$. Do you see a pattern?

The curve in Figure 31(b) or 31(c) is an example of a *limaçon with an inner loop*.

Limaçons with an inner loop are characterized by equations of the form

$$r = a + b\cos\theta \qquad r = a + b\sin\theta$$
$$r = a - b\cos\theta \qquad r = a - b\sin\theta$$

where $a > 0$, $b > 0$, and $a < b$. The graph of a limaçon with an inner loop will pass through the pole twice.

NOW WORK PROBLEM 51.

EXAMPLE 11 Graphing a Polar Equation (Rose)

Graph the equation: $r = 2\cos(2\theta)$

Solution We check for symmetry.

Polar Axis: If we replace θ by $-\theta$, the result is

$$r = 2\cos[2(-\theta)] = 2\cos(2\theta)$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: If we replace θ by $\pi - \theta$, we obtain

$$r = 2\cos[2(\pi - \theta)] = 2\cos(2\pi - 2\theta) = 2\cos(2\theta)$$

The test is satisfied, so the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Since the graph is symmetric with respect to both the polar axis and the line $\theta = \frac{\pi}{2}$, it must be symmetric with respect to the pole.

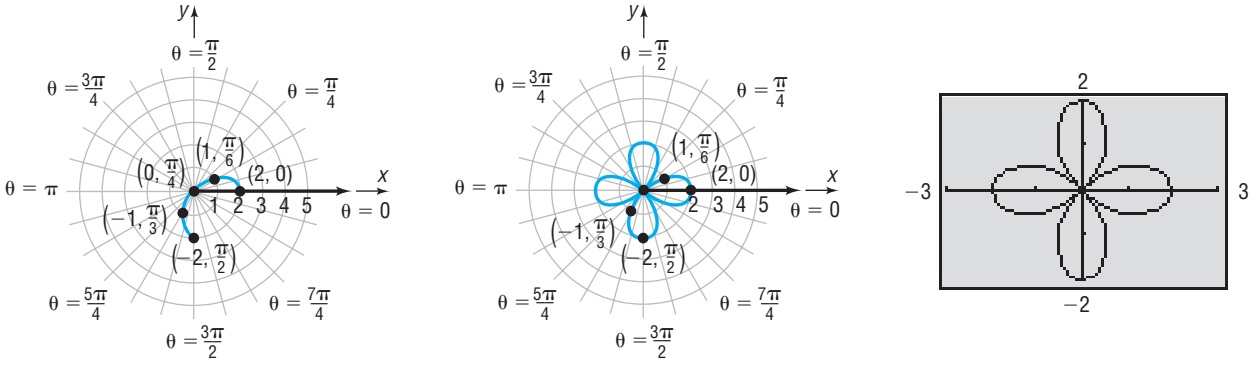
Table 4

θ	$r = 2\cos(2\theta)$
0	$2(1) = 2$
$\frac{\pi}{6}$	$2\left(\frac{1}{2}\right) = 1$
$\frac{\pi}{4}$	$2(0) = 0$
$\frac{\pi}{3}$	$2\left(-\frac{1}{2}\right) = -1$
$\frac{\pi}{2}$	$2(-1) = -2$

Next, we construct Table 4. Due to the symmetry with respect to the polar axis, the line $\theta = \frac{\pi}{2}$, and the pole, we consider only values of θ from 0 to $\frac{\pi}{2}$.

We plot and connect these points in Figure 32(a). Finally, because of symmetry, we reflect this portion of the graph first about the polar axis (the x-axis) and then about the line $\theta = \frac{\pi}{2}$ (the y-axis) to obtain the complete graph. See Figure 32(b). Figure 32(c) shows the graph using a graphing utility with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \frac{\pi}{24}$.

Figure 32



(a) (b) $r = 2\cos(2\theta)$ (c)

Exploration

Graph $r_1 = 2\cos(4\theta)$; clear the screen and graph $r_1 = 2\cos(6\theta)$. How many petals did each of these graphs have?

Clear the screen and graph, in order, each on a clear screen, $r_1 = 2\cos(3\theta)$, $r_1 = 2\cos(5\theta)$, and $r_1 = 2\cos(7\theta)$. What do you notice about the number of petals?

The curve in Figure 32(b) or (c) is called a *rose* with four petals.

Rose curves are characterized by equations of the form

$$r = a\cos(n\theta), \qquad r = a\sin(n\theta), \qquad a \neq 0$$

and have graphs that are rose shaped. If $n \neq 0$ is even, the rose has $2n$ petals; if $n \neq \pm 1$ is odd, the rose has n petals.

NOW WORK PROBLEM 55.

EXAMPLE 12 Graphing a Polar Equation (Lemniscate)

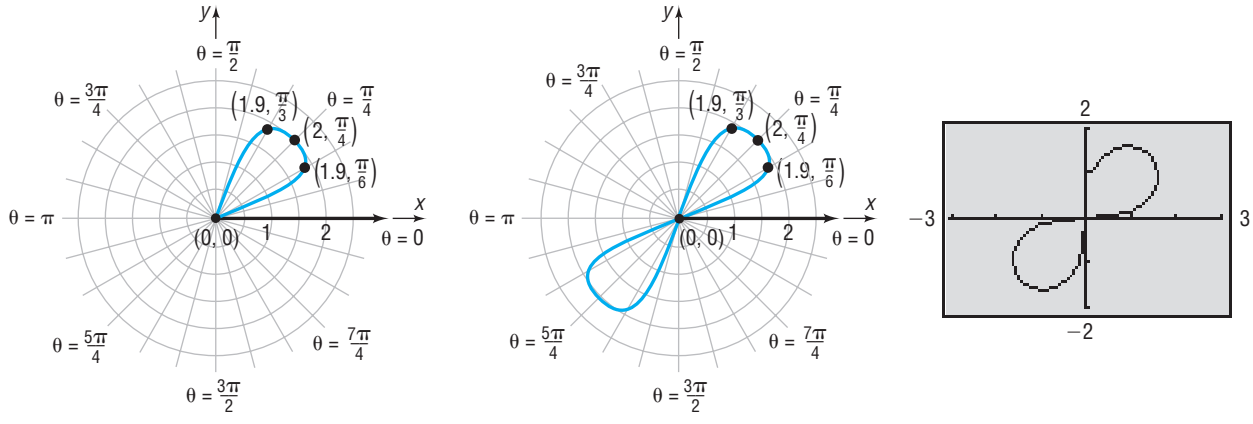
Graph the equation: $r^2 = 4\sin(2\theta)$

Table 5

θ	$r^2 = 4\sin(2\theta)$	r
0	$4(0) = 0$	0
$\frac{\pi}{6}$	$4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$	± 1.9
$\frac{\pi}{4}$	$4(1) = 4$	± 2
$\frac{\pi}{3}$	$4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$	± 1.9
$\frac{\pi}{2}$	$4(0) = 0$	0

Solution We leave it to you to verify that the graph is symmetric with respect to the pole. Table 5 lists points on the graph for values of $\theta = 0$ through $\theta = \frac{\pi}{2}$. Note that there are no points on the graph for $\frac{\pi}{2} < \theta < \pi$ (quadrant II), since $\sin(2\theta) < 0$ for such values. The points from Table 5 where $r \geq 0$ are plotted in Figure 33(a). The remaining points on the graph may be obtained by using symmetry. Figure 33(b) shows the final graph drawn by hand. Figure 33(c) shows the graph using a graphing utility with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \frac{\pi}{24}$.

Figure 33



(a) (b) $r^2 = 4\sin(2\theta)$ (c)

The curve in Figure 33(b) or (c) is an example of a *lemniscate* (from the Greek word *ribbon*).

Lemniscates are characterized by equations of the form

$$r^2 = a^2\sin(2\theta) \qquad r^2 = a^2\cos(2\theta)$$

where $a \neq 0$, and have graphs that are propeller shaped.

NOW WORK PROBLEM 59.

EXAMPLE 13 Graphing a Polar Equation (Spiral)

Graph the equation: $r = e^{\theta/5}$

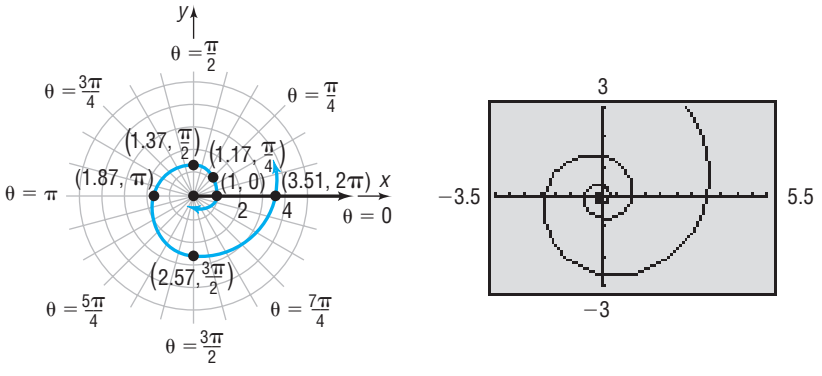
Solution The tests for symmetry with respect to the pole, the polar axis, and the line $\theta = \frac{\pi}{2}$ fail. Furthermore, there is no number θ for which $r = 0$, so the graph does not pass through the pole. We observe that r is positive for all θ, r increases as θ increases, $r \to 0$ as $\theta \to -\infty$, and $r \to \infty$ as $\theta \to \infty$. With the help of a calculator, we obtain

Table 6

θ	$r = e^{\theta/5}$
$-\frac{3\pi}{2}$	0.39
$-\pi$	0.53
$-\frac{\pi}{2}$	0.73
$-\frac{\pi}{4}$	0.85
0	1
$\frac{\pi}{4}$	1.17
$\frac{\pi}{2}$	1.37
π	1.87
$\frac{3\pi}{2}$	2.57
2π	3.51

the values in Table 6. See Figure 34(a) for the graph drawn by hand. Figure 34(b) shows the graph using a graphing utility with $\theta\text{min} = -4\pi$, $\theta\text{max} = 3\pi$, and $\theta\text{step} = \frac{\pi}{24}$.

Figure 34
$r = e^{\theta/5}$



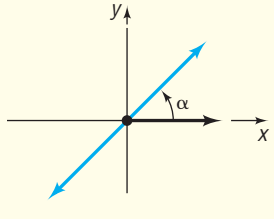
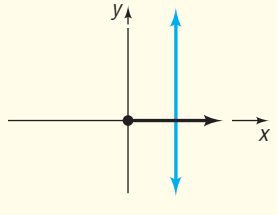
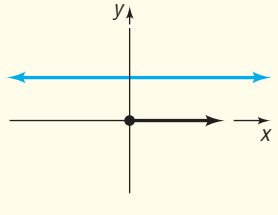
(a) (b)

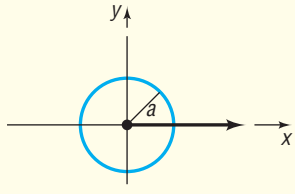
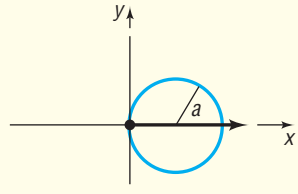
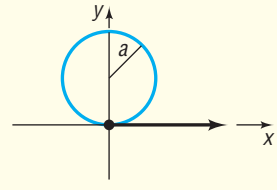
The curve in Figure 34 is called a **logarithmic spiral**, since its equation may be written as $\theta = 5 \ln r$ and it spirals infinitely both toward the pole and away from it.

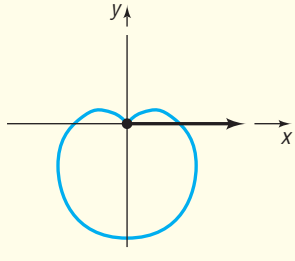
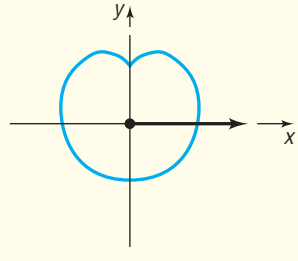
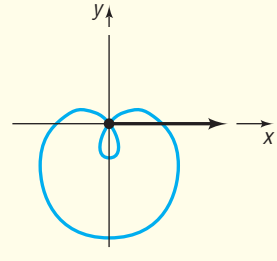
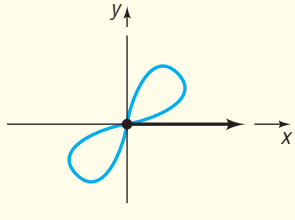
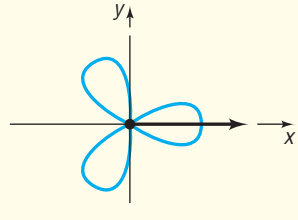
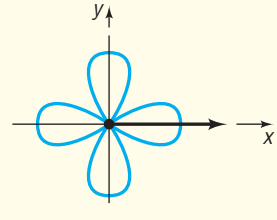
Classification of Polar Equations

The equations of some lines and circles in polar coordinates and their corresponding equations in rectangular coordinates are given in Table 7. Also included are the names and the graphs of a few of the more frequently encountered polar equations.

Table 7

	Lines		
Description	Line passing through the pole making an angle α with the polar axis	Vertical line	Horizontal line
Rectangular equation	$y = (\tan \alpha)x$	$x = a$	$y = b$
Polar equation	$\theta = \alpha$	$r \cos \theta = a$	$r \sin \theta = b$
Typical graph			

Circles			
Description	Center at the pole, radius a	Passing through the pole, tangent to the line $\theta = \frac{\pi}{2}$, center on the polar axis, radius a	Passing through the pole, tangent to the polar axis, center on the line $\theta = \frac{\pi}{2}$, radius a
Rectangular equation	$x^2 + y^2 = a^2, \quad a > 0$	$x^2 + y^2 = \pm 2ax, \quad a > 0$	$x^2 + y^2 = \pm 2ay, \quad a > 0$
Polar equation	$r = a, \quad a > 0$	$r = \pm 2a \cos\theta, \quad a > 0$	$r = \pm 2a \sin\theta, \quad a > 0$
Typical graph			

Other Equations			
Name	Cardioid	Limaçon without inner loop	Limaçon with inner loop
Polar equations	$r = a \pm a \cos\theta, \quad a > 0$ $r = a \pm a \sin\theta, \quad a > 0$	$r = a \pm b \cos\theta, \quad 0 < b < a$ $r = a \pm b \sin\theta, \quad 0 < b < a$	$r = a \pm b \cos\theta, \quad 0 < a < b$ $r = a \pm b \sin\theta, \quad 0 < a < b$
Typical graph			
Name	Lemniscate	Rose with three petals	Rose with four petals
Polar equations	$r^2 = a^2 \cos(2\theta), \quad a > 0$ $r^2 = a^2 \sin(2\theta), \quad a > 0$	$r = a \sin(3\theta), \quad a > 0$ $r = a \cos(3\theta), \quad a > 0$	$r = a \sin(2\theta), \quad a > 0$ $r = a \cos(2\theta), \quad a > 0$
Typical graph			

Sketching Quickly

If a polar equation involves only a sine (or cosine) function, you can quickly obtain a sketch of its graph by making use of Table 7, periodicity, and a short table.

EXAMPLE 14 Sketching the Graph of a Polar Equation Quickly by Hand

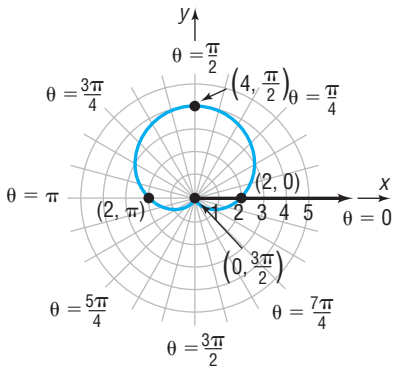
Graph the equation: $r = 2 + 2\sin\theta$

Solution We recognize the polar equation: Its graph is a cardioid. The period of $\sin\theta$ is 2π, so we form a table using $0 \le \theta \le 2\pi$, compute r, plot the points (r, θ), and sketch the graph of a cardioid as θ varies from 0 to 2π. See Table 8 and Figure 35.

Table 8

θ	$r = 2 + 2\sin\theta$
0	$2 + 2(0) = 2$
$\frac{\pi}{2}$	$2 + 2(1) = 4$
π	$2 + 2(0) = 2$
$\frac{3\pi}{2}$	$2 + 2(-1) = 0$
2π	$2 + 2(0) = 2$

Figure 35



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Calculus Comment

For those of you who are planning to study calculus, a comment about one important role of polar equations is in order.

In rectangular coordinates, the equation $x^2 + y^2 = 1$, whose graph is the unit circle, is not the graph of a function. In fact, it requires two functions to obtain the graph of the unit circle:

$$y_1 = \sqrt{1 - x^2} \quad \text{Upper semicircle} \qquad\qquad y_2 = -\sqrt{1 - x^2} \quad \text{Lower semicircle}$$

In polar coordinates, the equation $r = 1$, whose graph is also the unit circle, does define a function. That is, for each choice of θ there is only one corresponding value of r, that is, $r = 1$. Since many problems in calculus require the use of functions, the opportunity to express nonfunctions in rectangular coordinates as functions in polar coordinates becomes extremely useful.

Note also that the vertical-line test for functions is valid only for equations in rectangular coordinates.

HISTORICAL FEATURE

Jakob Bernoulli (1654–1705)

Polar coordinates seem to have been invented by Jakob Bernoulli (1654–1705) in about 1691, although, as with most such ideas, earlier traces of the notion exist. Early users of calculus remained committed to rectangular coordinates, and polar coordinates did not become widely used until the early 1800s. Even then, it was mostly geometers who used them for describing odd curves. Finally, about the mid-1800s, applied mathematicians realized the tremendous simplification that polar coordinates make possible in the description of objects with circular or cylindrical symmetry. From then on their use became widespread.

8.2 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. If the rectangular coordinates of a point are $(4, -6)$, the point symmetric to it with respect to the origin is _____. (pp. 17–19)
2. The difference formula for cosine is $\cos(\alpha - \beta) =$ _____. (p. 473)
3. The standard equation of a circle with center at $(-2, 5)$ and radius 3 is _____. (pp. 44–49)
4. Is the sine function even, odd, or neither? (pp. 398–399)
5. $\sin \dfrac{5\pi}{4} =$ _____. (pp. 380–381)
6. $\cos \dfrac{2\pi}{3} =$ _____. (pp. 380–381)

Concepts and Vocabulary

7. An equation whose variables are polar coordinates is called a _____ _____.
8. Using polar coordinates (r, θ), the circle $x^2 + y^2 = 2x$ takes the form _____.
9. A polar equation is symmetric with respect to the pole if an equivalent equation results when r is replaced by _____.
10. *True or False:* The tests for symmetry in polar coordinates are necessary, but not sufficient.
11. *True or False:* The graph of a cardioid never passes through the pole.
12. *True or False*: All polar equations have a symmetric feature.

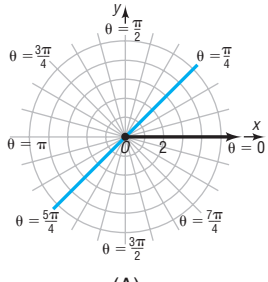
Skill Building

In Problems 13–28, transform each polar equation to an equation in rectangular coordinates. Then identify and graph the equation. Verify your graph using a graphing utility.

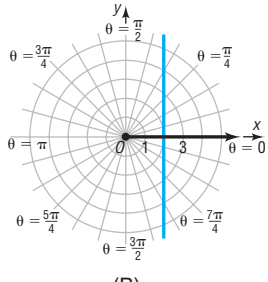
13. $r = 4$
14. $r = 2$
15. $\theta = \dfrac{\pi}{3}$
16. $\theta = -\dfrac{\pi}{4}$
17. $r \sin \theta = 4$
18. $r \cos \theta = 4$
19. $r \cos \theta = -2$
20. $r \sin \theta = -2$
21. $r = 2 \cos \theta$
22. $r = 2 \sin \theta$
23. $r = -4 \sin \theta$
24. $r = -4 \cos \theta$
25. $r \sec \theta = 4$
26. $r \csc \theta = 8$
27. $r \csc \theta = -2$
28. $r \sec \theta = -4$

In Problems 29–36, match each of the graphs (A) through (H) to one of the following polar equations.

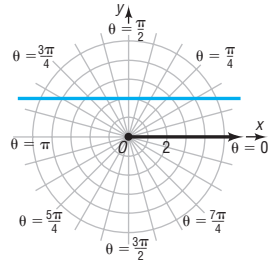
29. $r = 2$
30. $\theta = \dfrac{\pi}{4}$
31. $r = 2 \cos \theta$
32. $r \cos \theta = 2$
33. $r = 1 + \cos \theta$
34. $r = 2 \sin \theta$
35. $\theta = \dfrac{3\pi}{4}$
36. $r \sin \theta = 2$



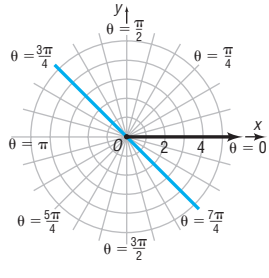
(A)



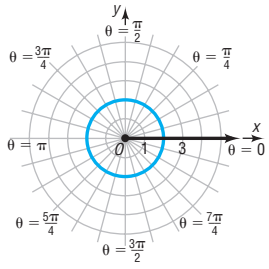
(B)



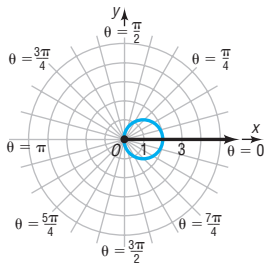
(C)



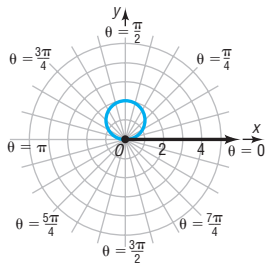
(D)



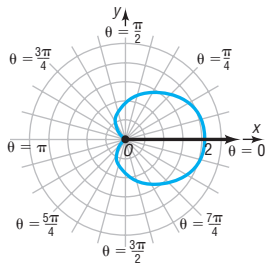
(E)



(F)



(G)

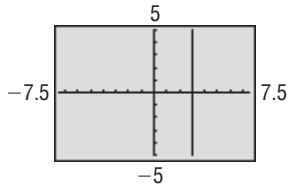


(H)

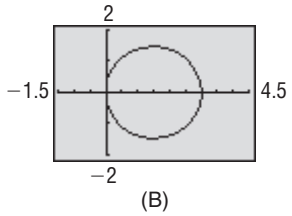
In Problems 37–42, match each of the graphs (A) through (F) to one of the following polar equations.

37. $r = 4$ **38.** $r = 3\cos\theta$ **39.** $r = 3\sin\theta$

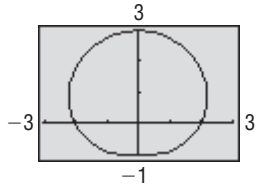
40. $r\sin\theta = 3$ **41.** $r\cos\theta = 3$ **42.** $r = 2 + \sin\theta$



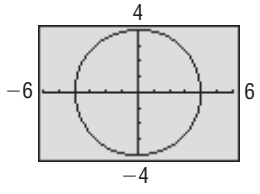
(A)



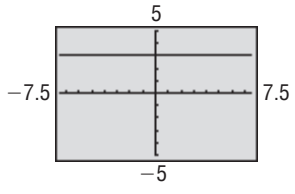
(B)



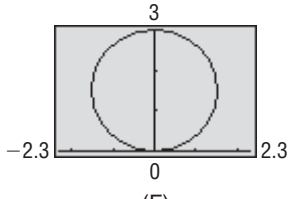
(C)



(D)



(E)



(F)

In Problems 43–66, identify and graph each polar equation. Verify your graph using a graphing utility.

43. $r = 2 + 2\cos\theta$ **44.** $r = 1 + \sin\theta$ **45.** $r = 3 - 3\sin\theta$ **46.** $r = 2 - 2\cos\theta$

47. $r = 2 + \sin\theta$ **48.** $r = 2 - \cos\theta$ **49.** $r = 4 - 2\cos\theta$ **50.** $r = 4 + 2\sin\theta$

51. $r = 1 + 2\sin\theta$ **52.** $r = 1 - 2\sin\theta$ **53.** $r = 2 - 3\cos\theta$ **54.** $r = 2 + 4\cos\theta$

55. $r = 3\cos(2\theta)$ **56.** $r = 2\sin(3\theta)$ **57.** $r = 4\sin(5\theta)$ **58.** $r = 3\cos(4\theta)$

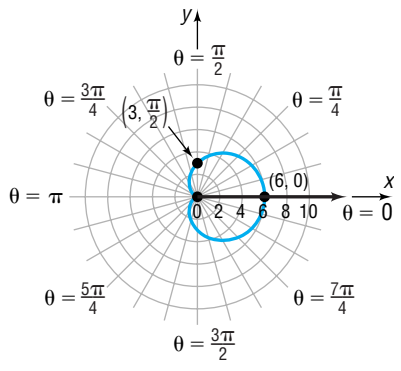
59. $r^2 = 9\cos(2\theta)$ **60.** $r^2 = \sin(2\theta)$ **61.** $r = 2^\theta$ **62.** $r = 3^\theta$

63. $r = 1 - \cos\theta$ **64.** $r = 3 + \cos\theta$ **65.** $r = 1 - 3\cos\theta$ **66.** $r = 4\cos(3\theta)$

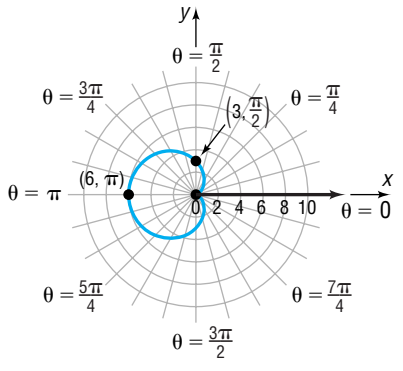
Applications and Extensions

In Problems 67–70, the polar equation for each graph is either $r = a + b\cos\theta$ or $r = a + b\sin\theta$, $a > 0, b > 0$. Select the correct equation and find the values of a and b.

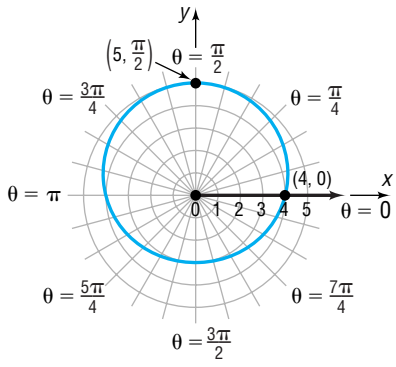
67.



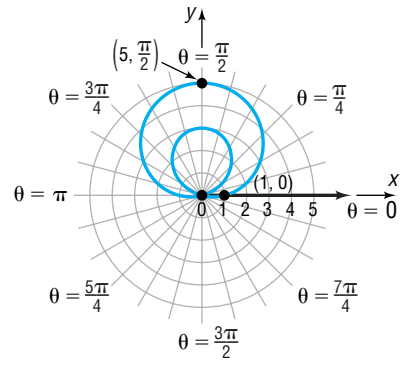
68.



69.



70.



In Problems 71–80, graph each polar equation. Verify your graph using a graphing utility.

71. $r = \dfrac{2}{1 - \cos\theta}$ (*parabola*)

72. $r = \dfrac{2}{1 - 2\cos\theta}$ (*hyperbola*)

73. $r = \dfrac{1}{3 - 2\cos\theta}$ (*ellipse*)

74. $r = \dfrac{1}{1 - \cos\theta}$ (*parabola*)

75. $r = \theta, \quad \theta \geq 0$ (*spiral of Archimedes*)

76. $r = \dfrac{3}{\theta}$ (*reciprocal spiral*)

77. $r = \csc\theta - 2, \quad 0 < \theta < \pi$ (*conchoid*)

78. $r = \sin\theta\tan\theta$ (*cissoid*)

79. $r = \tan\theta, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ (*kappa curve*)

80. $r = \cos\dfrac{\theta}{2}$

81. Show that the graph of the equation $r\sin\theta = a$ is a horizontal line a units above the pole if $a > 0$ and $|a|$ units below the pole if $a < 0$.

82. Show that the graph of the equation $r\cos\theta = a$ is a vertical line a units to the right of the pole if $a > 0$ and $|a|$ units to the left of the pole if $a < 0$.

83. Show that the graph of the equation $r = 2a\sin\theta$, $a > 0$, is a circle of radius a with center at $(0, a)$ in rectangular coordinates.

84. Show that the graph of the equation $r = -2a\sin\theta$, $a > 0$, is a circle of radius a with center at $(0, -a)$ in rectangular coordinates.

85. Show that the graph of the equation $r = 2a\cos\theta$, $a > 0$, is a circle of radius a with center at $(a, 0)$ in rectangular coordinates.

86. Show that the graph of the equation $r = -2a\cos\theta$, $a > 0$, is a circle of radius a with center at $(-a, 0)$ in rectangular coordinates.

Discussion and Writing

87. Explain why the following test for symmetry is valid: Replace r by $-r$ and θ by $-\theta$ in a polar equation. If an equivalent equation results, the graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$ (y-axis).

(a) Show that the test on page 587 fails for $r^2 = \cos\theta$, yet this new test works.

(b) Show that the test on page 587 works for $r^2 = \sin\theta$, yet this new test fails.

88. Develop a new test for symmetry with respect to the pole.

(a) Find a polar equation for which this new test fails, yet the test on page 587 works.

(b) Find a polar equation for which the test on page 587 fails, yet the new test works.

89. Write down two different tests for symmetry with respect to the polar axis. Find examples in which one test works and the other fails. Which test do you prefer to use? Justify your answer.

'Are You Prepared?' Answers

1. $(-4, 6)$ **2.** $\cos\alpha\cos\beta + \sin\alpha\sin\beta$ **3.** $(x + 2)^2 + (y - 5)^2 = 9$ **4.** odd **5.** $-\dfrac{\sqrt{2}}{2}$ **6.** $-\dfrac{1}{2}$

8.3 The Complex Plane; De Moivre's Theorem

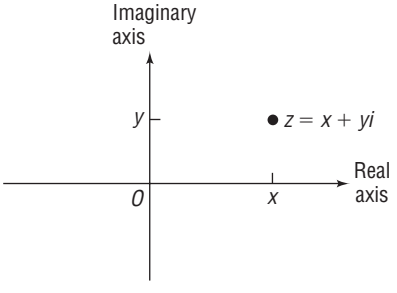
PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Complex Numbers (Appendix, Section A.6, pp. 1000–1005)
- Value of the Sine and Cosine Functions at Certain Angles (Section 5.2, pp. 374–381)
- Sum and Difference Formulas for Sine and Cosine (Section 6.4, pp. 473 and 476)

Now work the 'Are You Prepared?' problems on page 606.

OBJECTIVES
1. Convert a Complex Number from Rectangular Form to Polar Form
2. Plot Points in the Complex Plane
3. Find Products and Quotients of Complex Numbers in Polar Form
4. Use De Moivre's Theorem
5. Find Complex Roots

Figure 36
Complex plane



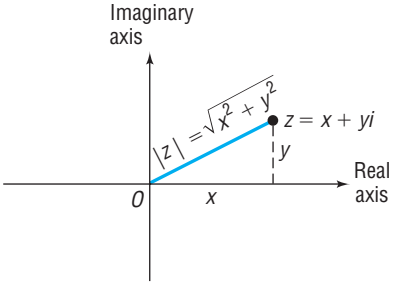
When we first introduced complex numbers, we were not prepared to give a geometric interpretation of a complex number. Now we are ready. Although we could give several interpretations, the one that follows is the easiest to understand.

A complex number $z = x + yi$ can be interpreted geometrically as the point (x, y) in the xy-plane. Each point in the plane corresponds to a complex number and, conversely, each complex number corresponds to a point in the plane. We shall refer to the collection of such points as the **complex plane**. The x-axis will be referred to as the **real axis**, because any point that lies on the real axis is of the form $z = x + 0i = x$, a real number. The y-axis is called the **imaginary axis**, because any point that lies on it is of the form $z = 0 + yi = yi$, a pure imaginary number. See Figure 36.

Let $z = x + yi$ be a complex number. The **magnitude** or **modulus** of z, denoted by $|z|$, is defined as the distance from the origin to the point (x, y). That is,

$$|z| = \sqrt{x^2 + y^2} \tag{1}$$

Figure 37



See Figure 37 for an illustration.

This definition for $|z|$ is consistent with the definition for the absolute value of a real number: If $z = x + yi$ is real, then $z = x + 0i$ and

$$|z| = \sqrt{x^2 + 0^2} = \sqrt{x^2} = |x|$$

For this reason, the magnitude of z is sometimes called the absolute value of z.

Recall that if $z = x + yi$ then its **conjugate**, denoted by $\bar{z}$, is $\bar{z} = x - yi$. Because $z\bar{z} = x^2 + y^2$, it follows from equation (1) that the magnitude of z can be written as

$$|z| = \sqrt{z\bar{z}} \tag{2}$$

1 Convert a Complex Number from Rectangular Form to Polar Form

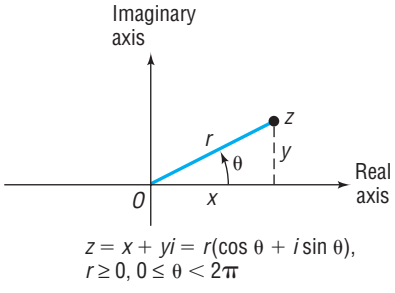
When a complex number is written in the standard form $z = x + yi$, we say that it is in **rectangular**, or **Cartesian, form** because (x, y) are the rectangular coordinates of the corresponding point in the complex plane. Suppose that (r, θ) are the polar coordinates of this point. Then

$$x = r\cos\theta \qquad y = r\sin\theta \tag{3}$$

If $r \geq 0$ and $0 \leq \theta < 2\pi$, the complex number $z = x + yi$ may be written in **polar form** as

$$z = x + yi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta) \tag{4}$$

Figure 38



See Figure 38.

If $z = r(\cos\theta + i\sin\theta)$ is the polar form of a complex number, the angle θ, $0 \leq \theta < 2\pi$, is called the **argument of z**.

Also, because $r \geq 0$, we have $r = \sqrt{x^2 + y^2}$. From equation (1) it follows that the magnitude of $z = r(\cos\theta + i\sin\theta)$ is

$$|z| = r$$

2 Plot Points in the Complex Plane

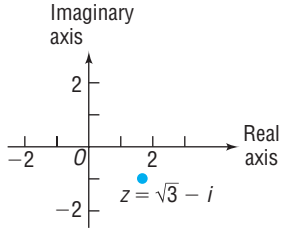
EXAMPLE 1

Plotting a Point in the Complex Plane and Writing a Complex Number in Polar Form

Plot the point corresponding to $z = \sqrt{3} - i$ in the complex plane, and write an expression for z in polar form.

Solution The point corresponding to $z = \sqrt{3} - i$ has the rectangular coordinates $(\sqrt{3}, -1)$. The point, located in quadrant IV, is plotted in Figure 39. Because $x = \sqrt{3}$ and $y = -1$, it follows that

Figure 39



$$r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$

So

$$\sin\theta = \frac{y}{r} = \frac{-1}{2}, \qquad \cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}, \qquad 0 \leq \theta < 2\pi$$

Then $\theta = \dfrac{11\pi}{6}$ and $r = 2$, so the polar form of $z = \sqrt{3} - i$ is

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

NOW WORK PROBLEM 11.

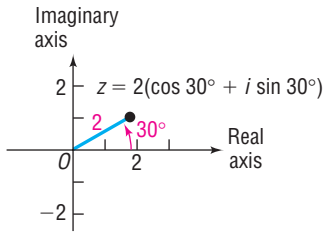
EXAMPLE 2 **Plotting a Point in the Complex Plane and Converting from Polar to Rectangular Form**

Plot the point corresponding to $z = 2(\cos 30° + i \sin 30°)$ in the complex plane, and write an expression for z in rectangular form.

Solution To plot the complex number $z = 2(\cos 30° + i \sin 30°)$, we plot the point whose polar coordinates are $(r, \theta) = (2, 30°)$, as shown in Figure 40. In rectangular form,

$$z = 2(\cos 30° + i \sin 30°) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

Figure 40



NOW WORK PROBLEM 23.

3 Find Products and Quotients of Complex Numbers in Polar Form

The polar form of a complex number provides an alternative method for finding products and quotients of complex numbers.

Theorem Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ be two complex numbers. Then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \tag{5}$$

If $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \tag{6}$$

In Words
The magnitude of a complex number z is r and its argument is θ, so when $z = r(\cos \theta + i \sin \theta)$, the magnitude of the product (quotient) of two complex numbers equals the product (quotient) of their magnitudes; the argument of the product (quotient) of two complex numbers is determined by the sum (difference) of their arguments.

Proof We will prove formula (5). The proof of formula (6) is left as an exercise (see Problem 66).

$$\begin{aligned} z_1 z_2 &= [r_1(\cos \theta_1 + i \sin \theta_1)][r_2(\cos \theta_2 + i \sin \theta_2)] \\ &= r_1 r_2[(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)] \\ &= r_1 r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

■

Let's look at an example of how this theorem can be used.

EXAMPLE 3 **Finding Products and Quotients of Complex Numbers in Polar Form**

If $z = 3(\cos 20° + i \sin 20°)$ and $w = 5(\cos 100° + i \sin 100°)$, find the following (leave your answers in polar form):

(a) zw (b) $\dfrac{z}{w}$

Solution (a)

$$\begin{aligned} zw &= [3(\cos 20° + i \sin 20°)][5(\cos 100° + i \sin 100°)] \\ &= (3 \cdot 5)[\cos(20° + 100°) + i \sin(20° + 100°)] \\ &= 15(\cos 120° + i \sin 120°) \end{aligned}$$

(b) $$\frac{z}{w} = \frac{3(\cos 20° + i \sin 20°)}{5(\cos 100° + i \sin 100°)}$$

$$= \frac{3}{5}[\cos(20° - 100°) + i \sin(20° - 100°)]$$

$$= \frac{3}{5}[\cos(-80°) + i \sin(-80°)]$$

$$= \frac{3}{5}(\cos 280° + i \sin 280°)$$

Argument must lie between 0° and 360°.

NOW WORK PROBLEM 33.

4 Use De Moivre's Theorem

De Moivre's Theorem, stated by Abraham De Moivre (1667–1754) in 1730, but already known to many people by 1710, is important for the following reason: The fundamental processes of algebra are the four operations of addition, subtraction, multiplication, and division, together with powers and the extraction of roots. De Moivre's Theorem allows these latter fundamental algebraic operations to be applied to complex numbers.

De Moivre's Theorem, in its most basic form, is a formula for raising a complex number z to the power n, where $n \geq 1$ is a positive integer. Let's see if we can guess the form of the result.

Let $z = r(\cos \theta + i \sin \theta)$ be a complex number. Then, based on equation (5), we have

$n = 2$: $z^2 = r^2[\cos(2\theta) + i \sin(2\theta)]$ Equation (5)

$n = 3$: $z^3 = z^2 \cdot z$

$= \{r^2[\cos(2\theta) + i \sin(2\theta)]\}[r(\cos \theta + i \sin \theta)]$

$= r^3[\cos(3\theta) + i \sin(3\theta)]$ Equation (5)

$n = 4$: $z^4 = z^3 \cdot z$

$= \{r^3[\cos(3\theta) + i \sin(3\theta)]\}[r(\cos \theta + i \sin \theta)]$

$= r^4[\cos(4\theta) + i \sin(4\theta)]$ Equation (5)

The pattern should now be clear.

Theorem

De Moivre's Theorem

If $z = r(\cos \theta + i \sin \theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i \sin(n\theta)] \tag{7}$$

where $n \geq 1$ is a positive integer.

We will not prove De Moivre's Theorem because the proof requires mathematical induction (which is not discussed until Section 11.4).

Let's look at some examples.

EXAMPLE 4 **Using De Moivre's Theorem**

Write $[2(\cos 20° + i \sin 20°)]^3$ in the standard form $a + bi$.

Solution

$$[2(\cos 20° + i \sin 20°)]^3 = 2^3[\cos(3 \cdot 20°) + i \sin(3 \cdot 20°)]$$
$$= 8(\cos 60° + i \sin 60°)$$
$$= 8\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 4 + 4\sqrt{3}i$$

◀

NOW WORK PROBLEM 41.

EXAMPLE 5 **Using De Moivre's Theorem**

Write $(1 + i)^5$ in the standard form $a + bi$.

Algebraic Solution

To apply De Moivre's Theorem, we must first write the complex number in polar form. Since the magnitude of $1 + i$ is $\sqrt{1^2 + 1^2} = \sqrt{2}$, we begin by writing

$$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$$

Now

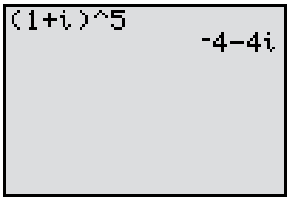
$$(1 + i)^5 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)\right]^5$$
$$= (\sqrt{2})^5\left[\cos\left(5 \cdot \frac{\pi}{4}\right) + i \sin\left(5 \cdot \frac{\pi}{4}\right)\right]$$
$$= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right)$$
$$= 4\sqrt{2}\left[-\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)i\right] = -4 - 4i$$

◀

Graphing Solution

Using a TI-84 Plus graphing calculator, we obtain the solution shown in Figure 41.

Figure 41



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5 Find Complex Roots

Let w be a given complex number, and let $n \geq 2$ denote a positive integer. Any complex number z that satisfies the equation

$$z^n = w$$

is called a **complex *n*th root** of w. In keeping with previous usage, if $n = 2$, the solutions of the equation $z^2 = w$ are called **complex square roots** of w, and if $n = 3$, the solutions of the equation $z^3 = w$ are called **complex cube roots** of w.

Theorem

Finding Complex Roots

Let $w = r(\cos\theta_0 + i\sin\theta_0)$ be a complex number and let $n \geq 2$ be an integer. If $w \neq 0$, there are n distinct complex roots of w, given by the formula

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad \textbf{(8)}$$

where $k = 0, 1, 2, \ldots, n-1$.

Proof (Outline) We will not prove this result in its entirety. Instead, we shall show only that each z_k in equation (8) satisfies the equation $z_k^n = w$, proving that each z_k is a complex nth root of w.

$$\begin{aligned}
z_k^n &= \left\{\sqrt[n]{r}\left[\cos\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]\right\}^n \\
&= (\sqrt[n]{r})^n\left\{\cos\left[n\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] + i\sin\left[n\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]\right\} && \text{De Moivre's Theorem} \\
&= r[\cos(\theta_0 + 2k\pi) + i\sin(\theta_0 + 2k\pi)] && \text{Simplify.} \\
&= r(\cos\theta_0 + i\sin\theta_0) = w && \text{Periodic Property}
\end{aligned}$$

So, each z_k, $k = 0, 1, \ldots, n-1$, is a complex nth root of w. To complete the proof, we would need to show that each z_k, $k = 0, 1, \ldots, n-1$, is, in fact, distinct and that there are no complex nth roots of w other than those given by equation (8). ■

EXAMPLE 6 Finding Complex Cube Roots

Find the complex cube roots of $-1 + \sqrt{3}i$. Leave your answers in polar form, with the argument in degrees.

Solution First, we express $-1 + \sqrt{3}i$ in polar form using degrees.

$$-1 + \sqrt{3}i = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2(\cos 120^\circ + i\sin 120^\circ)$$

So $r = 2$ and $\theta_0 = 120^\circ$. The three complex cube roots of $-1 + \sqrt{3}i = 2(\cos 120^\circ + i\sin 120^\circ)$ are

$$\begin{aligned}
z_k &= \sqrt[3]{2}\left[\cos\left(\frac{120^\circ}{3} + \frac{360^\circ k}{3}\right) + i\sin\left(\frac{120^\circ}{3} + \frac{360^\circ k}{3}\right)\right], \qquad k = 0, 1, 2 \\
&= \sqrt[3]{2}[\cos(40^\circ + 120^\circ k) + i\sin(40^\circ + 120^\circ k)], \qquad k = 0, 1, 2
\end{aligned}$$

So

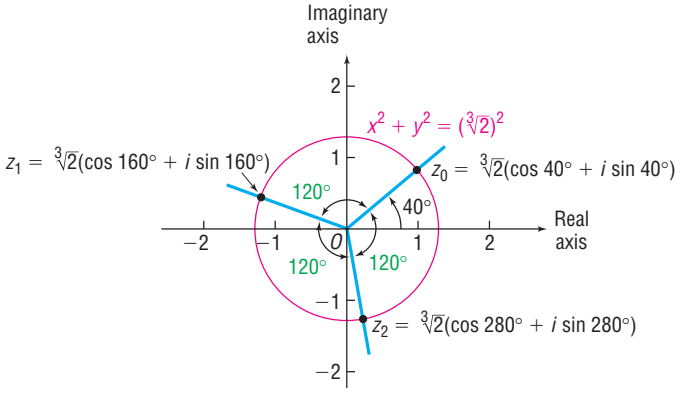
$$\begin{aligned}
z_0 &= \sqrt[3]{2}[\cos(40^\circ + 120^\circ\cdot 0) + i\sin(40^\circ + 120^\circ\cdot 0)] = \sqrt[3]{2}(\cos 40^\circ + i\sin 40^\circ) \\
z_1 &= \sqrt[3]{2}[\cos(40^\circ + 120^\circ\cdot 1) + i\sin(40^\circ + 120^\circ\cdot 1)] = \sqrt[3]{2}(\cos 160^\circ + i\sin 160^\circ) \\
z_2 &= \sqrt[3]{2}[\cos(40^\circ + 120^\circ\cdot 2) + i\sin(40^\circ + 120^\circ\cdot 2)] = \sqrt[3]{2}(\cos 280^\circ + i\sin 280^\circ)
\end{aligned}$$

◀

WARNING Most graphing utilities will only provide the answer z_0 to the calculation $(-1 + \sqrt{3}i)\wedge\left(\frac{1}{3}\right)$. The following paragraph explains how to obtain z_1 and z_2 from z_0. ■

Notice that each of the three complex roots of $-1 + \sqrt{3}i$ has the same magnitude, $\sqrt[3]{2}$. This means that the points corresponding to each cube root lie the same distance from the origin; that is, the three points lie on a circle with center at the origin and radius $\sqrt[3]{2}$. Furthermore, the arguments of these cube roots are 40°, 160°, and 280°, the difference of consecutive pairs being $120° = \frac{360°}{3}$. This means that the three points are equally spaced on the circle, as shown in Figure 42. These results are not coincidental. In fact, you are asked to show that these results hold for complex nth roots in Problems 63 through 65.

Figure 42



NOW WORK PROBLEM 53.

Historical Feature

John Wallis

The Babylonians, Greeks, and Arabs considered square roots of negative quantities to be impossible and equations with complex solutions to be unsolvable. The first hint that there was some connection between real solutions of equations and complex numbers came when Girolamo Cardano (1501–1576) and Tartaglia (1499–1557) found *real* roots of cubic equations by taking cube roots of *complex* quantities. For centuries thereafter, mathematicians worked with complex numbers without much belief in their actual existence. In 1673, John Wallis appears to have been the first to suggest the graphical representation of complex numbers, a truly significant idea that was not pursued further until about 1800. Several people, including Karl Friedrich Gauss (1777–1855), then rediscovered the idea, and graphical representation helped to establish complex numbers as equal members of the number family. In practical applications, complex numbers have found their greatest uses in the study of alternating current, where they are a commonplace tool, and in the field of subatomic physics.

Historical Problems

1. The quadratic formula will work perfectly well if the coefficients are complex numbers. Solve the following using De Moivre's Theorem where necessary.

[Hint: The answers are "nice."]

(a) $z^2 - (2 + 5i)z - 3 + 5i = 0$ **(b)** $z^2 - (1 + i)z - 2 - i = 0$

8.3 Assess your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The conjugate of $-4 - 3i$ is _____. (pp. 1000–1002)

2. The sum formula for sine is $\sin(\alpha + \beta) =$ _____. (p. 476)

3. The sum formula for cosine is $\cos(\alpha + \beta) =$ _____. (p. 473)

4. $\sin 120° =$ _____; $\cos 240° =$ _____. (pp. 380–381)

Concepts and Vocabulary

5. When a complex number z is written in the polar form $z = r(\cos\theta + i\sin\theta)$, the nonnegative number r is the _____ or _____ of z, and the angle θ, $0 \le \theta < 2\pi$, is the _____ of z.
6. _____ Theorem can be used to raise a complex number to a power.
7. A complex number will, in general, have _____ cube roots.
8. *True or False:* De Moivre's Theorem is useful for raising a complex number to a positive integer power.
9. *True or False:* Using De Moivre's Theorem, the square of a complex number will have two answers.
10. *True or False:* The polar form of a complex number is unique.

Skill Building

In Problems 11–22, plot each complex number in the complex plane and write it in polar form. Express the argument in degrees.

11. $1 + i$
12. $-1 + i$
13. $\sqrt{3} - i$
14. $1 - \sqrt{3}i$
15. $-3i$
16. -2
17. $4 - 4i$
18. $9\sqrt{3} + 9i$
19. $3 - 4i$
20. $2 + \sqrt{3}i$
21. $-2 + 3i$
22. $\sqrt{5} - i$

In Problems 23–32, write each complex number in rectangular form.

23. $2(\cos 120° + i\sin 120°)$
24. $3(\cos 210° + i\sin 210°)$
25. $4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$
26. $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
27. $3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$
28. $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
29. $0.2(\cos 100° + i\sin 100°)$
30. $0.4(\cos 200° + i\sin 200°)$
31. $2\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)$
32. $3\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

In Problems 33–40, find zw and $\frac{z}{w}$. Leave your answers in polar form.

33. $z = 2(\cos 40° + i\sin 40°)$
$w = 4(\cos 20° + i\sin 20°)$
34. $z = \cos 120° + i\sin 120°$
$w = \cos 100° + i\sin 100°$
35. $z = 3(\cos 130° + i\sin 130°)$
$w = 4(\cos 270° + i\sin 270°)$
36. $z = 2(\cos 80° + i\sin 80°)$
$w = 6(\cos 200° + i\sin 200°)$
37. $z = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$
$w = 2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$
38. $z = 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$
$w = 2\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$
39. $z = 2 + 2i$
$w = \sqrt{3} - i$
40. $z = 1 - i$
$w = 1 - \sqrt{3}i$

In Problems 41–52, write each expression in the standard form $a + bi$. Verify your answers using a graphing utility.

41. $[4(\cos 40° + i\sin 40°)]^3$
42. $[3(\cos 80° + i\sin 80°)]^3$
43. $\left[2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right]^5$
44. $\left[\sqrt{2}\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right)\right]^4$
45. $\left[\sqrt{3}(\cos 10° + i\sin 10°)\right]^6$
46. $\left[\frac{1}{2}(\cos 72° + i\sin 72°)\right]^5$
47. $\left[\sqrt{5}\left(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right)\right]^4$
48. $\left[\sqrt{3}\left(\cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18}\right)\right]^6$
49. $(1 - i)^5$
50. $(\sqrt{3} - i)^6$
51. $(\sqrt{2} - i)^6$
52. $(1 - \sqrt{5}i)^8$

In Problems 53–60, find all the complex roots. Leave your answers in polar form with the argument in degrees.

53. The complex cube roots of $1 + i$
54. The complex fourth roots of $\sqrt{3} - i$
55. The complex fourth roots of $4 - 4\sqrt{3}i$
56. The complex cube roots of $-8 - 8i$
57. The complex fourth roots of $-16i$
58. The complex cube roots of -8
59. The complex fifth roots of i
60. The complex fifth roots of $-i$

Applications and Extensions

61. Find the four complex fourth roots of unity (1) and plot them.

62. Find the six complex sixth roots of unity (1) and plot them.

63. Show that each complex nth root of a nonzero complex number w has the same magnitude.

64. Use the result of Problem 63 to draw the conclusion that each complex nth root lies on a circle with center at the origin. What is the radius of this circle?

65. Refer to Problem 64. Show that the complex nth roots of a nonzero complex number w are equally spaced on the circle.

66. Prove formula (6).

'Are You Prepared?' Answers

1. $-4 + 3i$ **2.** $\sin\alpha\cos\beta + \cos\alpha\sin\beta$ **3.** $\cos\alpha\cos\beta - \sin\alpha\sin\beta$ **4.** $\frac{\sqrt{3}}{2}; -\frac{1}{2}$

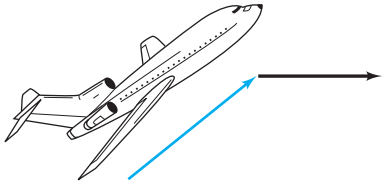
8.4 Vectors

OBJECTIVES
1. Graph Vectors
2. Find a Position Vector
3. Add and Subtract Vectors
4. Find a Scalar Product and the Magnitude of a Vector
5. Find a Unit Vector
6. Find a Vector from Its Direction and Magnitude
7. Work with Objects in Static Equilibrium

In simple terms, a **vector** (derived from the Latin *vehere*, meaning "to carry") is a quantity that has both magnitude and direction. It is customary to represent a vector by using an arrow. The length of the arrow represents the **magnitude** of the vector, and the arrowhead indicates the **direction** of the vector.

Many quantities in physics can be represented by vectors. For example, the velocity of an aircraft can be represented by an arrow that points in the direction of movement; the length of the arrow represents speed. If the aircraft speeds up, we lengthen the arrow; if the aircraft changes direction, we introduce an arrow in the new direction. See Figure 43. Based on this representation, it is not surprising that vectors and directed line segments are somehow related.

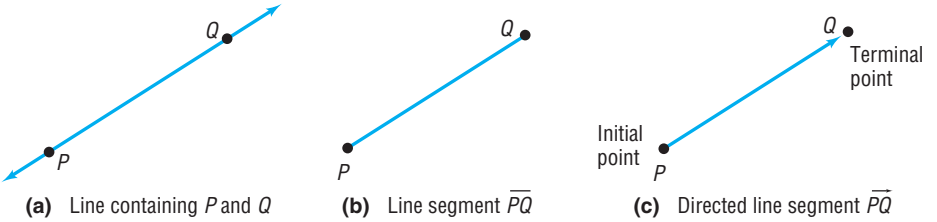
Figure 43



Geometric Vectors

If P and Q are two distinct points in the xy-plane, there is exactly one line containing both P and Q [Figure 44(a)]. The points on that part of the line that joins P to Q, including P and Q, form what is called the **line segment** $\overline{PQ}$ [Figure 44(b)]. If we order the points so that they proceed from P to Q, we have a **directed line segment** from P to Q, or a **geometric vector**, which we denote by $\overrightarrow{PQ}$. In a directed line segment $\overrightarrow{PQ}$, we call P the **initial point** and Q the **terminal point**, as indicated in Figure 44(c).

Figure 44



(a) Line containing P and Q **(b)** Line segment $\overline{PQ}$ **(c)** Directed line segment $\overrightarrow{PQ}$

The magnitude of the directed line segment $\overrightarrow{PQ}$ is the distance from the point P to the point Q; that is, it is the length of the line segment. The direction of $\overrightarrow{PQ}$ is from P to Q. If a vector $\mathbf{v}$* has the same magnitude and the same direction as the directed line segment $\overrightarrow{PQ}$, we write

$$\mathbf{v} = \overrightarrow{PQ}$$

The vector $\mathbf{v}$ whose magnitude is 0 is called the **zero vector, 0**. The zero vector is assigned no direction.

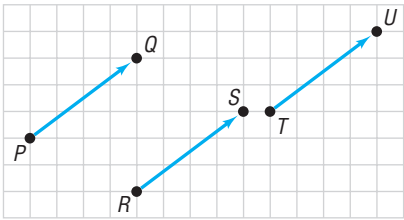
Two vectors $\mathbf{v}$ and $\mathbf{w}$ are **equal**, written

$$\mathbf{v} = \mathbf{w}$$

if they have the same magnitude and the same direction.

For example, the three vectors shown in Figure 45 have the same magnitude and the same direction, so they are equal, even though they have different initial points and different terminal points. As a result, we find it useful to think of a vector simply as an arrow, keeping in mind that two arrows (vectors) are equal if they have the same direction and the same magnitude (length).

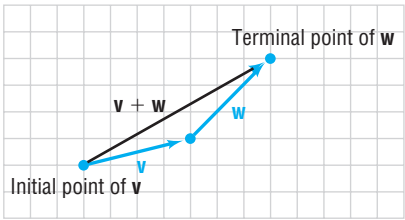
Figure 45



Adding Vectors

The **sum** $\mathbf{v} + \mathbf{w}$ of two vectors is defined as follows: We position the vectors $\mathbf{v}$ and $\mathbf{w}$ so that the terminal point of $\mathbf{v}$ coincides with the initial point of $\mathbf{w}$, as shown in Figure 46. The vector $\mathbf{v} + \mathbf{w}$ is then the unique vector whose initial point coincides with the initial point of $\mathbf{v}$ and whose terminal point coincides with the terminal point of $\mathbf{w}$.

Figure 46

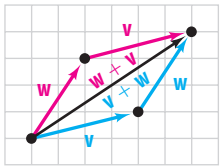


Vector addition is **commutative**. That is, if $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

Figure 47 illustrates this fact. (Observe that the commutative property is another way of saying that opposite sides of a parallelogram are equal and parallel.)

Figure 47

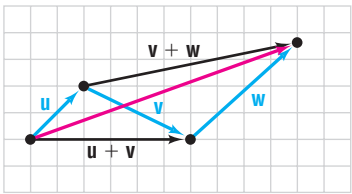


Vector addition is also **associative**. That is, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, then

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

Figure 48 illustrates the associative property for vectors.

Figure 48
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$



The zero vector has the property that

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

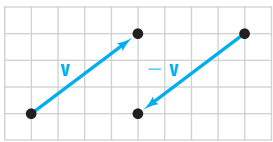
for any vector $\mathbf{v}$.

If $\mathbf{v}$ is a vector, then $-\mathbf{v}$ is the vector having the same magnitude as $\mathbf{v}$, but whose direction is opposite to $\mathbf{v}$, as shown in Figure 49.

Furthermore,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

Figure 49



If $\mathbf{v}$ and $\mathbf{w}$ are two vectors, we define the **difference** $\mathbf{v} - \mathbf{w}$ as

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$

*Boldface letters will be used to denote vectors, to distinguish them from numbers. For handwritten work, an arrow is placed over the letter to signify a vector.

Figure 50

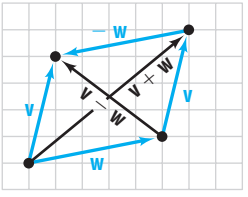


Figure 50 illustrates the relationships among **v**, **w**, **v** + **w**, and **v** − **w**.

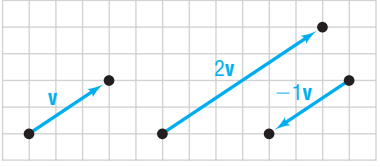
Multiplying Vectors by Numbers

When dealing with vectors, we refer to real numbers as **scalars**. Scalars are quantities that have only magnitude. Examples from physics of scalar quantities are temperature, speed, and time. We now define how to multiply a vector by a scalar.

If α is a scalar and **v** is a vector, the **scalar product** $\alpha\mathbf{v}$ is defined as follows:

1. If $\alpha > 0$, the product $\alpha\mathbf{v}$ is the vector whose magnitude is α times the magnitude of **v** and whose direction is the same as **v**.
2. If $\alpha < 0$, the product $\alpha\mathbf{v}$ is the vector whose magnitude is $|\alpha|$ times the magnitude of **v** and whose direction is opposite that of **v**.
3. If $\alpha = 0$ or if $\mathbf{v} = \mathbf{0}$, then $\alpha\mathbf{v} = \mathbf{0}$.

Figure 51



See Figure 51 for some illustrations.

For example, if **a** is the acceleration of an object of mass m due to a force **F** being exerted on it, then, by Newton's second law of motion, $\mathbf{F} = m\mathbf{a}$. Here, $m\mathbf{a}$ is the product of the scalar m and the vector **a**.

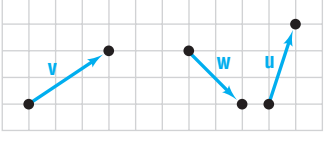
Scalar products have the following properties:

$$0\mathbf{v} = \mathbf{0} \qquad 1\mathbf{v} = \mathbf{v} \qquad -1\mathbf{v} = -\mathbf{v}$$
$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v} \qquad \alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$$
$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$$

1 Graph Vectors

EXAMPLE 1 **Graphing Vectors**

Figure 52

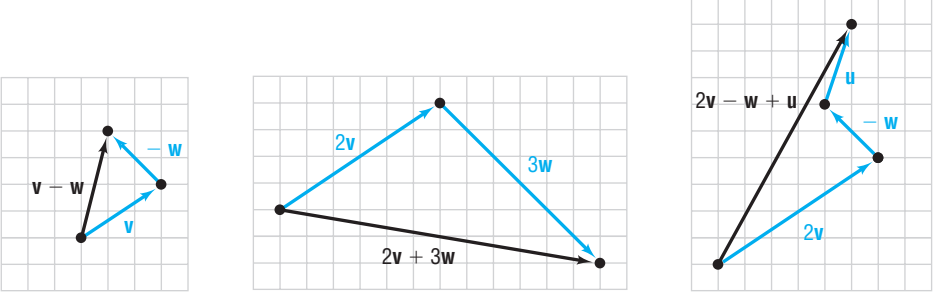


Use the vectors illustrated in Figure 52 to graph each of the following vectors:

(a) $\mathbf{v} - \mathbf{w}$ (b) $2\mathbf{v} + 3\mathbf{w}$ (c) $2\mathbf{v} - \mathbf{w} + \mathbf{u}$

Solution Figure 53 illustrates each graph.

Figure 53



(a) $\mathbf{v} - \mathbf{w}$ (b) $2\mathbf{v} + 3\mathbf{w}$ (c) $2\mathbf{v} - \mathbf{w} + \mathbf{u}$

NOW WORK PROBLEMS 7 AND 9.

Magnitudes of Vectors

If $\mathbf{v}$ is a vector, we use the symbol $\|\mathbf{v}\|$ to represent the **magnitude** of $\mathbf{v}$. Since $\|\mathbf{v}\|$ equals the length of a directed line segment, it follows that $\|\mathbf{v}\|$ has the following properties:

Theorem

Properties of $\|\mathbf{v}\|$

If $\mathbf{v}$ is a vector and if α is a scalar, then

(a) $\|\mathbf{v}\| \geq 0$ (b) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

(c) $\|-\mathbf{v}\| = \|\mathbf{v}\|$ (d) $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$

Property (a) is a consequence of the fact that distance is a nonnegative number. Property (b) follows, because the length of the directed line segment $\overrightarrow{PQ}$ is positive unless P and Q are the same point, in which case the length is 0. Property (c) follows because the length of the line segment $\overline{PQ}$ equals the length of the line segment $\overline{QP}$. Property (d) is a direct consequence of the definition of a scalar product.

A vector $\mathbf{u}$ for which $\|\mathbf{u}\| = 1$ is called a **unit vector**.

2 Find a Position Vector

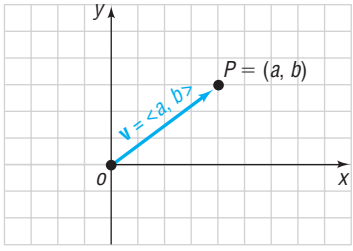
To compute the magnitude and direction of a vector, we need an algebraic way of representing vectors.

An **algebraic vector** $\mathbf{v}$ is represented as

$$\mathbf{v} = \langle a, b \rangle$$

where a and b are real numbers (scalars) called the **components** of the vector $\mathbf{v}$.

Figure 54



We use a rectangular coordinate system to represent algebraic vectors in the plane. If $\mathbf{v} = \langle a, b \rangle$ is an algebraic vector whose initial point is at the origin, then $\mathbf{v}$ is called a **position vector**. See Figure 54. Notice that the terminal point of the position vector $\mathbf{v} = \langle a, b \rangle$ is $P = (a, b)$.

The next result states that any vector whose initial point is not at the origin is equal to a unique position vector.

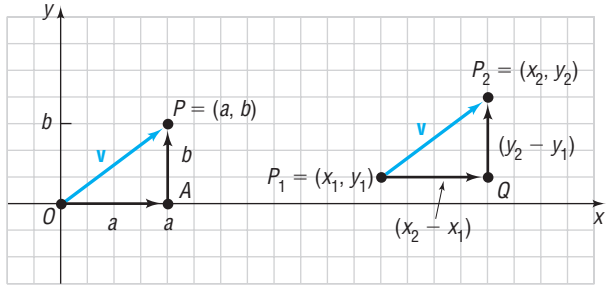
Theorem

Suppose that $\mathbf{v}$ is a vector with initial point $P_1 = (x_1, y_1)$, not necessarily the origin, and terminal point $P_2 = (x_2, y_2)$. If $\mathbf{v} = \overrightarrow{P_1P_2}$, then $\mathbf{v}$ is equal to the position vector

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle \tag{1}$$

To see why this is true, look at Figure 55. Triangle OPA and triangle P_1P_2Q are congruent. [Do you see why? The line segments have the same magnitude, so

Figure 55
$\langle a, b \rangle = \langle x_2 - x_1, y_2 - y_1 \rangle$



$d(O, P) = d(P_1, P_2)$; and they have the same direction, so $\angle POA = \angle P_2P_1Q$. Since the triangles are right triangles, we have angle–side–angle.] It follows that corresponding sides are equal. As a result, $x_2 - x_1 = a$ and $y_2 - y_1 = b$, so $\mathbf{v}$ may be written as

$$\mathbf{v} = \langle a, b\rangle = \langle x_2 - x_1, y_2 - y_1\rangle$$

Because of this result, we can replace any algebraic vector by a unique position vector, and vice versa. This flexibility is one of the main reasons for the wide use of vectors.

EXAMPLE 2 Finding a Position Vector

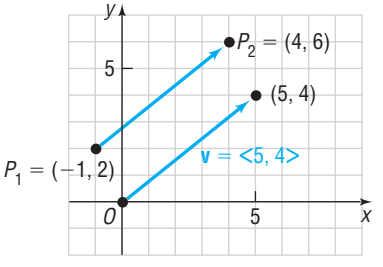
Find the position vector of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ if $P_1 = (-1, 2)$ and $P_2 = (4, 6)$.

Solution By equation (1), the position vector equal to $\mathbf{v}$ is

$$\mathbf{v} = \langle 4 - (-1), 6 - 2\rangle = \langle 5, 4\rangle$$

See Figure 56.

Figure 56



Two position vectors $\mathbf{v}$ and $\mathbf{w}$ are equal if and only if the terminal point of $\mathbf{v}$ is the same as the terminal point of $\mathbf{w}$. This leads to the following result:

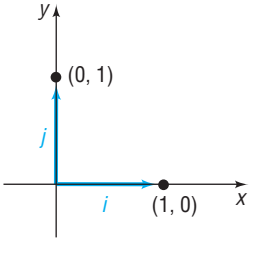
Theorem

Equality of Vectors

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are equal if and only if their corresponding components are equal. That is,

$$\text{If } \mathbf{v} = \langle a_1, b_1\rangle \quad \text{and} \quad \mathbf{w} = \langle a_2, b_2\rangle$$
$$\text{then} \quad \mathbf{v} = \mathbf{w} \quad \text{if and only if} \quad a_1 = a_2 \quad \text{and} \quad b_1 = b_2.$$

Figure 57



We now present an alternative representation of a vector in the plane that is common in the physical sciences. Let $\mathbf{i}$ denote the unit vector whose direction is along the positive x-axis; let $\mathbf{j}$ denote the unit vector whose direction is along the positive y-axis. Then $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$, as shown in Figure 57. Any vector $\mathbf{v} = \langle a, b\rangle$ can be written using the unit vectors $\mathbf{i}$ and $\mathbf{j}$ as follows:

$$\mathbf{v} = \langle a, b\rangle = a\langle 1, 0\rangle + b\langle 0, 1\rangle = a\mathbf{i} + b\mathbf{j}$$

We call a and b the **horizontal** and **vertical components** of $\mathbf{v}$, respectively. For example, if $\mathbf{v} = \langle 5, 4\rangle = 5\mathbf{i} + 4\mathbf{j}$, then 5 is the horizontal component and 4 is the vertical component.

NOW WORK PROBLEM 27.

3 Add and Subtract Vectors

We define addition, subtraction, scalar product, and magnitude in terms of the components of a vector.

In Words
To add two vectors, add corresponding components. To subtract two vectors, subtract corresponding components.

Let $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} = \langle a_1, b_1\rangle$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} = \langle a_2, b_2\rangle$ be two vectors, and let α be a scalar. Then

$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} = \langle a_1 + a_2, b_1 + b_2\rangle \quad \textbf{(2)}$$

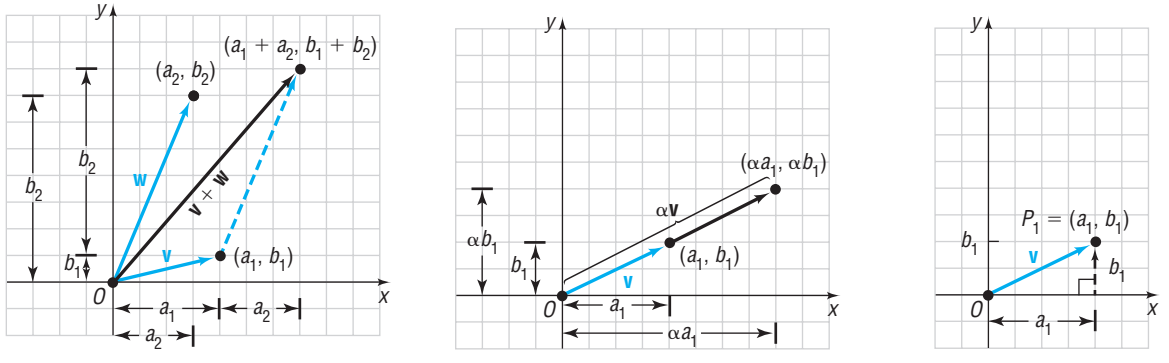
$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} = \langle a_1 - a_2, b_1 - b_2\rangle \quad \textbf{(3)}$$

$$\alpha\mathbf{v} = (\alpha a_1)\mathbf{i} + (\alpha b_1)\mathbf{j} = \langle \alpha a_1, \alpha b_1\rangle \quad \textbf{(4)}$$

$$\|\mathbf{v}\| = \sqrt{a_1^2 + b_1^2} \quad \textbf{(5)}$$

These definitions are compatible with the geometric definitions given earlier in this section. See Figure 58.

Figure 58



(a) Illustration of property (2)

(b) Illustration of property (4), $\alpha > 0$

(c) Illustration of property (5): $\|\mathbf{v}\|$ = Distance from O to P_1, $\|\mathbf{v}\| = \sqrt{a_1^2 + b_1^2}$

EXAMPLE 3 Adding and Subtracting Vectors

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} = \langle 2, 3\rangle$ and $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} = \langle 3, -4\rangle$, find:

(a) $\mathbf{v} + \mathbf{w}$ (b) $\mathbf{v} - \mathbf{w}$

Solution

(a) $\mathbf{v} + \mathbf{w} = (2\mathbf{i} + 3\mathbf{j}) + (3\mathbf{i} - 4\mathbf{j}) = (2 + 3)\mathbf{i} + (3 - 4)\mathbf{j} = 5\mathbf{i} - \mathbf{j}$

or

$\mathbf{v} + \mathbf{w} = \langle 2, 3\rangle + \langle 3, -4\rangle = \langle 2 + 3, 3 + (-4)\rangle = \langle 5, -1\rangle$

(b) $\mathbf{v} - \mathbf{w} = (2\mathbf{i} + 3\mathbf{j}) - (3\mathbf{i} - 4\mathbf{j}) = (2 - 3)\mathbf{i} + [3 - (-4)]\mathbf{j} = -\mathbf{i} + 7\mathbf{j}$

or

$\mathbf{v} - \mathbf{w} = \langle 2, 3\rangle - \langle 3, -4\rangle = \langle 2 - 3, 3 - (-4)\rangle = \langle -1, 7\rangle$ ◀

4 Find a Scalar Product and the Magnitude of a Vector

EXAMPLE 4 **Finding Scalar Products and Magnitudes**

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} = \langle 2, 3\rangle$ and $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} = \langle 3, -4\rangle$, find:

(a) $3\mathbf{v}$ (b) $2\mathbf{v} - 3\mathbf{w}$ (c) $\|\mathbf{v}\|$

Solution (a) $3\mathbf{v} = 3(2\mathbf{i} + 3\mathbf{j}) = 6\mathbf{i} + 9\mathbf{j}$

or

$3\mathbf{v} = 3\langle 2, 3\rangle = \langle 6, 9\rangle$

(b) $2\mathbf{v} - 3\mathbf{w} = 2(2\mathbf{i} + 3\mathbf{j}) - 3(3\mathbf{i} - 4\mathbf{j}) = 4\mathbf{i} + 6\mathbf{j} - 9\mathbf{i} + 12\mathbf{j}$
$= -5\mathbf{i} + 18\mathbf{j}$

or

$2\mathbf{v} - 3\mathbf{w} = 2\langle 2, 3\rangle - 3\langle 3, -4\rangle = \langle 4, 6\rangle - \langle 9, -12\rangle$
$= \langle 4 - 9, 6 - (-12)\rangle = \langle -5, 18\rangle$

(c) $\|\mathbf{v}\| = \|2\mathbf{i} + 3\mathbf{j}\| = \sqrt{2^2 + 3^2} = \sqrt{13}$ ◀

NOW WORK PROBLEMS 33 AND 39.

For the remainder of the section, we will express a vector $\mathbf{v}$ in the form $a\mathbf{i} + b\mathbf{j}$.

5 Find a Unit Vector

Recall that a unit vector $\mathbf{u}$ is a vector for which $\|\mathbf{u}\| = 1$. In many applications, it is useful to be able to find a unit vector $\mathbf{u}$ that has the same direction as a given vector $\mathbf{v}$.

Theorem **Unit Vector in the Direction of v**

For any nonzero vector $\mathbf{v}$, the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

is a unit vector that has the same direction as $\mathbf{v}$.

Proof Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. Then $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ and

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{a\mathbf{i} + b\mathbf{j}}{\sqrt{a^2 + b^2}} = \frac{a}{\sqrt{a^2 + b^2}}\mathbf{i} + \frac{b}{\sqrt{a^2 + b^2}}\mathbf{j}$$

The vector $\mathbf{u}$ is in the same direction as $\mathbf{v}$, since $\|\mathbf{v}\| > 0$. Furthermore,

$$\|\mathbf{u}\| = \sqrt{\frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}} = \sqrt{\frac{a^2 + b^2}{a^2 + b^2}} = 1$$

That is, $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$. ■

As a consequence of this theorem, if **u** is a unit vector in the same direction as a vector **v**, then **v** may be expressed as

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{u} \tag{6}$$

This way of expressing a vector is useful in many applications.

EXAMPLE 5 **Finding a Unit Vector**

Find a unit vector in the same direction as $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

Solution We find $\|\mathbf{v}\|$ first.

$$\|\mathbf{v}\| = \|4\mathbf{i} - 3\mathbf{j}\| = \sqrt{16 + 9} = 5$$

Now we multiply **v** by the scalar $\frac{1}{\|\mathbf{v}\|} = \frac{1}{5}$. A unit vector in the same direction as **v** is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{4\mathbf{i} - 3\mathbf{j}}{5} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$$

✔ **CHECK:** This vector is, in fact, a unit vector because

$$\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

◀

NOW WORK PROBLEM 49.

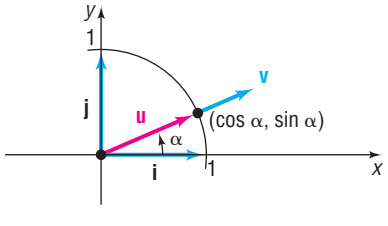
6 Find a Vector from Its Direction and Magnitude

If a vector represents the speed and direction of an object, it is called a **velocity vector**. If a vector represents the direction and amount of a force acting on an object, it is called a **force vector**. In many applications, a vector is described in terms of its magnitude and direction, rather than in terms of its components. For example, a ball thrown with an initial speed of 25 miles per hour at an angle 30° to the horizontal is a velocity vector.

Suppose that we are given the magnitude $\|\mathbf{v}\|$ of a nonzero vector **v** and the angle α, $0° \le \alpha < 360°$, between **v** and **i**. To express **v** in terms of $\|\mathbf{v}\|$ and α, we first find the unit vector **u** having the same direction as **v**.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \quad \text{or} \quad \mathbf{v} = \|\mathbf{v}\|\mathbf{u} \tag{7}$$

Figure 59



Look at Figure 59. The coordinates of the terminal point of **u** are $(\cos\alpha, \sin\alpha)$. Then $\mathbf{u} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$ and, from (7),

$$\mathbf{v} = \|\mathbf{v}\|(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}) \tag{8}$$

where α is the angle between **v** and **i**.

EXAMPLE 6 Writing a Vector When Its Magnitude and Direction Are Given

A ball is thrown with an initial speed of 25 miles per hour in a direction that makes an angle of 30° with the positive x-axis. Express the velocity vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$. What is the initial speed in the horizontal direction? What is the initial speed in the vertical direction?

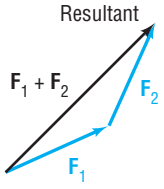
Solution The magnitude of $\mathbf{v}$ is $\|\mathbf{v}\| = 25$ miles per hour, and the angle between the direction of $\mathbf{v}$ and $\mathbf{i}$, the positive x-axis, is $\alpha = 30°$. By equation (8),

$$\mathbf{v} = \|\mathbf{v}\|(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}) = 25(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = 25\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{25\sqrt{3}}{2}\mathbf{i} + \frac{25}{2}\mathbf{j}$$

The initial speed of the ball in the horizontal direction is the horizontal component of $\mathbf{v}$, $\frac{25\sqrt{3}}{2} \approx 21.65$ miles per hour. The initial speed in the vertical direction is the vertical component of $\mathbf{v}$, $\frac{25}{2} = 12.5$ miles per hour. ◀

NOW WORK PROBLEM 61.

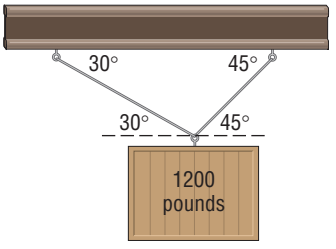
Figure 60



7 Work with Objects in Static Equilibrium

Because forces can be represented by vectors, two forces "combine" the way that vectors "add." If $\mathbf{F}_1$ and $\mathbf{F}_2$ are two forces simultaneously acting on an object, the vector sum $\mathbf{F}_1 + \mathbf{F}_2$ is the **resultant force**. The resultant force produces the same effect on the object as that obtained when the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the object. See Figure 60. An application of this concept is *static equilibrium.* An object is said to be in **static equilibrium** if (1) the object is at rest and (2) the sum of all forces acting on the object is zero, that is, if the resultant force is 0.

Figure 61

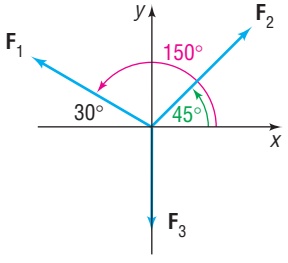


EXAMPLE 7 An Object in Static Equilibrium

A box of supplies that weighs 1200 pounds is suspended by two cables attached to the ceiling, as shown in Figure 61. What is the tension in the two cables?

Solution We draw a force diagram using the vectors shown in Figure 62. The tensions in the cables are the magnitudes $\|\mathbf{F}_1\|$ and $\|\mathbf{F}_2\|$ of the force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$. The magnitude of the force vector $\mathbf{F}_3$ equals 1200 pounds, the weight of the box. Now write each force vector in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. For $\mathbf{F}_1$ and $\mathbf{F}_2$, we use equation (8). Remember that α is the angle between the vector and the positive x-axis.

Figure 62



$$\mathbf{F}_1 = \|\mathbf{F}_1\|(\cos 150°\mathbf{i} + \sin 150°\mathbf{j}) = \|\mathbf{F}_1\|\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = -\frac{\sqrt{3}}{2}\|\mathbf{F}_1\|\mathbf{i} + \frac{1}{2}\|\mathbf{F}_1\|\mathbf{j}$$

$$\mathbf{F}_2 = \|\mathbf{F}_2\|(\cos 45°\mathbf{i} + \sin 45°\mathbf{j}) = \|\mathbf{F}_2\|\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right) = \frac{\sqrt{2}}{2}\|\mathbf{F}_2\|\mathbf{i} + \frac{\sqrt{2}}{2}\|\mathbf{F}_2\|\mathbf{j}$$

$$\mathbf{F}_3 = -1200\mathbf{j}$$

For static equilibrium, the sum of the force vectors must equal zero.

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = -\frac{\sqrt{3}}{2}\|\mathbf{F}_1\|\mathbf{i} + \frac{1}{2}\|\mathbf{F}_1\|\mathbf{j} + \frac{\sqrt{2}}{2}\|\mathbf{F}_2\|\mathbf{i} + \frac{\sqrt{2}}{2}\|\mathbf{F}_2\|\mathbf{j} - 1200\mathbf{j} = \mathbf{0}$$

The **i** component and **j** component will each equal zero. This results in the two equations

$$-\frac{\sqrt{3}}{2}\|\mathbf{F}_1\| + \frac{\sqrt{2}}{2}\|\mathbf{F}_2\| = 0 \tag{9}$$

$$\frac{1}{2}\|\mathbf{F}_1\| + \frac{\sqrt{2}}{2}\|\mathbf{F}_2\| - 1200 = 0 \tag{10}$$

We solve equation (9) for $\|\mathbf{F}_2\|$ and obtain

$$\|\mathbf{F}_2\| = \frac{\sqrt{3}}{\sqrt{2}}\|\mathbf{F}_1\| \tag{11}$$

Substituting into equation (10) and solving for $\|\mathbf{F}_1\|$, we obtain

$$\frac{1}{2}\|\mathbf{F}_1\| + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{\sqrt{2}}\|\mathbf{F}_1\|\right) - 1200 = 0$$

$$\frac{1}{2}\|\mathbf{F}_1\| + \frac{\sqrt{3}}{2}\|\mathbf{F}_1\| - 1200 = 0$$

$$\frac{1+\sqrt{3}}{2}\|\mathbf{F}_1\| = 1200$$

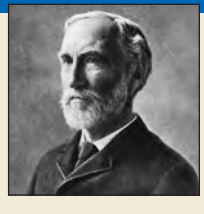
$$\|\mathbf{F}_1\| = \frac{2400}{1+\sqrt{3}} \approx 878.5 \text{ pounds}$$

Substituting this value into equation (11) yields $\|\mathbf{F}_2\|$.

$$\|\mathbf{F}_2\| = \frac{\sqrt{3}}{\sqrt{2}}\|\mathbf{F}_1\| = \frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{2400}{1+\sqrt{3}} \approx 1075.9 \text{ pounds}$$

The left cable has tension of approximately 878.5 pounds and the right cable has tension of approximately 1075.9 pounds. ◀

Historical Feature



Josiah Gibbs (1839–1903)

The history of vectors is surprisingly complicated for such a natural concept. In the *xy*-plane, complex numbers do a good job of imitating vectors. About 1840, mathematicians became interested in finding a system that would do for three dimensions what the complex numbers do for two dimensions. Hermann Grassmann (1809–1877), in Germany, and William Rowan Hamilton (1805–1865), in Ireland, both attempted to find solutions.

Hamilton's system was the *quaternions*, which are best thought of as a real number plus a vector, and do for four dimensions what complex numbers do for two dimensions. In this system the order of multiplication matters; that is, **ab** ≠ **ba.** Also, two products of vectors emerged, the scalar (or dot) product and the vector (or cross) product.

Grassmann's abstract style, although easily read today, was almost impenetrable during the previous century, and only a few of his ideas were appreciated. Among those few were the same scalar and vector products that Hamilton had found.

About 1880, the American physicist Josiah Willard Gibbs (1839–1903) worked out an algebra involving only the simplest concepts: the vectors and the two products. He then added some calculus, and the resulting system was simple, flexible, and well adapted to expressing a large number of physical laws. This system remains in use essentially unchanged. Hamilton's and Grassmann's more extensive systems each gave birth to much interesting mathematics, but little of this mathematics is seen at elementary levels.

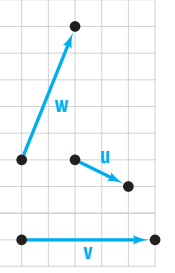
8.4 Assess Your Understanding

Concepts and Vocabulary

1. A vector whose magnitude is 1 is called a(n) _____ vector.
2. The product of a vector by a number is called a(n) _____ product.
3. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then a is called the _____ component of $\mathbf{v}$ and b is the _____ component of $\mathbf{v}$.
4. *True or False:* Vectors are quantities that have magnitude and direction.
5. *True or False:* Force is a physical example of a vector.
6. *True or False:* Mass is a physical example of a vector.

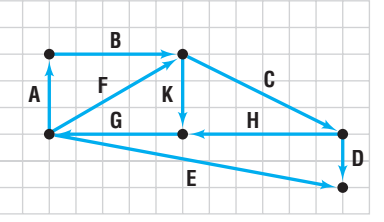
Skill Building

In Problems 7–14, use the vectors in the figure at the right to graph each of the following vectors.



7. $\mathbf{v} + \mathbf{w}$
8. $\mathbf{u} + \mathbf{v}$
9. $3\mathbf{v}$
10. $4\mathbf{w}$
11. $\mathbf{v} - \mathbf{w}$
12. $\mathbf{u} - \mathbf{v}$
13. $3\mathbf{v} + \mathbf{u} - 2\mathbf{w}$
14. $2\mathbf{u} - 3\mathbf{v} + \mathbf{w}$

In Problems 15–22, use the figure at the right. Determine whether the given statement is true or false.



15. $\mathbf{A} + \mathbf{B} = \mathbf{F}$
16. $\mathbf{K} + \mathbf{G} = \mathbf{F}$
17. $\mathbf{C} = \mathbf{D} - \mathbf{E} + \mathbf{F}$
18. $\mathbf{G} + \mathbf{H} + \mathbf{E} = \mathbf{D}$
19. $\mathbf{E} + \mathbf{D} = \mathbf{G} + \mathbf{H}$
20. $\mathbf{H} - \mathbf{C} = \mathbf{G} - \mathbf{F}$
21. $\mathbf{A} + \mathbf{B} + \mathbf{K} + \mathbf{G} = \mathbf{0}$
22. $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{H} + \mathbf{G} = \mathbf{0}$
23. If $\|\mathbf{v}\| = 4$, what is $\|3\mathbf{v}\|$?
24. If $\|\mathbf{v}\| = 2$, what is $\|-4\mathbf{v}\|$?

In Problems 25–32, the vector $\mathbf{v}$ *has initial point* P *and terminal point* Q. *Write* $\mathbf{v}$ *in the form* $a\mathbf{i} + b\mathbf{j}$; *that is, find its position vector.*

25. $P = (0, 0)$; $Q = (3, 4)$
26. $P = (0, 0)$; $Q = (-3, -5)$
27. $P = (3, 2)$; $Q = (5, 6)$
28. $P = (-3, 2)$; $Q = (6, 5)$
29. $P = (-2, -1)$; $Q = (6, -2)$
30. $P = (-1, 4)$; $Q = (6, 2)$
31. $P = (1, 0)$; $Q = (0, 1)$
32. $P = (1, 1)$; $Q = (2, 2)$

In Problems 33–38, find $\|\mathbf{v}\|$.

33. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$
34. $\mathbf{v} = -5\mathbf{i} + 12\mathbf{j}$
35. $\mathbf{v} = \mathbf{i} - \mathbf{j}$
36. $\mathbf{v} = -\mathbf{i} - \mathbf{j}$
37. $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$
38. $\mathbf{v} = 6\mathbf{i} + 2\mathbf{j}$

In Problems 39–44, find each quantity if $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}$ *and* $\mathbf{w} = -2\mathbf{i} + 3\mathbf{j}$.

39. $2\mathbf{v} + 3\mathbf{w}$
40. $3\mathbf{v} - 2\mathbf{w}$
41. $\|\mathbf{v} - \mathbf{w}\|$
42. $\|\mathbf{v} + \mathbf{w}\|$
43. $\|\mathbf{v}\| - \|\mathbf{w}\|$
44. $\|\mathbf{v}\| + \|\mathbf{w}\|$

In Problems 45–50, find the unit vector having the same direction as $\mathbf{v}$.

45. $\mathbf{v} = 5\mathbf{i}$
46. $\mathbf{v} = -3\mathbf{j}$
47. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$
48. $\mathbf{v} = -5\mathbf{i} + 12\mathbf{j}$
49. $\mathbf{v} = \mathbf{i} - \mathbf{j}$
50. $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$

51. Find a vector **v** whose magnitude is 4 and whose component in the **i** direction is twice the component in the **j** direction.

52. Find a vector **v** whose magnitude is 3 and whose component in the **i** direction is equal to the component in the **j** direction.

53. If $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = x\mathbf{i} + 3\mathbf{j}$, find all numbers x for which $\|\mathbf{v} + \mathbf{w}\| = 5$.

54. If $P = (-3, 1)$ and $Q = (x, 4)$, find all numbers x such that the vector represented by $\overrightarrow{PQ}$ has length 5.

In Problems 55–60, write the vector **v** *in the form* $a\mathbf{i} + b\mathbf{j}$, *given its magnitude* $\|\mathbf{v}\|$ *and the angle* α *it makes with the positive x-axis.*

55. $\|\mathbf{v}\| = 5, \quad \alpha = 60°$

56. $\|\mathbf{v}\| = 8, \quad \alpha = 45°$

57. $\|\mathbf{v}\| = 14, \quad \alpha = 120°$

58. $\|\mathbf{v}\| = 3, \quad \alpha = 240°$

59. $\|\mathbf{v}\| = 25, \quad \alpha = 330°$

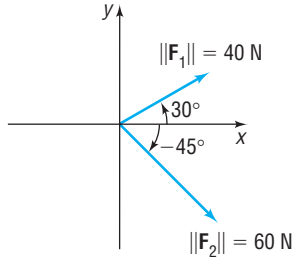
60. $\|\mathbf{v}\| = 15, \quad \alpha = 315°$

Applications and Extensions

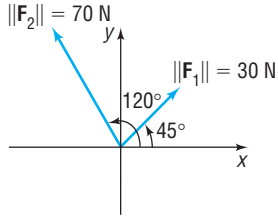
61. A child pulls a wagon with a force of 40 pounds. The handle of the wagon makes an angle of 30° with the ground. Express the force vector **F** in terms of **i** and **j**.

62. A man pushes a wheelbarrow up an incline of 20° with a force of 100 pounds. Express the force vector **F** in terms of **i** and **j**.

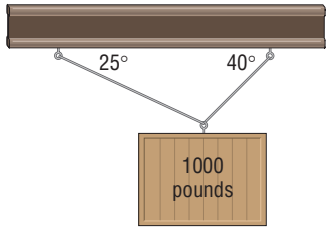
63. Resultant Force Two forces of magnitude 40 newtons (N) and 60 newtons act on an object at angles of 30° and −45° with the positive x-axis as shown in the figure. Find the direction and magnitude of the resultant force; that is, find $\mathbf{F}_1 + \mathbf{F}_2$.



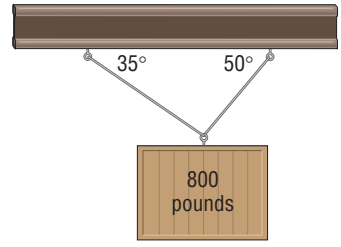
64. Resultant Force Two forces of magnitude 30 newtons (N) and 70 newtons act on an object at angles of 45° and 120° with the positive x-axis as shown in the figure. Find the direction and magnitude of the resultant force; that is, find $\mathbf{F}_1 + \mathbf{F}_2$.



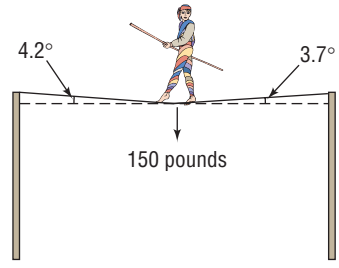
65. Static Equilibrium A weight of 1000 pounds is suspended from two cables as shown in the figure. What is the tension in the two cables?



66. Static Equilibrium A weight of 800 pounds is suspended from two cables as shown in the figure. What is the tension in the two cables?

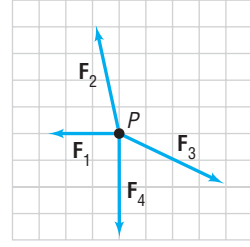


67. Static Equilibrium A tightrope walker located at a certain point deflects the rope as indicated in the figure. If the weight of the tightrope walker is 150 pounds, how much tension is in each part of the rope?



68. Static Equilibrium Repeat Problem 67 if the left angle is 3.8°, the right angle is 2.6°, and the weight of the tightrope walker is 135 pounds.

69. Show on the following graph the force needed for the object at P to be in static equilibrium.



Discussion and Writing

70. Explain in your own words what a vector is. Give an example of a vector.

71. Write a brief paragraph comparing the algebra of complex numbers and the algebra of vectors.

8.5 The Dot Product

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Law of Cosines (Section 7.3, p. 543)

Now work the 'Are You Prepared?' problem on page 626.

OBJECTIVES
1. Find the Dot Product of Two Vectors
2. Find the Angle between Two Vectors
3. Determine Whether Two Vectors Are Parallel
4. Determine Whether Two Vectors Are Orthogonal
5. Decompose a Vector into Two Orthogonal Vectors
6. Compute Work

1 Find the Dot Product of Two Vectors

The definition for a product of two vectors is somewhat unexpected. However, such a product has meaning in many geometric and physical applications.

If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$ are two vectors, the **dot product** $\mathbf{v} \cdot \mathbf{w}$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2 \tag{1}$$

EXAMPLE 1 **Finding Dot Products**

If $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i} + 3\mathbf{j}$, find:

(a) $\mathbf{v} \cdot \mathbf{w}$ (b) $\mathbf{w} \cdot \mathbf{v}$ (c) $\mathbf{v} \cdot \mathbf{v}$
(d) $\mathbf{w} \cdot \mathbf{w}$ (e) $\|\mathbf{v}\|$ (f) $\|\mathbf{w}\|$

Solution

(a) $\mathbf{v} \cdot \mathbf{w} = 2(5) + (-3)3 = 1$ (b) $\mathbf{w} \cdot \mathbf{v} = 5(2) + 3(-3) = 1$
(c) $\mathbf{v} \cdot \mathbf{v} = 2(2) + (-3)(-3) = 13$ (d) $\mathbf{w} \cdot \mathbf{w} = 5(5) + 3(3) = 34$
(e) $\|\mathbf{v}\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$ (f) $\|\mathbf{w}\| = \sqrt{5^2 + 3^2} = \sqrt{34}$ ◀

Since the dot product $\mathbf{v} \cdot \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is a real number (scalar), we sometimes refer to it as the **scalar product**.

Properties

The results obtained in Example 1 suggest some general properties.

Theorem **Properties of the Dot Product**

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, then

Commutative Property

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \tag{2}$$

Distributive Property

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w} \tag{3}$$

$$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2 \tag{4}$$

$$\mathbf{0}\cdot\mathbf{v} = 0 \tag{5}$$

Proof We will prove properties (2) and (4) here and leave properties (3) and (5) as exercises (see Problems 39 and 40).

To prove property (2), we let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$. Then

$$\mathbf{u}\cdot\mathbf{v} = a_1a_2 + b_1b_2 = a_2a_1 + b_2b_1 = \mathbf{v}\cdot\mathbf{u}$$

To prove property (4), we let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. Then

$$\mathbf{v}\cdot\mathbf{v} = a^2 + b^2 = \|\mathbf{v}\|^2$$

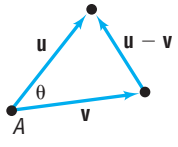
■

2 Find the Angle between Two Vectors

One use of the dot product is to calculate the angle between two vectors. We proceed as follows.

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors with the same initial point A. Then the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ form a triangle. The angle θ at vertex A of the triangle is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. See Figure 63. We wish to find a formula for calculating the angle θ.

Figure 63



The sides of the triangle have lengths $\|\mathbf{v}\|$, $\|\mathbf{u}\|$, and $\|\mathbf{u} - \mathbf{v}\|$, and θ is the included angle between the sides of length $\|\mathbf{v}\|$ and $\|\mathbf{u}\|$. The Law of Cosines (Section 7.3) can be used to find the cosine of the included angle.

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Now we use property (4) to rewrite this equation in terms of dot products.

$$(\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{6}$$

Then we apply the distributive property (3) twice on the left side of (6) to obtain

$$\begin{aligned}(\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u} - \mathbf{v}) - \mathbf{v}\cdot(\mathbf{u} - \mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v} \qquad (7)\end{aligned}$$

↑ Property (2)

Combining equations (6) and (7), we have

$$\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

We have proved the following result:

Theorem

Angle between Vectors

If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors, the angle θ, $0 \le \theta \le \pi$, between $\mathbf{u}$ and $\mathbf{v}$ is determined by the formula

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \tag{8}$$

EXAMPLE 2 Finding the Angle θ between Two Vectors

Find the angle θ between $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Solution We compute the quantities $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, and $\|\mathbf{v}\|$.

$$\mathbf{u} \cdot \mathbf{v} = 4(2) + (-3)(5) = -7$$

$$\|\mathbf{u}\| = \sqrt{4^2 + (-3)^2} = 5$$

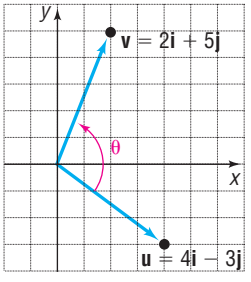
$$\|\mathbf{v}\| = \sqrt{2^2 + 5^2} = \sqrt{29}$$

By formula (8), if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-7}{5\sqrt{29}} \approx -0.26$$

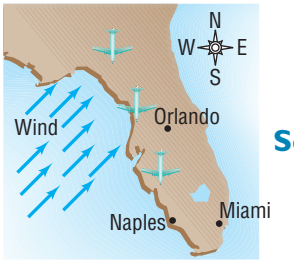
We find that $\theta \approx 105°$. See Figure 64. ◀

Figure 64



NOW WORK PROBLEMS 7(a) AND (b).

EXAMPLE 3 Finding the Actual Speed and Direction of an Aircraft



A Boeing 737 aircraft maintains a constant airspeed of 500 miles per hour in the direction due south. The velocity of the jet stream is 80 miles per hour in a northeasterly direction. Find the actual speed and direction of the aircraft relative to the ground.

Solution We set up a coordinate system in which north (N) is along the positive y-axis. See Figure 65. Let

$$\mathbf{v}_a = \text{velocity of aircraft relative to the air} = -500\mathbf{j}$$
$$\mathbf{v}_w = \text{velocity of jet stream}$$
$$\mathbf{v}_g = \text{velocity of aircraft relative to ground}$$

The velocity of the jet stream $\mathbf{v}_w$ has magnitude 80 and direction NE (northeast), so the angle between $\mathbf{v}_w$ and $\mathbf{i}$ is 45°. We express $\mathbf{v}_w$ in terms of $\mathbf{i}$ and $\mathbf{j}$ as

$$\mathbf{v}_w = 80(\cos 45°\mathbf{i} + \sin 45°\mathbf{j}) = 80\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right) = 40\sqrt{2}(\mathbf{i} + \mathbf{j})$$

The velocity of the aircraft relative to the ground is

$$\mathbf{v}_g = \mathbf{v}_a + \mathbf{v}_w = -500\mathbf{j} + 40\sqrt{2}(\mathbf{i} + \mathbf{j}) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 500\right)\mathbf{j}$$

The actual speed of the aircraft is

$$\|\mathbf{v}_g\| = \sqrt{\left(40\sqrt{2}\right)^2 + \left(40\sqrt{2} - 500\right)^2} \approx 447 \text{ miles per hour}$$

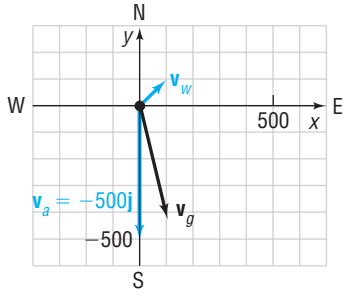
The angle θ between $\mathbf{v}_g$ and the vector $\mathbf{v}_a = -500\mathbf{j}$ (the velocity of the aircraft relative to the air) is determined by the equation

$$\cos\theta = \frac{\mathbf{v}_g \cdot \mathbf{v}_a}{\|\mathbf{v}_g\|\|\mathbf{v}_a\|} \approx \frac{\left(40\sqrt{2} - 500\right)(-500)}{(447)(500)} \approx 0.9920$$

$$\theta \approx 7.3°$$

The direction of the aircraft relative to the ground is approximately S7.3°E (about 7.3° east of south). ◀

Figure 65



NOW WORK PROBLEM 25.

3 Determine Whether Two Vectors Are Parallel

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be **parallel** if there is a nonzero scalar α so that $\mathbf{v} = \alpha\mathbf{w}$. In this case, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is 0 or π.

EXAMPLE 4 Determining Whether Vectors Are Parallel

The vectors $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 2\mathbf{j}$ are parallel, since $\mathbf{v} = \frac{1}{2}\mathbf{w}$. Furthermore, since

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{18 + 2}{\sqrt{10}\sqrt{40}} = \frac{20}{\sqrt{400}} = 1$$

the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is 0. ◀

4 Determine Whether Two Vectors are Orthogonal

If the angle θ between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ is $\frac{\pi}{2}$, the vectors $\mathbf{v}$ and $\mathbf{w}$ are called **orthogonal**.* See Figure 66.

Figure 66
v is orthogonal to w.

Since $\cos\frac{\pi}{2} = 0$, it follows from formula (8) that if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal then $\mathbf{v}\cdot\mathbf{w} = 0$.

On the other hand, if $\mathbf{v}\cdot\mathbf{w} = 0$, then either $\mathbf{v} = 0$ or $\mathbf{w} = 0$ or $\cos\theta = 0$. In the latter case, $\theta = \frac{\pi}{2}$, and $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. If $\mathbf{v}$ or $\mathbf{w}$ is the zero vector, then, since the zero vector has no specific direction, we adopt the convention that the zero vector is orthogonal to every vector.

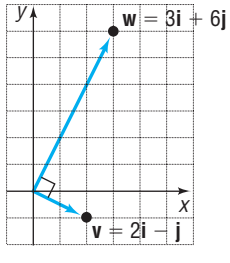
Theorem

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if

$$\boxed{\mathbf{v}\cdot\mathbf{w} = 0}$$

EXAMPLE 5 Determining Whether Two Vectors Are Orthogonal

Figure 67



The vectors

$$\mathbf{v} = 2\mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{w} = 3\mathbf{i} + 6\mathbf{j}$$

are orthogonal, since

$$\mathbf{v}\cdot\mathbf{w} = 6 - 6 = 0$$

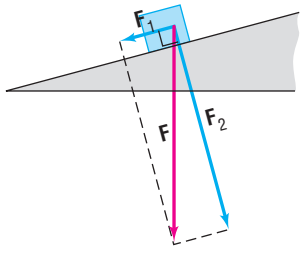
See Figure 67. ◀

NOW WORK PROBLEM 7(c).

Orthogonal, perpendicular, and *normal* are all terms that mean "meet at a right angle." It is customary to refer to two vectors as being *orthogonal*, two lines as being *perpendicular*, and a line and a plane or a vector and a plane as being *normal*.

5 Decompose a Vector into Two Orthogonal Vectors

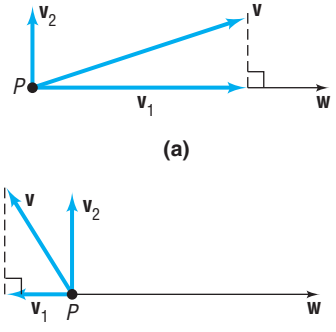
Figure 68



In many physical applications, it is necessary to find "how much" of a vector is applied in a given direction. Look at Figure 68. The force **F** due to gravity is pulling straight down (toward the center of Earth) on the block. To study the effect of gravity on the block, it is necessary to determine how much of **F** is actually pushing the block down the incline ($\mathbf{F}_1$) and how much is pressing the block against the incline ($\mathbf{F}_2$), at a right angle to the incline. Knowing the **decomposition** of **F** often will allow us to determine when friction is overcome and the block will slide down the incline.

Suppose that **v** and **w** are two nonzero vectors with the same initial point P. We seek to decompose **v** into two vectors: $\mathbf{v}_1$, which is parallel to **w**, and $\mathbf{v}_2$, which is orthogonal to **w**. See Figure 69(a) and (b). The vector $\mathbf{v}_1$ is called the **vector projection of v onto w**.

Figure 69



The vector $\mathbf{v}_1$ is obtained as follows: From the terminal point of **v**, drop a perpendicular to the line containing **w**. The vector $\mathbf{v}_1$ is the vector from P to the foot of this perpendicular. The vector $\mathbf{v}_2$ is given by $\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$. Note that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1$ is parallel to **w**, and $\mathbf{v}_2$ is orthogonal to **w**. This is the decomposition of **v** that we wanted.

Now we seek a formula for $\mathbf{v}_1$ that is based on a knowledge of the vectors **v** and **w**. Since $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, we have

$$\mathbf{v} \cdot \mathbf{w} = (\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = \mathbf{v}_1 \cdot \mathbf{w} + \mathbf{v}_2 \cdot \mathbf{w} \tag{9}$$

Since $\mathbf{v}_2$ is orthogonal to **w**, we have $\mathbf{v}_2 \cdot \mathbf{w} = 0$. Since $\mathbf{v}_1$ is parallel to **w**, we have $\mathbf{v}_1 = \alpha\mathbf{w}$ for some scalar α. Equation (9) can be written as

$$\mathbf{v} \cdot \mathbf{w} = \alpha\mathbf{w} \cdot \mathbf{w} = \alpha\|\mathbf{w}\|^2 \qquad \mathbf{v}_1 = \alpha\mathbf{w};\ \mathbf{v}_2 \cdot \mathbf{w} = 0$$

$$\alpha = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}$$

Then

$$\mathbf{v}_1 = \alpha\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$$

Theorem

If **v** and **w** are two nonzero vectors, the vector projection of **v** onto **w** is

$$\boxed{\mathbf{v}_1 = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}} \tag{10}$$

The decomposition of **v** into $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to **w** and $\mathbf{v}_2$ is perpendicular to **w**, is

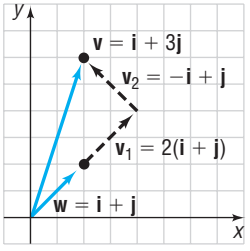
$$\boxed{\mathbf{v}_1 = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} \qquad \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1} \tag{11}$$

EXAMPLE 6 Decomposing a Vector into Two Orthogonal Vectors

Find the vector projection of $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ onto $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Decompose **v** into two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to **w** and $\mathbf{v}_2$ is orthogonal to **w**.

Solution We use formulas (10) and (11).

Figure 70



$$\mathbf{v}_1 = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{1+3}{(\sqrt{2})^2}\mathbf{w} = 2\mathbf{w} = 2(\mathbf{i}+\mathbf{j})$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (\mathbf{i}+3\mathbf{j}) - 2(\mathbf{i}+\mathbf{j}) = -\mathbf{i}+\mathbf{j}$$

See Figure 70. ◀

NOW WORK PROBLEM 19.

6 Compute Work

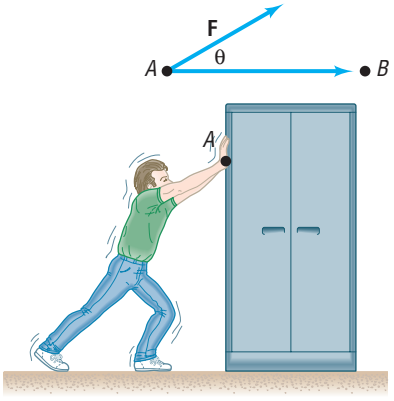
In elementary physics, the **work** W done by a constant force $\mathbf{F}$ in moving an object from a point A to a point B is defined as

$$W = (\text{magnitude of force})(\text{distance}) = \|\mathbf{F}\|\|\overrightarrow{AB}\|$$

Work is commonly measured in foot-pounds or in newton-meters (joules).

In this definition, it is assumed that the force $\mathbf{F}$ is applied along the line of motion. If the constant force $\mathbf{F}$ is not along the line of motion, but, instead, is at an angle θ to the direction of motion, as illustrated in Figure 71, then the **work** W **done by** $\mathbf{F}$ in moving an object from A to B is defined as

Figure 71



$$W = \mathbf{F}\cdot\overrightarrow{AB} \tag{12}$$

This definition is compatible with the force times distance definition given above, since

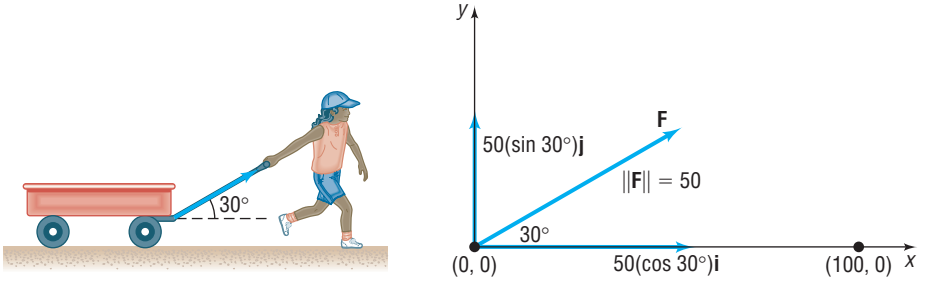
$$W = (\text{amount of force in the direction of } \overrightarrow{AB})(\text{distance})$$

$$= \|\text{projection of } \mathbf{F} \text{ on } AB\|\|\overrightarrow{AB}\| = \frac{\mathbf{F}\cdot\overrightarrow{AB}}{\|\overrightarrow{AB}\|^2}\|\overrightarrow{AB}\|\|\overrightarrow{AB}\| = \mathbf{F}\cdot\overrightarrow{AB}$$

EXAMPLE 7 Computing Work

Figure 72(a) shows a girl pulling a wagon with a force of 50 pounds. How much work is done in moving the wagon 100 feet if the handle makes an angle of 30° with the ground?

Figure 72



(a) (b)

Solution We position the vectors in a coordinate system in such a way that the wagon is moved from $(0, 0)$ to $(100, 0)$. The motion is from $A = (0, 0)$ to $B = (100, 0)$, so $\overrightarrow{AB} = 100\mathbf{i}$. The force vector $\mathbf{F}$, as shown in Figure 72(b), is

$$\mathbf{F} = 50(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = 50\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 25\left(\sqrt{3}\mathbf{i} + \mathbf{j}\right)$$

By formula (12), the work done is

$$W = \mathbf{F} \cdot \overrightarrow{AB} = 25\left(\sqrt{3}\mathbf{i} + \mathbf{j}\right) \cdot 100\mathbf{i} = 2500\sqrt{3} \text{ foot-pounds}$$ ◀

NOW WORK PROBLEM 35.

Historical Feature

1. We stated in an earlier Historical Feature that complex numbers were used as vectors in the plane before the general notion of a vector was clarified. Suppose that we make the correspondence

Vector ↔ Complex number

$$a\mathbf{i} + b\mathbf{j} \leftrightarrow a + bi$$
$$c\mathbf{i} + d\mathbf{j} \leftrightarrow c + di$$

Show that

$$(a\mathbf{i} + b\mathbf{j}) \cdot (c\mathbf{i} + d\mathbf{j}) = \text{real part}[\overline{(a + bi)}(c + di)]$$

This is how the dot product was found originally. The imaginary part is also interesting. It is a determinant (see Section 10.3) and represents the area of the parallelogram whose edges are the vectors. This is close to some of Hermann Grassmann's ideas and is also connected with the scalar triple product of three-dimensional vectors.

8.5 Assess Your Understanding

'Are You Prepared?'

Answer is given at the end of these exercises. If you get the wrong answer, read the page listed in red.

1. In a triangle with sides a, b, c and angles α, β, γ, the Law of Cosines states that _____. (p. 543)

Concepts and Vocabulary

2. If $\mathbf{v} \cdot \mathbf{w} = 0$, then the two vectors $\mathbf{v}$ and $\mathbf{w}$ are _____.
3. If $\mathbf{v} = 3\mathbf{w}$, then the two vectors $\mathbf{v}$ and $\mathbf{w}$ are _____.
4. *True or False:* If $\mathbf{v}$ and $\mathbf{w}$ are parallel vectors, then $\mathbf{v} \cdot \mathbf{w} = 0$.
5. *True or False:* Given two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, it is always possible to decompose $\mathbf{v}$ into two vectors, one parallel to $\mathbf{w}$ and the other perpendicular to $\mathbf{w}$.
6. *True or False:* Work is a physical example of a vector.

Skill Building

In Problems 7–16, (a) find the dot product $\mathbf{v} \cdot \mathbf{w}$; (b) find the angle between $\mathbf{v}$ and $\mathbf{w}$; (c) state whether the vectors are parallel, orthogonal, or neither.

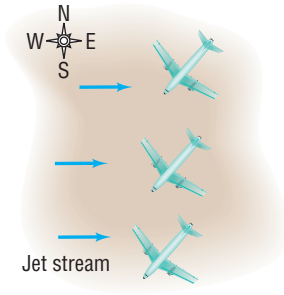
7. $\mathbf{v} = \mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + \mathbf{j}$
8. $\mathbf{v} = \mathbf{i} + \mathbf{j}, \quad \mathbf{w} = -\mathbf{i} + \mathbf{j}$
9. $\mathbf{v} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = \mathbf{i} - 2\mathbf{j}$
10. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = \mathbf{i} + 2\mathbf{j}$
11. $\mathbf{v} = \sqrt{3}\mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + \mathbf{j}$
12. $\mathbf{v} = \mathbf{i} + \sqrt{3}\mathbf{j}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j}$
13. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} + 3\mathbf{j}$
14. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} - 3\mathbf{j}$
15. $\mathbf{v} = 4\mathbf{i}, \quad \mathbf{w} = \mathbf{j}$
16. $\mathbf{v} = \mathbf{i}, \quad \mathbf{w} = -3\mathbf{j}$
17. Find a so that the vectors $\mathbf{v} = \mathbf{i} - a\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} + 3\mathbf{j}$ are orthogonal.
18. Find b so that the vectors $\mathbf{v} = \mathbf{i} + \mathbf{j}$ and $\mathbf{w} = i + b\mathbf{j}$ are orthogonal.

In Problems 19–24, decompose $\mathbf{v}$ into two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

19. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j}$
20. $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = 2\mathbf{i} + \mathbf{j}$
21. $\mathbf{v} = \mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} - 2\mathbf{j}$
22. $\mathbf{v} = 2\mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} - 2\mathbf{j}$
23. $\mathbf{v} = 3\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = -2\mathbf{i} - \mathbf{j}$
24. $\mathbf{v} = \mathbf{i} - 3\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} - \mathbf{j}$

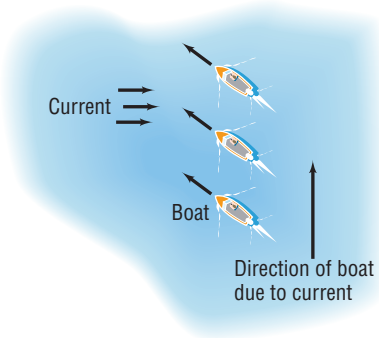
Applications and Extensions

25. **Finding the Actual Speed and Direction of an Aircraft** A Boeing 747 jumbo jet maintains an airspeed of 550 miles per hour in a southwesterly direction. The velocity of the jet stream is a constant 80 miles per hour from the west. Find the actual speed and direction of the aircraft.



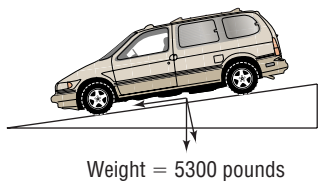
26. **Finding the Correct Compass Heading** The pilot of an aircraft wishes to head directly east, but is faced with a wind speed of 40 miles per hour from the northwest. If the pilot maintains an airspeed of 250 miles per hour, what compass heading should be maintained? What is the actual speed of the aircraft?

27. **Correct Direction for Crossing a River** A river has a constant current of 3 kilometers per hour. At what angle to a boat dock should a motorboat, capable of maintaining a constant speed of 20 kilometers per hour, be headed in order to reach a point directly opposite the dock? If the river is $\frac{1}{2}$ kilometer wide, how long will it take to cross?



28. **Correct Direction for Crossing a River** Repeat Problem 27 if the current is 5 kilometers per hour.

29. **Braking Load** A Toyota Sienna with a gross weight of 5300 pounds is parked on a street with a slope of 8°. See the figure. Find the force required to keep the Sienna from rolling down the hill. What is the force perpendicular to the hill?



30. **Braking Load** A Pontiac Bonneville with a gross weight of 4500 pounds is parked on a street with a slope of 10°. Find the force required to keep the Bonneville from rolling down the hill. What is the force perpendicular to the hill?

31. **Ground Speed and Direction of an Airplane** An airplane has an airspeed of 500 kilometers per hour bearing N45°E. The wind velocity is 60 kilometers per hour in the direction N30°W. Find the resultant vector representing the path of the plane relative to the ground. What is the ground speed of the plane? What is its direction?

32. **Ground Speed and Direction of an Airplane** An airplane has an airspeed of 600 kilometers per hour bearing S30°E. The wind velocity is 40 kilometers per hour in the direction S45°E. Find the resultant vector representing the path of the plane relative to the ground. What is the ground speed of the plane? What is its direction?

33. **Crossing a River** A small motorboat in still water maintains a speed of 20 miles per hour. In heading directly across a river (that is, perpendicular to the current) whose current is 3 miles per hour, find a vector representing the speed and direction of the motorboat. What is the true speed of the motorboat? What is its direction?

34. **Crossing a River** A small motorboat in still water maintains a speed of 10 miles per hour. In heading directly across a river (that is, perpendicular to the current) whose current is 4 miles per hour, find a vector representing the speed and direction of the motorboat. What is the true speed of the motorboat? What is its direction?

35. **Computing Work** Find the work done by a force of 3 pounds acting in the direction 60° to the horizontal in moving an object 2 feet from $(0, 0)$ to $(2, 0)$.

36. **Computing Work** Find the work done by a force of 1 pound acting in the direction 45° to the horizontal in moving an object 5 feet from $(0, 0)$ to $(5, 0)$.

37. **Computing Work** A wagon is pulled horizontally by exerting a force of 20 pounds on the handle at an angle of 30° with the horizontal. How much work is done in moving the wagon 100 feet?

38. Find the acute angle that a constant unit force vector makes with the positive x-axis if the work done by the force in moving a particle from $(0, 0)$ to $(4, 0)$ equals 2.

39. Prove the distributive property:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

40. Prove property (5), $\mathbf{0} \cdot \mathbf{v} = 0$.

41. If $\mathbf{v}$ is a unit vector and the angle between $\mathbf{v}$ and $\mathbf{i}$ is α, show that $\mathbf{v} = \cos \alpha \mathbf{i} + \sin \alpha \mathbf{j}$.

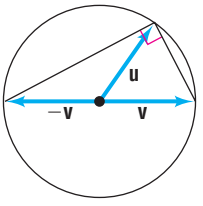
42. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are unit vectors. If the angle between $\mathbf{v}$ and $\mathbf{i}$ is α and that between $\mathbf{w}$ and $\mathbf{i}$ is β, use the idea of the dot product $\mathbf{v} \cdot \mathbf{w}$ to prove that

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

43. Show that the projection of $\mathbf{v}$ onto $\mathbf{i}$ is $(\mathbf{v} \cdot \mathbf{i})\mathbf{i}$. In fact, show that we can always write a vector $\mathbf{v}$ as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j}$$

44. (a) If $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude, show that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal.
(b) Use this to prove that an angle inscribed in a semicircle is a right angle (see the figure).



45. Let $\mathbf{v}$ and $\mathbf{w}$ denote two nonzero vectors. Show that the vector $\mathbf{v} - \alpha\mathbf{w}$ is orthogonal to $\mathbf{w}$ if $\alpha = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}$.

46. Let $\mathbf{v}$ and $\mathbf{w}$ denote two nonzero vectors. Show that the vectors $\|\mathbf{w}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{w}$ and $\|\mathbf{w}\|\mathbf{v} - \|\mathbf{v}\|\mathbf{w}$ are orthogonal.

47. In the definition of work given in this section, what is the work done if $\mathbf{F}$ is orthogonal to $\overrightarrow{AB}$?

48. Prove the **polarization identity**,

$$\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4(\mathbf{u} \cdot \mathbf{v})$$

Discussion and Writing

49. Create an application different from any found in the text that requires the dot product.

'Are You Prepared?' Answer

1. $c^2 = a^2 + b^2 - 2ab \cos \gamma$

8.6 Vectors in Space

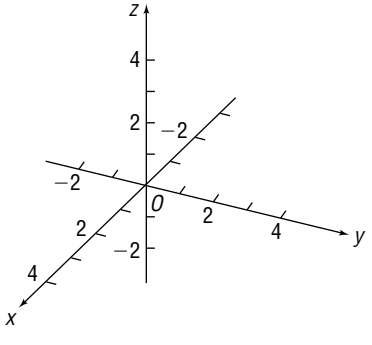
PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Distance Formula (Section 1.1, p. 5)

Now work the 'Are You Prepared?' problem on page 637.

OBJECTIVES
1. Find the Distance between Two Points in Space
2. Find Position Vectors in Space
3. Perform Operations on Vectors
4. Find the Dot Product
5. Find the Angle between Two Vectors
6. Find the Direction Angles of a Vector

Figure 73



Rectangular Coordinates in Space

In the plane, each point is associated with an ordered pair of real numbers. In space, each point is associated with an ordered triple of real numbers. Through a fixed point, called the **origin** O, draw three mutually perpendicular lines, the x-axis, the y-axis, and the z-axis. On each of these axes, select an appropriate scale and the positive direction. See Figure 73.

The direction chosen for the positive z-axis in Figure 73 makes the system *right-handed*. This conforms to the *right-hand rule*, which states that if the index finger of the right hand points in the direction of the positive x-axis and the middle finger points in the direction of the positive y-axis then the thumb will point in the direction of the positive z-axis. See Figure 74.

Figure 74

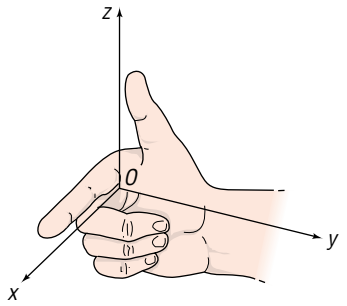
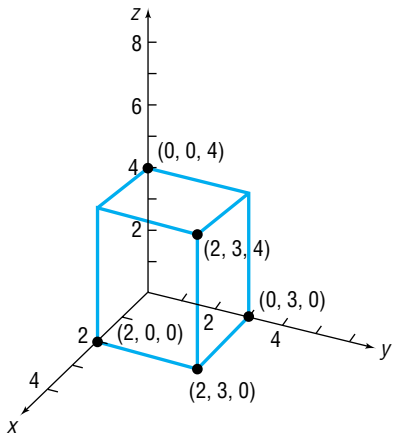


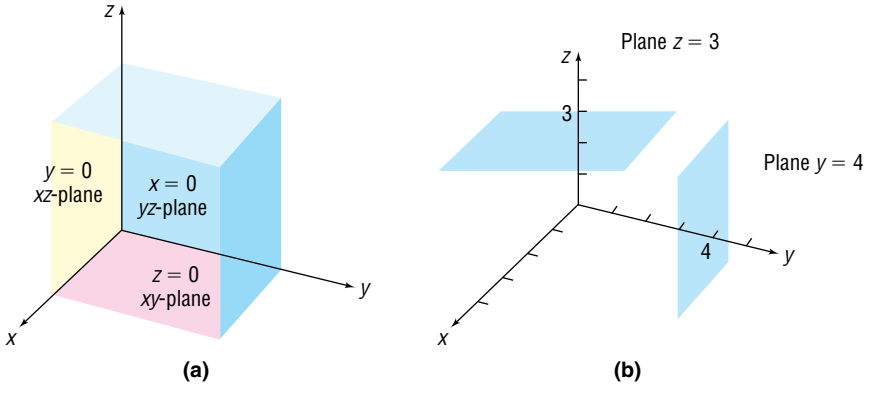
Figure 75



We associate with each point P an ordered triple (x, y, z) of real numbers, the **coordinates of *P*.** For example, the point $(2, 3, 4)$ is located by starting at the origin and moving 2 units along the positive x-axis, 3 units in the direction of the positive y-axis, and 4 units in the direction of the positive z-axis. See Figure 75.

Figure 75 also shows the location of the points $(2, 0, 0)$, $(0, 3, 0)$, $(0, 0, 4)$, $(2, 3, 0)$, and $(2, 3, 4)$. Points of the form $(x, 0, 0)$ lie on the x-axis, while points of the form $(0, y, 0)$ and $(0, 0, z)$ lie on the y-axis and z-axis, respectively. Points of the form $(x, y, 0)$ lie in a plane, called the ***xy*-plane.** Its equation is $z = 0$. Similarly, points of the form $(x, 0, z)$ lie in the ***xz*-plane** (equation $y = 0$) and points of the form $(0, y, z)$ lie in the ***yz*-plane** (equation $x = 0$). See Figure 76(a). By extension of these ideas, all points obeying the equation $z = 3$ will lie in a plane parallel to and 3 units above the xy-plane. The equation $y = 4$ represents a plane parallel to the xz-plane and 4 units to the right of the plane $y = 0$. See Figure 76(b).

Figure 76



NOW WORK PROBLEM 9.

1 Find the Distance between Two Points in Space

The formula for the distance between two points in space is an extension of the Distance Formula for points in the plane given in Chapter 1.

Theorem

Distance Formula in Space

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points in space, the distance d from P_1 to P_2 is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \textbf{(1)}$$

The proof, which we omit, utilizes a double application of the Pythagorean Theorem.

EXAMPLE 1 **Using the Distance Formula**

Find the distance from $P_1 = (-1, 3, 2)$ to $P_2 = (4, -2, 5)$.

Solution $d = \sqrt{[4 - (-1)]^2 + [-2 - 3]^2 + [5 - 2]^2} = \sqrt{25 + 25 + 9} = \sqrt{59}$ ◀

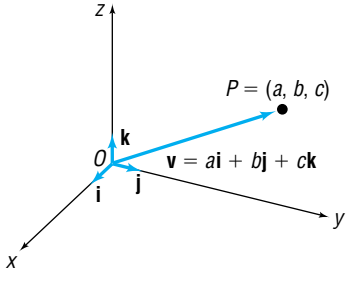
NOW WORK PROBLEM 15.

2 Find Position Vectors in Space

To represent vectors in space, we introduce the unit vectors **i, j,** and **k** whose directions are along the positive x-axis, positive y-axis, and positive z-axis, respectively. If **v** is a vector with initial point at the origin O and terminal point at $P = (a, b, c)$, then we can represent **v** in terms of the vectors **i, j,** and **k** as

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Figure 77



See Figure 77.

The scalars a, b, and c are called the **components** of the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, with a being the component in the direction **i**, b the component in the direction **j**, and c the component in the direction **k**.

A vector whose initial point is at the origin is called a **position vector.** The next result states that any vector whose initial point is not at the origin is equal to a unique position vector.

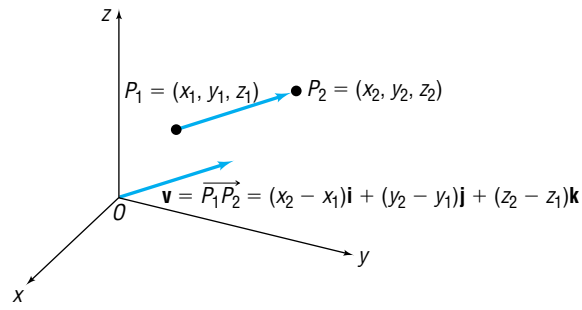
Theorem

Suppose that **v** is a vector with initial point $P_1 = (x_1, y_1, z_1)$, not necessarily the origin, and terminal point $P_2 = (x_2, y_2, z_2)$. If $\mathbf{v} = \overrightarrow{P_1P_2}$, then **v** is equal to the position vector

$$\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k} \qquad \textbf{(2)}$$

Figure 78 illustrates this result.

Figure 78



EXAMPLE 2 Finding a Position Vector

Find the position vector of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ if $P_1 = (-1, 2, 3)$ and $P_2 = (4, 6, 2)$.

Solution By equation (2), the position vector equal to $\mathbf{v}$ is

$$\mathbf{v} = [4 - (-1)]\mathbf{i} + (6 - 2)\mathbf{j} + (2 - 3)\mathbf{k} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$ ◀

NOW WORK PROBLEM 29.

3 Perform Operations on Vectors

Next, we define equality, addition, subtraction, scalar product, and magnitude in terms of the components of a vector.

Let $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ be two vectors, and let α be a scalar. Then

$$\begin{aligned}
\mathbf{v} &= \mathbf{w} \quad \text{if and only if } a_1 = a_2, b_1 = b_2, \text{ and } c_1 = c_2 \\
\mathbf{v} + \mathbf{w} &= (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k} \\
\mathbf{v} - \mathbf{w} &= (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} + (c_1 - c_2)\mathbf{k} \\
\alpha\mathbf{v} &= (\alpha a_1)\mathbf{i} + (\alpha b_1)\mathbf{j} + (\alpha c_1)\mathbf{k} \\
\|\mathbf{v}\| &= \sqrt{a_1^2 + b_1^2 + c_1^2}
\end{aligned}$$

These definitions are compatible with the geometric ones given earlier in Section 8.4.

EXAMPLE 3 Adding and Subtracting Vectors

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, find:

(a) $\mathbf{v} + \mathbf{w}$ (b) $\mathbf{v} - \mathbf{w}$

Solution (a)
$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + (3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) \\
&= (2 + 3)\mathbf{i} + (3 - 4)\mathbf{j} + (-2 + 5)\mathbf{k} \\
&= 5\mathbf{i} - \mathbf{j} + 3\mathbf{k}
\end{aligned}$$

(b)
$$\begin{aligned}
\mathbf{v} - \mathbf{w} &= (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - (3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) \\
&= (2 - 3)\mathbf{i} + [3 - (-4)]\mathbf{j} + [-2 - 5]\mathbf{k} \\
&= -\mathbf{i} + 7\mathbf{j} - 7\mathbf{k}
\end{aligned}$$ ◀

EXAMPLE 4 **Finding Scalar Products and Magnitudes**

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, find:

(a) $3\mathbf{v}$ (b) $2\mathbf{v} - 3\mathbf{w}$ (c) $\|\mathbf{v}\|$

Solution

(a) $3\mathbf{v} = 3(2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 6\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$

(b) $2\mathbf{v} - 3\mathbf{w} = 2(2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - 3(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$
$= 4\mathbf{i} + 6\mathbf{j} - 4\mathbf{k} - 9\mathbf{i} + 12\mathbf{j} - 15\mathbf{k} = -5\mathbf{i} + 18\mathbf{j} - 19\mathbf{k}$

(c) $\|\mathbf{v}\| = \|2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}\| = \sqrt{2^2 + 3^2 + (-2)^2} = \sqrt{17}$ ◀

NOW WORK PROBLEMS 33 AND 39.

Recall that a unit vector $\mathbf{u}$ is one for which $\|\mathbf{u}\| = 1$. In many applications, it is useful to be able to find a unit vector $\mathbf{u}$ that has the same direction as a given vector $\mathbf{v}$.

Theorem **Unit Vector in the Direction of v**

For any nonzero vector $\mathbf{v}$, the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

is a unit vector that has the same direction as $\mathbf{v}$.

As a consequence of this theorem, if $\mathbf{u}$ is a unit vector in the same direction as a vector $\mathbf{v}$, then $\mathbf{v}$ may be expressed as

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$$

This way of expressing a vector is useful in many applications.

EXAMPLE 5 **Finding a Unit Vector**

Find the unit vector in the same direction as $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$.

Solution We find $\|\mathbf{v}\|$ first.

$$\|\mathbf{v}\| = \|2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}\| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

Now we multiply $\mathbf{v}$ by the scalar $\frac{1}{\|\mathbf{v}\|} = \frac{1}{7}$. The result is the unit vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}}{7} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$$ ◀

NOW WORK PROBLEM 47.

4 Find the Dot Product

The definition of *dot product* is an extension of the definition given for vectors in the plane.

If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two vectors, the **dot product** $\mathbf{v} \cdot \mathbf{w}$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2 + c_1c_2 \qquad (3)$$

EXAMPLE 6 **Finding Dot Products**

If $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{w} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, find:

(a) $\mathbf{v} \cdot \mathbf{w}$ (b) $\mathbf{w} \cdot \mathbf{v}$ (c) $\mathbf{v} \cdot \mathbf{v}$
(d) $\mathbf{w} \cdot \mathbf{w}$ (e) $\|\mathbf{v}\|$ (f) $\|\mathbf{w}\|$

Solution

(a) $\mathbf{v} \cdot \mathbf{w} = 2(5) + (-3)3 + 6(-1) = -5$

(b) $\mathbf{w} \cdot \mathbf{v} = 5(2) + 3(-3) + (-1)(6) = -5$

(c) $\mathbf{v} \cdot \mathbf{v} = 2(2) + (-3)(-3) + 6(6) = 49$

(d) $\mathbf{w} \cdot \mathbf{w} = 5(5) + 3(3) + (-1)(-1) = 35$

(e) $\|\mathbf{v}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7$

(f) $\|\mathbf{w}\| = \sqrt{5^2 + 3^2 + (-1)^2} = \sqrt{35}$ ◀

The dot product in space has the same properties as the dot product in the plane.

Theorem

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, then

Commutative Property

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

Distributive Property

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$
$$\mathbf{0} \cdot \mathbf{v} = 0$$

5 Find the Angle Between Two Vectors

The angle θ between two vectors in space follows the same formula as for two vectors in the plane.

Theorem

Angle between Vectors

If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors, the angle θ, $0 \le \theta \le \pi$, between $\mathbf{u}$ and $\mathbf{v}$ is determined by the formula

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \qquad \textbf{(4)}$$

EXAMPLE 7 **Finding the Angle θ between Two Vectors**

Find the angle θ between $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$.

Solution We compute the quantities $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, and $\|\mathbf{v}\|$.

$$\mathbf{u} \cdot \mathbf{v} = 2(2) + (-3)(5) + 6(-1) = -17$$
$$\|\mathbf{u}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7$$
$$\|\mathbf{v}\| = \sqrt{2^2 + 5^2 + (-1)^2} = \sqrt{30}$$

By formula (4), if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-17}{7\sqrt{30}} \approx -0.443$$

We find that $\theta \approx 116.3°$.

NOW WORK PROBLEM 51.

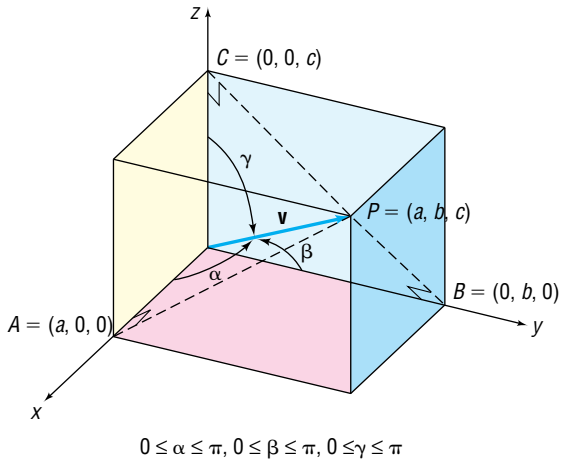
6 Find the Direction Angles of a Vector

A nonzero vector $\mathbf{v}$ in space can be described by specifying its magnitude and its three **direction angles** α, β, and γ. These direction angles are defined as

$$\alpha = \text{angle between } \mathbf{v} \text{ and } \mathbf{i}\text{, the positive } x\text{-axis, } 0 \le \alpha \le \pi$$
$$\beta = \text{angle between } \mathbf{v} \text{ and } \mathbf{j}\text{, the positive } y\text{-axis, } 0 \le \beta \le \pi$$
$$\gamma = \text{angle between } \mathbf{v} \text{ and } \mathbf{k}\text{, the positive } z\text{-axis, } 0 \le \gamma \le \pi$$

See Figure 79.

Figure 79



Our first goal is to find expressions for α, β, and γ in terms of the components of a vector. Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ denote a nonzero vector. The angle α between $\mathbf{v}$ and $\mathbf{i}$, the positive x-axis, obeys

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\,\|\mathbf{i}\|} = \frac{a}{\|\mathbf{v}\|}$$

Similarly,

$$\cos\beta = \frac{b}{\|\mathbf{v}\|} \qquad \cos\gamma = \frac{c}{\|\mathbf{v}\|}$$

Since $\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$, we have the following result:

Theorem

Direction Angles

If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a nonzero vector in space, the direction angles α, β, and γ obey

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{a}{\|\mathbf{v}\|} \qquad \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{b}{\|\mathbf{v}\|}$$
$$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{c}{\|\mathbf{v}\|} \tag{5}$$

The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector $\mathbf{v}$. They play the same role in space as slope does in the plane.

EXAMPLE 8 Finding the Direction Angles of a Vector

Find the direction angles of $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

Solution

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + (-6)^2} = \sqrt{49} = 7$$

Using the formulas in equation (5), we have

$$\cos\alpha = \frac{-3}{7} \qquad \cos\beta = \frac{2}{7} \qquad \cos\gamma = \frac{-6}{7}$$
$$\alpha \approx 115.4^\circ \qquad \beta \approx 73.4^\circ \qquad \gamma \approx 149.0^\circ$$

◀

Theorem

Property of the Direction Cosines

If α, β, and γ are the direction angles of a nonzero vector $\mathbf{v}$ in space, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{6}$$

The proof is a direct consequence of the equations in (5).

Based on equation (6), when two direction cosines are known, the third is determined up to its sign. Knowing two direction cosines is not sufficient to uniquely determine the direction of a vector in space.

EXAMPLE 9 Finding the Direction Angle of a Vector

The vector $\mathbf{v}$ makes an angle of $\alpha = \frac{\pi}{3}$ with the positive x-axis, an angle of $\beta = \frac{\pi}{3}$ with the positive y-axis, and an acute angle γ with the positive z-axis. Find γ.

Solution By equation (6), we have

$$\cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) + \cos^2\gamma = 1 \qquad 0 < \gamma < \frac{\pi}{2}$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$$

$$\cos^2\gamma = \frac{1}{2}$$

$$\cos\gamma = \frac{\sqrt{2}}{2} \quad \text{or} \quad \cos\gamma = -\frac{\sqrt{2}}{2}$$

$$\gamma = \frac{\pi}{4} \quad \text{or} \quad \gamma = \frac{3\pi}{4}$$

Since we are requiring that γ be acute, the answer is $\gamma = \frac{\pi}{4}$. ◀

The direction cosines of a vector give information about only the direction of the vector; they provide no information about its magnitude. For example, *any* vector parallel to the xy-plane and making an angle of $\frac{\pi}{4}$ radian with the positive x-axis and y-axis has direction cosines

$$\cos\alpha = \frac{\sqrt{2}}{2} \qquad \cos\beta = \frac{\sqrt{2}}{2} \qquad \cos\gamma = 0$$

However, if the direction angles *and* the magnitude of a vector are known, then the vector is uniquely determined.

EXAMPLE 10 Writing a Vector in Terms of Its Magnitude and Direction Cosines

Show that any nonzero vector $\mathbf{v}$ in space can be written in terms of its magnitude and direction cosines as

$$\mathbf{v} = \|\mathbf{v}\|[(\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}] \tag{7}$$

Solution Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. From the equations in (5), we see that

$$a = \|\mathbf{v}\|\cos\alpha \qquad b = \|\mathbf{v}\|\cos\beta \qquad c = \|\mathbf{v}\|\cos\gamma$$

Substituting, we find that

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \|\mathbf{v}\|(\cos\alpha)\mathbf{i} + \|\mathbf{v}\|(\cos\beta)\mathbf{j} + \|\mathbf{v}\|(\cos\gamma)\mathbf{k}$$

$$= \|\mathbf{v}\|[(\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}]$$ ◀

NOW WORK PROBLEM 59.

Example 10 shows that the direction cosines of a vector $\mathbf{v}$ are also the components of the unit vector in the direction of $\mathbf{v}$.

8.6 Assess Your Understanding

'Are You Prepared?'

Answer is given at the end of these exercises. If you get the wrong answer, read the page listed in red.

1. The distance d from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is $d =$ _____ (p. 5)

Concepts and Vocabulary

2. In space, points of the form $(x, y, 0)$ lie in a plane called the _____.

3. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a vector in space, the scalars a, b, c are called the _____ of $\mathbf{v}$.

4. The sum of the squares of the direction cosines of a vector in space add up to _____.

5. *True or False:* In space, the dot product of two vectors is a positive number.

6. *True or False:* A vector in space may be described by specifying its magnitude and its direction angles.

Skill Building

In Problems 7–14, describe the set of points (x, y, z) defined by the equation.

7. $y = 0$ **8.** $x = 0$ **9.** $z = 2$ **10.** $y = 3$

11. $x = -4$ **12.** $z = -3$ **13.** $x = 1$ and $y = 2$ **14.** $x = 3$ and $z = 1$

In Problems 15–20, find the distance from P_1 to P_2.

15. $P_1 = (0, 0, 0)$ and $P_2 = (4, 1, 2)$ **16.** $P_1 = (0, 0, 0)$ and $P_2 = (1, -2, 3)$

17. $P_1 = (-1, 2, -3)$ and $P_2 = (0, -2, 1)$ **18.** $P_1 = (-2, 2, 3)$ and $P_2 = (4, 0, -3)$

19. $P_1 = (4, -2, -2)$ and $P_2 = (3, 2, 1)$ **20.** $P_1 = (2, -3, -3)$ and $P_2 = (4, 1, -1)$

In Problems 21–26, opposite vertices of a rectangular box whose edges are parallel to the coordinate axes are given. List the coordinates of the other six vertices of the box.

21. $(0, 0, 0)$; $(2, 1, 3)$ **22.** $(0, 0, 0)$; $(4, 2, 2)$ **23.** $(1, 2, 3)$; $(3, 4, 5)$

24. $(5, 6, 1)$; $(3, 8, 2)$ **25.** $(-1, 0, 2)$; $(4, 2, 5)$ **26.** $(-2, -3, 0)$; $(-6, 7, 1)$

In Problems 27–32, the vector $\mathbf{v}$ has initial point P and terminal point Q. Write $\mathbf{v}$ in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$; that is, find its position vector.

27. $P = (0, 0, 0)$; $Q = (3, 4, -1)$ **28.** $P = (0, 0, 0)$; $Q = (-3, -5, 4)$

29. $P = (3, 2, -1)$; $Q = (5, 6, 0)$ **30.** $P = (-3, 2, 0)$; $Q = (6, 5, -1)$

31. $P = (-2, -1, 4)$; $Q = (6, -2, 4)$ **32.** $P = (-1, 4, -2)$; $Q = (6, 2, 2)$

In Problems 33–38, find $\|\mathbf{v}\|$.

33. $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ **34.** $\mathbf{v} = -6\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$ **35.** $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

36. $\mathbf{v} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ **37.** $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ **38.** $\mathbf{v} = 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

In Problems 39–44, find each quantity if $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

39. $2\mathbf{v} + 3\mathbf{w}$ **40.** $3\mathbf{v} - 2\mathbf{w}$ **41.** $\|\mathbf{v} - \mathbf{w}\|$

42. $\|\mathbf{v} + \mathbf{w}\|$ **43.** $\|\mathbf{v}\| - \|\mathbf{w}\|$ **44.** $\|\mathbf{v}\| + \|\mathbf{w}\|$

In Problems 45–50, find the unit vector having the same direction as $\mathbf{v}$.

45. $\mathbf{v} = 5\mathbf{i}$ **46.** $\mathbf{v} = -3\mathbf{j}$ **47.** $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$

48. $\mathbf{v} = -6\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$ **49.** $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ **50.** $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

In Problems 51–58, find the dot product $\mathbf{v} \cdot \mathbf{w}$ and the angle between $\mathbf{v}$ and $\mathbf{w}$.

51. $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ **52.** $\mathbf{v} = \mathbf{i} + \mathbf{j}$, $\mathbf{w} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$

53. $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ **54.** $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

55. $\mathbf{v} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ **56.** $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{w} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

57. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $\mathbf{w} = 6\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$ **58.** $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{w} = 6\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$

In Problems 59–66, find the direction angles of each vector. Write each vector in the form of equation (7).

59. $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ **60.** $\mathbf{v} = -6\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$ **61.** $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ **62.** $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$

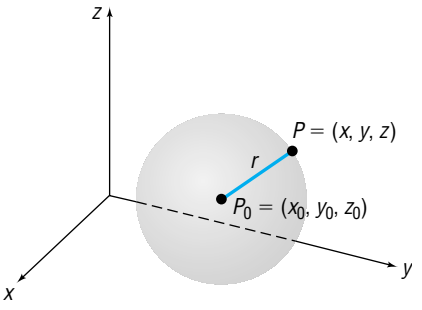
63. $\mathbf{v} = \mathbf{i} + \mathbf{j}$ **64.** $\mathbf{v} = \mathbf{j} + \mathbf{k}$ **65.** $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ **66.** $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

Applications and Extensions

67. The Sphere In space, the collection of all points that are the same distance from some fixed point is called a **sphere.** See the illustration. The constant distance is called the **radius,** and the fixed point is the **center** of the sphere. Show that the equation of a sphere with center at (x_0, y_0, z_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

[**Hint:** Use the Distance Formula (1).]



In Problems 68–70, find the equation of a sphere with radius r and center P_0.

68. $r = 1;\quad P_0 = (3, 1, 1)$ **69.** $r = 2;\quad P_0 = (1, 2, 2)$ **70.** $r = 3;\quad P_0 = (-1, 1, 2)$

In Problems 71–76, find the radius and center of each sphere.

[**Hint**: *Complete the square in each variable.*]

71. $x^2 + y^2 + z^2 + 2x - 2y = 2$ **72.** $x^2 + y^2 + z^2 + 2x - 2z = -1$

73. $x^2 + y^2 + z^2 - 4x + 4y + 2z = 0$ **74.** $x^2 + y^2 + z^2 - 4x = 0$

75. $2x^2 + 2y^2 + 2z^2 - 8x + 4z = -1$ **76.** $3x^2 + 3y^2 + 3z^2 + 6x - 6y = 3$

The **work** *W done by a constant force* **F** *in moving an object from a point A in space to a point B in space is defined as* $W = \mathbf{F} \cdot \overrightarrow{AB}$. *Use this definition in Problems 77–79.*

77. Work Find the work done by a force of 3 newtons acting in the direction $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ in moving an object 2 meters from $(0, 0, 0)$ to $(0, 2, 0)$.

78. Work Find the work done by a force of 1 newton acting in the direction $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ in moving an object 3 meters from $(0, 0, 0)$ to $(1, 2, 2)$.

79. Work Find the work done in moving an object along a vector $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ if the applied force is $\mathbf{F} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

'Are You Prepared?' Answer

1. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

8.7 The Cross Product

OBJECTIVES
1. Find the Cross Product of Two Vectors
2. Know Algebraic Properties of the Cross Product
3. Know Geometric Properties of the Cross Product
4. Find a Vector Orthogonal to Two Given Vectors
5. Find the Area of a Parallelogram

1 Find the Cross Product of Two Vectors

For vectors in space, and only for vectors in space, a second product of two vectors is defined, called the *cross product*. The cross product of two vectors in space is, in fact, also a vector that has applications in both geometry and physics.

If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two vectors in space, the **cross product** $\mathbf{v} \times \mathbf{w}$ is defined as the vector

$$\boxed{\mathbf{v} \times \mathbf{w} = (b_1c_2 - b_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \qquad \textbf{(1)}}$$

Notice that the cross product $\mathbf{v} \times \mathbf{w}$ of two vectors is a vector. Because of this, it is sometimes referred to as the **vector product.**

EXAMPLE 1 Finding Cross Products Using Equation (1)

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, then an application of equation (1) gives

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= (3\cdot 3 - 2\cdot 5)\mathbf{i} - (2\cdot 3 - 1\cdot 5)\mathbf{j} + (2\cdot 2 - 1\cdot 3)\mathbf{k}\\ &= (9 - 10)\mathbf{i} - (6 - 5)\mathbf{j} + (4 - 3)\mathbf{k}\\ &= -\mathbf{i} - \mathbf{j} + \mathbf{k}\end{aligned}$$

◀

Determinants* may be used as an aid in computing cross products. A **2 by 2 determinant,** symbolized by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

has the value $a_1b_2 - a_2b_1$; that is,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

A **3 by 3 determinant** has the value

$$\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} A - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} B + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} C$$

EXAMPLE 2 Evaluating Determinants

(a) $\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 2\cdot 2 - 1\cdot 3 = 4 - 3 = 1$

(b) $$\begin{aligned}\begin{vmatrix} A & B & C \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} &= \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} A - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} B + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} C\\ &= (9 - 10)A - (6 - 5)B + (4 - 3)C\\ &= -A - B + C\end{aligned}$$

◀

NOW WORK PROBLEM 7.

The cross product of the vectors $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, that is,

$$\mathbf{v} \times \mathbf{w} = (b_1c_2 - b_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

*Determinants are discussed in detail in Section 10.3.

may be written symbolically using determinants as

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\mathbf{k}$$

EXAMPLE 3 Using Determinants to Find Cross Products

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, find:

(a) $\mathbf{v} \times \mathbf{w}$ (b) $\mathbf{w} \times \mathbf{v}$ (c) $\mathbf{v} \times \mathbf{v}$ (d) $\mathbf{w} \times \mathbf{w}$

Solution

(a) $$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\mathbf{k} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$$

(b) $$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}\mathbf{k} = \mathbf{i} + \mathbf{j} - \mathbf{k}$$

(c) $$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 5 \\ 3 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix}\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

(d) $$\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$ ◀

NOW WORK PROBLEM 15.

2 Know Algebraic Properties of the Cross Product

Notice in Examples 3(a) and 3(b) that $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{v}$ are negatives of one another. From Examples 3(c) and 3(d), we might conjecture that the cross product of a vector with itself is the zero vector. These and other algebraic properties of the cross product are given next.

Theorem

Algebraic Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in space and if α is a scalar, then

$$\mathbf{u} \times \mathbf{u} = \mathbf{0} \tag{2}$$

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \tag{3}$$

$$\alpha(\mathbf{u} \times \mathbf{v}) = (\alpha\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (\alpha\mathbf{v}) \tag{4}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \tag{5}$$

Proof We will prove properties (2) and (4) here and leave properties (3) and (5) as exercises (see Problems 55 and 56).

To prove property (2), we let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_1 & c_1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_1 & c_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix}\mathbf{k}$$
$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

To prove property (4), we let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Then

$$\alpha(\mathbf{u} \times \mathbf{v}) = \alpha[(b_1c_2 - b_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}]$$

Apply (1).

$$= \alpha(b_1c_2 - b_2c_1)\mathbf{i} - \alpha(a_1c_2 - a_2c_1)\mathbf{j} + \alpha(a_1b_2 - a_2b_1)\mathbf{k} \qquad \textbf{(6)}$$

Since $\alpha\mathbf{u} = \alpha a_1\mathbf{i} + \alpha b_1\mathbf{j} + \alpha c_1\mathbf{k}$, we have

$$(\alpha\mathbf{u}) \times \mathbf{v} = (\alpha b_1c_2 - b_2\alpha c_1)\mathbf{i} - (\alpha a_1c_2 - a_2\alpha c_1)\mathbf{j} + (\alpha a_1b_2 - a_2\alpha b_1)\mathbf{k}$$
$$= \alpha(b_1c_2 - b_2c_1)\mathbf{i} - \alpha(a_1c_2 - a_2c_1)\mathbf{j} + \alpha(a_1b_2 - a_2b_1)\mathbf{k} \qquad \textbf{(7)}$$

Based on equations (6) and (7), the first part of property (4) follows. The second part can be proved in like fashion. ■

NOW WORK PROBLEM 17.

3 Know Geometric Properties of the Cross Product

The cross product has several interesting geometric properties.

Theorem

Geometric Properties of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in space.

$\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. **(8)**

$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$, **(9)**
where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

$\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram **(10)**
having $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$ as adjacent sides.

$\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel. **(11)**

Proof of Property (8) Let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = (b_1c_2 - b_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Now we compute the dot product $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$.

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \cdot [(b_1c_2 - b_2c_1)\mathbf{i} - (a_1c_2 - a_2c_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}]$$
$$= a_1(b_1c_2 - b_2c_1) - b_1(a_1c_2 - a_2c_1) + c_1(a_1b_2 - a_2b_1) = 0$$

Since two vectors are orthogonal if their dot product is zero, it follows that $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ are orthogonal. Similarly, $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, so $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ are orthogonal. ■

4 Find a Vector Orthogonal to Two Given Vectors

As long as the vectors **u** and **v** are not parallel, they will form a plane in space. See Figure 80. Based on property (8), the vector $\mathbf{u} \times \mathbf{v}$ is normal to this plane. As Figure 80 illustrates, there are two vectors normal to the plane containing **u** and **v**. It can be shown that the vector $\mathbf{u} \times \mathbf{v}$ is the one determined by the thumb of the right hand when the other fingers of the right hand are cupped so that they point in a direction from **u** to **v**. See Figure 81.*

Figure 80

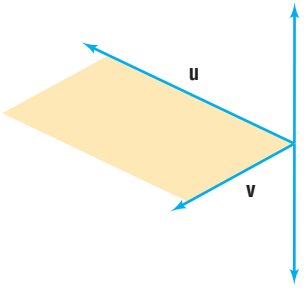
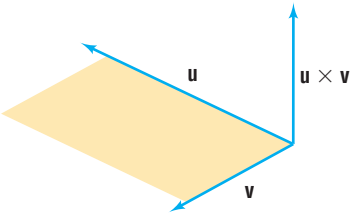


Figure 81



EXAMPLE 4 Finding a Vector Orthogonal to Two Given Vectors

Find a vector that is orthogonal to $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Solution Based on property (8), such a vector is $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -1 & 3 & -1 \end{vmatrix} = (2 - 3)\mathbf{i} - [-3 - (-1)]\mathbf{j} + (9 - 2)\mathbf{k} = -\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$$

The vector $-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$ is orthogonal to both **u** and **v**.

CHECK: Two vectors are orthogonal if their dot product is zero.

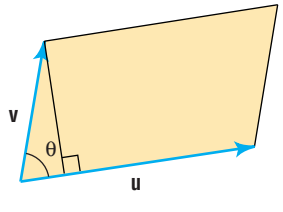
$$\mathbf{u} \cdot (-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) = -3 - 4 + 7 = 0$$

$$\mathbf{v} \cdot (-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) = (-\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}) = 1 + 6 - 7 = 0$$ ◀

NOW WORK PROBLEM 41.

The proof of property (9) is left as an exercise. See Problem 58.

Figure 82



Proof of Property (10) Suppose that **u** and **v** are adjacent sides of a parallelogram. See Figure 82. Then the lengths of these sides are $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$. If θ is the angle between **u** and **v**, then the height of the parallelogram is $\|\mathbf{v}\| \sin \theta$ and its area is

$$\text{Area of parallelogram} = \text{Base} \times \text{Height} = \|\mathbf{u}\|[\|\mathbf{v}\| \sin \theta] \underset{\text{Property (9)}}{=} \|\mathbf{u} \times \mathbf{v}\|$$ ■

*This is a consequence of using a right-handed coordinate system.

5 Find the Area of a Parallelogram

EXAMPLE 5 **Finding the Area of a Parallelogram**

Find the area of the parallelogram whose vertices are $P_1 = (0, 0, 0)$, $P_2 = (3, -2, 1)$, $P_3 = (-1, 3, -1)$, and $P_4 = (2, 1, 0)$.

Solution Two adjacent sides of this parallelogram are

$$\mathbf{u} = \overrightarrow{P_1P_2} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{v} = \overrightarrow{P_1P_3} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

Since $\mathbf{u} \times \mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$ (Example 4), the area of the parallelogram is

$$\text{Area of parallelogram} = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{1 + 4 + 49} = \sqrt{54} = 3\sqrt{6}$$

WARNING: Not all pairs of vertices give rise to a side. For example, $\overrightarrow{P_1P_4}$ is a diagonal of the parallelogram since $\overrightarrow{P_1P_3} + \overrightarrow{P_3P_4} = \overrightarrow{P_1P_4}$. Also, $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_2P_4}$ are not adjacent sides; they are parallel sides. ■

NOW WORK PROBLEM 49.

Proof of Property (11) The proof requires two parts. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then there is a scalar α such that $\mathbf{u} = \alpha\mathbf{v}$. Then

$$\mathbf{u} \times \mathbf{v} = (\alpha\mathbf{v}) \times \mathbf{v} \underset{\text{Property (4)}}{=} \alpha(\mathbf{v} \times \mathbf{v}) \underset{\text{Property (2)}}{=} \mathbf{0}$$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then, by property (9), we have

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = 0$$

Since $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, then we must have $\sin\theta = 0$, so $\theta = 0$ or $\theta = \pi$. In either case, since θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel. ■

8.7 Assess Your Understanding

Concepts and Vocabulary

1. *True or False:* If $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. *True or False:* For any vector $\mathbf{v}$, $\mathbf{v} \times \mathbf{v} = \mathbf{0}$.
3. *True or False:* If $\mathbf{u}$ and $\mathbf{v}$ are vectors, then $\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{u} = \mathbf{0}$.
4. *True or False:* $\mathbf{u} \times \mathbf{v}$ is a vector that is parallel to both $\mathbf{u}$ and $\mathbf{v}$.
5. *True or False:* $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
6. *True or False:* The area of the parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides is the magnitude of the cross product of $\mathbf{u}$ and $\mathbf{v}$.

Skill Building

In Problems 7–14, find the value of each determinant.

7. $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$
8. $\begin{vmatrix} -2 & 5 \\ 2 & -3 \end{vmatrix}$
9. $\begin{vmatrix} 6 & 5 \\ -2 & -1 \end{vmatrix}$
10. $\begin{vmatrix} -4 & 0 \\ 5 & 3 \end{vmatrix}$
11. $\begin{vmatrix} A & B & C \\ 2 & 1 & 4 \\ 1 & 3 & 1 \end{vmatrix}$
12. $\begin{vmatrix} A & B & C \\ 0 & 2 & 4 \\ 3 & 1 & 3 \end{vmatrix}$
13. $\begin{vmatrix} A & B & C \\ -1 & 3 & 5 \\ 5 & 0 & -2 \end{vmatrix}$
14. $\begin{vmatrix} A & B & C \\ 1 & -2 & -3 \\ 0 & 2 & -2 \end{vmatrix}$

In Problems 15–22, find (a) $\mathbf{v} \times \mathbf{w}$, *(b)* $\mathbf{w} \times \mathbf{v}$, *(c)* $\mathbf{w} \times \mathbf{w}$, *and (d)* $\mathbf{v} \times \mathbf{v}$.

15. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
$\mathbf{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

16. $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
$\mathbf{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

17. $\mathbf{v} = \mathbf{i} + \mathbf{j}$
$\mathbf{w} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

18. $\mathbf{v} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$
$\mathbf{w} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

19. $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
$\mathbf{w} = \mathbf{j} - \mathbf{k}$

20. $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$
$\mathbf{w} = \mathbf{i} - \mathbf{k}$

21. $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$
$\mathbf{w} = 4\mathbf{i} - 3\mathbf{k}$

22. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$
$\mathbf{w} = 3\mathbf{j} - 2\mathbf{k}$

In Problems 23–44, use the vectors $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *given below to find each expression.*

$$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \qquad \mathbf{v} = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \qquad \mathbf{w} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

23. $\mathbf{u} \times \mathbf{v}$

24. $\mathbf{v} \times \mathbf{w}$

25. $\mathbf{v} \times \mathbf{u}$

26. $\mathbf{w} \times \mathbf{v}$

27. $\mathbf{v} \times \mathbf{v}$

28. $\mathbf{w} \times \mathbf{w}$

29. $(3\mathbf{u}) \times \mathbf{v}$

30. $\mathbf{v} \times (4\mathbf{w})$

31. $\mathbf{u} \times (2\mathbf{v})$

32. $(-3\mathbf{v}) \times \mathbf{w}$

33. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$

34. $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$

35. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

36. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

37. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$

38. $(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}$

39. $\mathbf{u} \times (\mathbf{v} \times \mathbf{v})$

40. $(\mathbf{w} \times \mathbf{w}) \times \mathbf{v}$

41. Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

42. Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{w}$.

43. Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{i} + \mathbf{j}$.

44. Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{j} + \mathbf{k}$.

In Problems 45–48, find the area of the parallelogram with one corner at P_1 *and adjacent sides* $\overrightarrow{P_1P_2}$ *and* $\overrightarrow{P_1P_3}$.

45. $P_1 = (0, 0, 0)$, $P_2 = (1, 2, 3)$, $P_3 = (-2, 3, 0)$

46. $P_1 = (0, 0, 0)$, $P_2 = (2, 3, 1)$, $P_3 = (-2, 4, 1)$

47. $P_1 = (1, 2, 0)$, $P_2 = (-2, 3, 4)$, $P_3 = (0, -2, 3)$

48. $P_1 = (-2, 0, 2)$, $P_2 = (2, 1, -1)$, $P_3 = (2, -1, 2)$

In Problems 49–52, find the area of the parallelogram with vertices P_1, P_2, P_3, *and* P_4.

49. $P_1 = (1, 1, 2)$, $P_2 = (1, 2, 3)$, $P_3 = (-2, 3, 0)$, $P_4 = (-2, 4, 1)$

50. $P_1 = (2, 1, 1)$, $P_2 = (2, 3, 1)$, $P_3 = (-2, 4, 1)$, $P_4 = (-2, 6, 1)$

51. $P_1 = (1, 2, -1)$, $P_2 = (4, 2, -3)$, $P_3 = (6, -5, 2)$, $P_4 = (9, -5, 0)$

52. $P_1 = (-1, 1, 1)$, $P_2 = (-1, 2, 2)$, $P_3 = (-3, 4, -5)$, $P_4 = (-3, 5, -4)$

Applications and Extensions

53. Find a unit vector normal to the plane containing $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

54. Find a unit vector normal to the plane containing $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$.

55. Prove property (3).

56. Prove property (5).

57. Prove for vectors $\mathbf{u}$ and $\mathbf{v}$ that

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

[**Hint:** Proceed as in the proof of property (4), computing first the left side and then the right side.]

58. Prove property (9).

[**Hint:** Use the result of Problem 57 and the fact that if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos \theta$.]

59. Show that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal then

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

60. Show that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors then so is $\mathbf{u} \times \mathbf{v}$ a unit vector.

Discussion and Writing

61. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, what, if anything, can you conclude about $\mathbf{u}$ and $\mathbf{v}$?

Chapter Review

Things to Know

Polar Coordinates (p. 572–575)

Relationship between polar coordinates (r, θ) and rectangular coordinates (x, y) (pp. 575 and 578)	$x = r\cos\theta,\ y = r\sin\theta$ $r^2 = x^2 + y^2,\ \tan\theta = \dfrac{y}{x},\quad x \neq 0$
Polar form of a complex number (p. 601)	If $z = x + yi$, then $z = r(\cos\theta + i\sin\theta)$, where $r = \lvert z\rvert = \sqrt{x^2 + y^2},\quad \sin\theta = \dfrac{y}{r},\quad \cos\theta = \dfrac{x}{r},\quad 0 \le \theta < 2\pi.$
De Moivre's Theorem (p. 603)	If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$, where $n \ge 1$ is a positive integer.
nth root of a complex number $z = r(\cos\theta_0 + i\sin\theta_0)$ (p. 605)	$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\dfrac{\theta_0}{n} + \dfrac{2k\pi}{n}\right) + i\sin\left(\dfrac{\theta_0}{n} + \dfrac{2k\pi}{n}\right)\right],\quad k = 0, \ldots, n-1,$ where $n \ge 2$ is an integer.

Vector (pp. 608–610) — Quantity having magnitude and direction; equivalent to a directed line segment $\overrightarrow{PQ}$

Position vector (p. 611)	Vector whose initial point is at the origin
Unit vector (pp. 611 and 632)	Vector whose magnitude is 1
Dot product (pp. 620 and 632)	If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$, then $\mathbf{v}\cdot\mathbf{w} = a_1a_2 + b_1b_2$. If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}_2$, then $\mathbf{v}\cdot\mathbf{w} = a_1a_2 + b_1b_2 + c_1c_2$.
Angle θ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ (pp. 621 and 633)	$\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\lVert\mathbf{u}\rVert\,\lVert\mathbf{v}\rVert}$
Direction angles of a vector in space (p. 636)	If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $\mathbf{v} = \lVert\mathbf{v}\rVert[(\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}]$, where $\cos\alpha = \dfrac{a}{\lVert\mathbf{v}\rVert},\quad \cos\beta = \dfrac{b}{\lVert\mathbf{v}\rVert},\quad \cos\gamma = \dfrac{c}{\lVert\mathbf{v}\rVert}.$
Cross product (p. 639)	If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, then $\mathbf{v}\times\mathbf{w} = [b_1c_2 - b_2c_1]\mathbf{i} - [a_1c_2 - a_2c_1]\mathbf{j} + [a_1b_2 - a_2b_1]\mathbf{k}$.
Area of a parallelogram (p. 641)	$\lVert\mathbf{u}\times\mathbf{v}\rVert = \lVert\mathbf{u}\rVert\,\lVert\mathbf{v}\rVert\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Objectives

Section		You should be able to . . .	Review Exercises
8.1	1	Plot points using polar coordinates (p. 572)	1–6
	2	Convert from polar coordinates to rectangular coordinates (p. 575)	1–6
	3	Convert from rectangular coordinates to polar coordinates (p. 576)	7–12
8.2	1	Graph and identify polar equations by converting to rectangular equations (p. 582)	13–18
	2	Graph polar equations using a graphing utility (p. 583)	13–24
	3	Test polar equations for symmetry (p. 587)	19–24
	4	Graph polar equations by plotting points (p. 588)	19–24
8.3	1	Convert a complex number from rectangular form to polar form (p. 601)	25–28
	2	Plot points in the complex plane (p. 601)	29–34
	3	Find products and quotients of complex numbers in polar form (p. 602)	35–40

	4	Use De Moivre's Theorem (p. 603)	41–48
	5	Find complex roots (p. 604)	49–50
8.4	1	Graph vectors (p. 610)	51–54
	2	Find a position vector (p. 611)	55–58
	3	Add and subtract vectors (p. 613)	59, 60
	4	Find a scalar product and the magnitude of a vector (p. 614)	61–66
	5	Find a unit vector (p. 614)	67, 68
	6	Find a vector from its direction and magnitude (p. 615)	69, 70
	7	Work with objects in static equilibrium (p. 616)	111
8.5	1	Find the dot product of two vectors (p. 620)	85–88
	2	Find the angle between two vectors (p. 621)	85–88, 109, 110, 112
	3	Determine whether two vectors are parallel (p. 623)	93–96
	4	Determine whether two vectors are orthogonal (p. 623)	93–96
	5	Decompose a vector into two orthogonal vectors (p. 624)	99–102
	6	Compute work (p. 625)	113
8.6	1	Find the distance between two points in space (p. 630)	71, 72
	2	Find position vectors in space (p. 630)	73, 74
	3	Perform operations on vectors (p. 631)	75–80
	4	Find the dot product (p. 632)	89–92
	5	Find the angle between two vectors (p. 633)	89–92
	6	Find the direction angles of a vector (p. 634)	103, 104
8.7	1	Find the cross product of two vectors (p. 638)	81, 82
	2	Know algebraic properties of the cross product (p. 640)	107, 108
	3	Know geometric properties of the cross product (p. 641)	105, 106
	4	Find a vector orthogonal to two given vectors (p. 642)	84
	5	Find the area of a parallelogram (p. 643)	105, 106

Review Exercises

In Problems 1–6, plot each point given in polar coordinates, and find its rectangular coordinates.

1. $\left(3, \frac{\pi}{6}\right)$ **2.** $\left(4, \frac{2\pi}{3}\right)$ **3.** $\left(-2, \frac{4\pi}{3}\right)$ **4.** $\left(-1, \frac{5\pi}{4}\right)$ **5.** $\left(-3, -\frac{\pi}{2}\right)$ **6.** $\left(-4, -\frac{\pi}{4}\right)$

In Problems 7–12, the rectangular coordinates of a point are given. Find two pairs of polar coordinates (r, θ) for each point, one with $r > 0$ and the other with $r < 0$. Express θ in radians.

7. $(-3, 3)$ **8.** $(1, -1)$ **9.** $(0, -2)$ **10.** $(2, 0)$ **11.** $(3, 4)$ **12.** $(-5, 12)$

In Problems 13–18, the letters r and θ represent polar coordinates. Write each polar equation as an equation in rectangular coordinates (x, y). Identify the equation and graph it by hand. Verify your graph using a graphing utility.

13. $r = 2 \sin \theta$ **14.** $3r = \sin \theta$ **15.** $r = 5$

16. $\theta = \frac{\pi}{4}$ **17.** $r \cos \theta + 3r \sin \theta = 6$ **18.** $r^2 + 4r \sin \theta - 8r \cos \theta = 5$

In Problems 19–24, sketch by hand the graph of each polar equation. Be sure to test for symmetry. Verify your graph using a graphing utility.

19. $r = 4\cos\theta$ **20.** $r = 3\sin\theta$ **21.** $r = 3 - 3\sin\theta$

22. $r = 2 + \cos\theta$ **23.** $r = 4 - \cos\theta$ **24.** $r = 1 - 2\sin\theta$

In Problems 25–28, write each complex number in polar form. Express each argument in degrees.

25. $-1 - i$ **26.** $-\sqrt{3} + i$ **27.** $4 - 3i$ **28.** $3 - 2i$

In Problems 29–34, write each complex number in the standard form $a + bi$ and plot each in the complex plane.

29. $2(\cos 150° + i\sin 150°)$ **30.** $3(\cos 60° + i\sin 60°)$ **31.** $3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

32. $4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ **33.** $0.1(\cos 350° + i\sin 350°)$ **34.** $0.5(\cos 160° + i\sin 160°)$

In Problems 35–40, find zw and $\frac{z}{w}$. Leave your answers in polar form.

35. $z = \cos 80° + i\sin 80°$
$w = \cos 50° + i\sin 50°$

36. $z = \cos 205° + i\sin 205°$
$w = \cos 85° + i\sin 85°$

37. $z = 3\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$
$w = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

38. $z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$
$w = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

39. $z = 5(\cos 10° + i\sin 10°)$
$w = \cos 355° + i\sin 355°$

40. $z = 4(\cos 50° + i\sin 50°)$
$w = \cos 340° + i\sin 340°$

In Problems 41–48, write each expression in the standard form $a + bi$.

41. $[3(\cos 20° + i\sin 20°)]^3$ **42.** $[2(\cos 50° + i\sin 50°)]^3$

43. $\left[\sqrt{2}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)\right]^4$ **44.** $\left[2\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right)\right]^4$

45. $\left(1 - \sqrt{3}i\right)^6$ **46.** $(2 - 2i)^8$

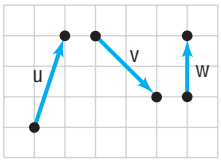
47. $(3 + 4i)^4$ **48.** $(1 - 2i)^4$

49. Find all the complex cube roots of 27. **50.** Find all the complex fourth roots of -16.

In Problems 51–54, use the figure to graph each of the following:

51. $\mathbf{u} + \mathbf{v}$ **52.** $\mathbf{v} + \mathbf{w}$

53. $2\mathbf{u} + 3\mathbf{v}$ **54.** $5\mathbf{v} - 2\mathbf{w}$



In Problems 55–58, the vector $\mathbf{v}$ is represented by the directed line segment $\overrightarrow{PQ}$. Write $\mathbf{v}$ in the form $a\mathbf{i} + b\mathbf{j}$ and find $\|\mathbf{v}\|$.

55. $P = (1, -2)$; $Q = (3, -6)$ **56.** $P = (-3, 1)$; $Q = (4, -2)$

57. $P = (0, -2)$; $Q = (-1, 1)$ **58.** $P = (3, -4)$; $Q = (-2, 0)$

In Problems 59–68, use the vectors $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 3\mathbf{j}$ to find:

59. $\mathbf{v} + \mathbf{w}$ **60.** $\mathbf{v} - \mathbf{w}$ **61.** $4\mathbf{v} - 3\mathbf{w}$ **62.** $-\mathbf{v} + 2\mathbf{w}$

63. $\|\mathbf{v}\|$ **64.** $\|\mathbf{v} + \mathbf{w}\|$ **65.** $\|\mathbf{v}\| + \|\mathbf{w}\|$ **66.** $\|2\mathbf{v}\| - 3\|\mathbf{w}\|$

67. Find a unit vector in the same direction as $\mathbf{v}$. **68.** Find a unit vector in the opposite direction of $\mathbf{w}$.

69. Find the vector $\mathbf{v}$ in the xy-plane with magnitude 3 if the angle between $\mathbf{v}$ and $\mathbf{i}$ is 60°.

70. Find the vector $\mathbf{v}$ in the xy-plane with magnitude 5 if the angle between $\mathbf{v}$ and $\mathbf{i}$ is 150°.

71. Find the distance from $P_1 = (1, 3, -2)$ to $P_2 = (4, -2, 1)$.

72. Find the distance from $P_1 = (0, -4, 3)$ to $P_2 = (6, -5, -1)$.

73. A vector $\mathbf{v}$ has initial point $P = (1, 3, -2)$ and terminal point $Q = (4, -2, 1)$. Write $\mathbf{v}$ in the form $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

74. A vector $\mathbf{v}$ has initial point $P = (0, -4, 3)$ and terminal point $Q = (6, -5, -1)$. Write $\mathbf{v}$ in the form $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

In Problems 75–84, use the vectors $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ *and* $\mathbf{w} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ *to find each expression.*

75. $4\mathbf{v} - 3\mathbf{w}$

76. $-\mathbf{v} + 2\mathbf{w}$

77. $\|\mathbf{v} - \mathbf{w}\|$

78. $\|\mathbf{v} + \mathbf{w}\|$

79. $\|\mathbf{v}\| - \|\mathbf{w}\|$

80. $\|\mathbf{v}\| + \|\mathbf{w}\|$

81. $\mathbf{v} \times \mathbf{w}$

82. $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$

83. Find a unit vector in the same direction as $\mathbf{v}$ and then in the opposite direction of $\mathbf{v}$.

84. Find a unit vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

In Problems 85–92, find the dot product $\mathbf{v} \cdot \mathbf{w}$ *and the angle between* $\mathbf{v}$ *and* $\mathbf{w}$.

85. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} - 3\mathbf{j}$

86. $\mathbf{v} = 3\mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + \mathbf{j}$

87. $\mathbf{v} = \mathbf{i} - 3\mathbf{j}, \quad \mathbf{w} = -\mathbf{i} + \mathbf{j}$

88. $\mathbf{v} = \mathbf{i} + 4\mathbf{j}, \quad \mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$

89. $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

90. $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{w} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

91. $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{w} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

92. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{w} = 5\mathbf{i} + \mathbf{j} + \mathbf{k}$

In Problems 93–98, determine whether $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel, orthogonal, or neither.*

93. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}; \quad \mathbf{w} = -4\mathbf{i} - 6\mathbf{j}$

94. $\mathbf{v} = -2\mathbf{i} - \mathbf{j}; \quad \mathbf{w} = 2\mathbf{i} + \mathbf{j}$

95. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}; \quad \mathbf{w} = -3\mathbf{i} + 4\mathbf{j}$

96. $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j}; \quad \mathbf{w} = -3\mathbf{i} + 2\mathbf{j}$

97. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}; \quad \mathbf{w} = 4\mathbf{i} + 6\mathbf{j}$

98. $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j}; \quad \mathbf{w} = 2\mathbf{i} + 4\mathbf{j}$

In Problems 99 and 100, decompose $\mathbf{v}$ *into two vectors, one parallel to* $\mathbf{w}$ *and the other orthogonal to* $\mathbf{w}$.

99. $\mathbf{v} = 2\mathbf{i} + \mathbf{j}; \quad \mathbf{w} = -4\mathbf{i} + 3\mathbf{j}$

100. $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}; \quad \mathbf{w} = -2\mathbf{i} + \mathbf{j}$

101. Decompose $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ into two vectors, one parallel to $\mathbf{w} = 3\mathbf{i} + \mathbf{j}$, the other perpendicular to $\mathbf{w}$.

102. Decompose $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$ into two vectors, one parallel to $\mathbf{w} = 3\mathbf{i} - \mathbf{j}$. the other perpendicular to $\mathbf{w}$.

103. Find the direction angles of the vector $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

104. Find the direction angles of the vector $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

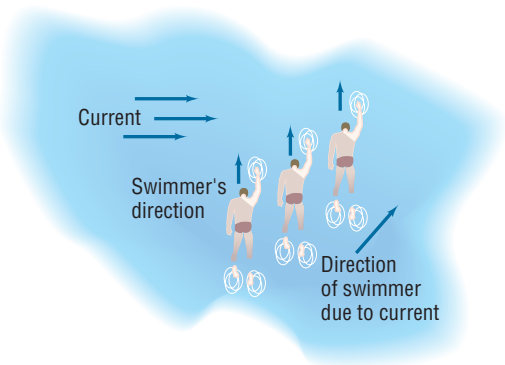
105. Find the area of the parallelogram with vertices $P_1 = (1, 1, 1)$, $P_2 = (2, 3, 4)$, $P_3 = (6, 5, 2)$, and $P_4 = (7, 7, 5)$.

106. Find the area of the parallelogram with vertices $P_1 = (2, -1, 1)$, $P_2 = (5, 1, 4)$, $P_3 = (0, 1, 1)$, and $P_4 = (3, 3, 4)$.

107. If $\mathbf{u} \times \mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, what is $\mathbf{v} \times \mathbf{u}$?

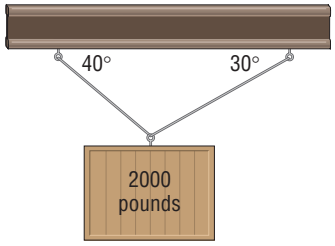
108. Suppose that $\mathbf{u} = 3\mathbf{v}$. What is $\mathbf{u} \times \mathbf{v}$?

109. Actual Speed and Direction of a Swimmer A swimmer can maintain a constant speed of 5 miles per hour. If the swimmer heads directly across a river that has a current moving at the rate of 2 miles per hour, what is the actual speed of the swimmer? (See the figure.) If the river is 1 mile wide, how far downstream will the swimmer end up from the point directly across the river from the starting point?



110. Actual Speed and Direction of an Airplane An airplane has an airspeed of 500 kilometers per hour in a northerly direction. The wind velocity is 60 kilometers per hour in a southeasterly direction. Find the actual speed and direction of the plane relative to the ground.

111. Static Equilibrium A weight of 2000 pounds is suspended from two cables as shown in the figure. What are the tensions in each cable?



112. Actual Speed and Distance of a Motorboat A small motorboat is moving at a true speed of 11 miles per hour in a southerly direction. The current is known to be from the northeast at 3 miles per hour. What is the speed of the motorboat relative to the water? In what direction does the compass indicate that the boat is headed?

113. Computing Work Find the work done by a force of 5 pounds acting in the direction 60° to the horizontal in moving an object 20 feet from $(0, 0)$ to $(20, 0)$.

Chapter Test

In Problems 1–3, plot each point given in polar coordinates.

1. $\left(2, \frac{3\pi}{4}\right)$ **2.** $\left(3, -\frac{\pi}{6}\right)$ **3.** $\left(-4, \frac{\pi}{3}\right)$

4. Convert $\left(2, 2\sqrt{3}\right)$ from rectangular coordinates to polar coordinates (r, θ), where $r > 0$ and $0 \le \theta < 2\pi$.

In Problems 5–7, convert the polar equation to a rectangular equation. Graph the equation by hand.

5. $r = 7$ **6.** $\tan\theta = 3$ **7.** $r\sin^2\theta + 8\sin\theta = r$

In Problems 8–9, test each of the polar equations for symmetry with respect to the pole, the polar axis, and the line $\theta = \frac{\pi}{2}$.

8. $r^2\cos\theta = 5$ **9.** $r = 5\sin\theta\cos^2\theta$

In Problems 10–12, perform the given operation, given $z = 2(\cos 85° + i\sin 85°)$ *and* $w = 3(\cos 22° + i\sin 22°)$. *Write your answer in polar form.*

10. $z \cdot w$ **11.** $\frac{w}{z}$ **12.** w^5

13. Find all the cube roots of $-8 + 8\sqrt{3}i$. Write all answers in the form $a + bi$ and then plot them in rectangular coordinates.

In Problems 14–18, $P_1 = \left(3\sqrt{2}, 7\sqrt{2}\right)$ *and* $P_2 = \left(8\sqrt{2}, 2\sqrt{2}\right)$.

14. Find the position vector $\mathbf{v}$ equal to $\overrightarrow{P_1P_2}$.

15. Find $\|\mathbf{v}\|$.

16. Find the unit vector in the direction of $\mathbf{v}$.

17. Find the angle between $\mathbf{v}$ and $\mathbf{i}$.

18. Decompose $\mathbf{v}$ into its vertical and horizontal components.

In Problems 19–22, $\mathbf{v}_1 = \langle 4, 6\rangle$, $\mathbf{v}_2 = \langle -3, -6\rangle$, $\mathbf{v}_3 = \langle -8, 4\rangle$, $\mathbf{v}_4 = \langle 10, 15\rangle$.

19. Find the vector $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$

20. Which two vectors are parallel?

21. Which two vectors are orthogonal?

22. Find the angle between vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

In Problems 23–25, use the vectors $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

23. Find $\mathbf{u} \times \mathbf{v}$.

24. Find the direction angles for $\mathbf{u}$.

25. Find the area of the parallelgram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

26. A 1200 pound chandelier is to be suspended over a large ballroom; the chandelier will be hung on a cable whose ends will be attached to the ceiling, 16 feet apart. The chandelier will be free hanging so that the ends of the cable will make equal angles with the ceiling. If the top of the chandelier is to be 16 feet from the ceiling, what is the minimum tension the cable must be able to endure?

Chapter Projects

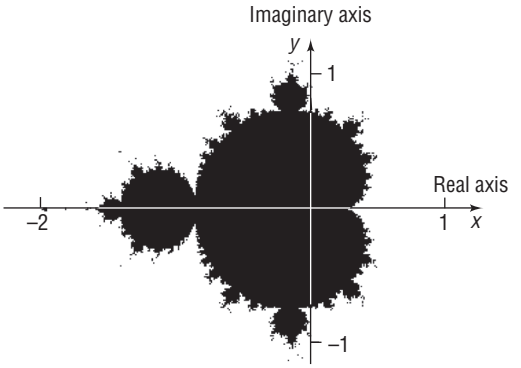
1. Mandelbrot Sets

(a) Draw a complex plane and plot the points $z_1 = 3 + 4i$, $z_2 = -2 + i$, $z_3 = 0 - 2i$, and $z_4 = -2$.

(b) Consider the expression $a_n = (a_{n-1})^2 + z$, where z is some complex number (called the **seed**) and $a_0 = z$. Compute $a_1\ (= a_0^2 + z)$, $a_2\ (= a_1^2 + z)$, $a_3\ (= a_2^2 + z)$, a_4, a_5 and a_6 for the following seeds: $z_1 = 0.1 - 0.4i$, $z_2 = 0.5 + 0.8i$, $z_3 = -0.9 + 0.7i$, $z_4 = -1.1 + 0.1i$, $z_5 = 0 - 1.3i$, and $z_6 = 1 + 1i$.

(c) The dark portion of the graph represents the set of all values $z = x + yi$ that are in the Mandelbrot set. Determine which complex numbers in part (b) are in this set by plotting them on the graph. Do the complex numbers that are not in the Mandelbrot set have any common characteristics regarding the values of a_6 found in part (b)?

(d) Compute $|z| = \sqrt{x^2 + y^2}$ for each of the complex numbers in part (b). Now compute $|a_6|$ for each of the complex numbers in part (b). For which complex numbers is $|a_6| \geq |z|$ and $|z| > 2$? Conclude that the criterion for a complex number to be in the Mandelbrot set is that $|a_n| \geq |z|$ and $|z| > 2$.



The following projects are available at the Instructor's Resource Center (IRC):

2. Project at Motorola *Signal Fades Due to Interference?*

3. Compound Interest

4. Complex Equations

Cumulative Review

1. Find the real solutions, if any, of the equation $e^{x^2-9} = 1$.

2. Find an equation for the line containing the origin that makes an angle of 30° with the positive x-axis.

3. Find an equation for the circle with center at the point $(0, 1)$ and radius 3. Graph this circle.

4. What is the domain of the function $f(x) = \ln(1 - 2x)$?

5. Test the equation $x^2 + y^3 = 2x^4$ for symmetry with respect to the x-axis, the y-axis, and the origin.

6. Graph the function $y = |\ln x|$.

7. Graph the function $y = |\sin x|$.

8. Graph the function $y = \sin|x|$.

9. Find the exact value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

10. Graph the equations $x = 3$ and $y = 4$ on the same set of rectangular coordinates.

11. Graph the equations $r = 2$ and $\theta = \frac{\pi}{3}$ on the same set of polar coordinates.