Analytic Trigonometry

6

Tremor Brought First Hint of Doom

Underwater earthquakes that triggered the devastating tsunami in PNG would have been felt by villagers about 30 minutes before the waves struck, scientists said yesterday. "The tremor was felt by coastal residents who may not have realized its significance or did not have time to retreat," said associate Professor Ted Bryant, a geoscientist at the University of Wollongong.

The tsunami would have sounded like a fleet of bombers as it crashed into a 30-kilometer stretch of coast. "The tsunami would have been caused by a rapid uplift or drop of the sea floor," said an applied mathematician and cosmologist from Monash University, Professor Joe Monaghan, who is one of Australia's leading experts on tsunamis.

Tsunamis, ridges of water hundreds of kilometers long and stretching from front to back for several kilometers, line up parallel to the beach. "We are talking about a huge volume of water moving very fast—300 kilometers per hour would be typical," Professor Monaghan said.

SOURCE: Peter Spinks, *The Age*, Tuesday, July 21, 1998.

—See Chapter Project 1.

A LOOK BACK In Chapter 4, we defined inverse functions and developed their properties, particularly the relationship between the domain and range of a function and its inverse. We learned that the graph of a function and its inverse are symmetric with respect to the line $y = x$.

We continued in Chapter 4 by defining the exponential function and the inverse of the exponential function, the logarithmic function.

A LOOK AHEAD In the first two sections of this chapter, we define the six inverse trigonometric functions and investigate their properties. In Sections 6.3 through 6.6 of this chapter, we continue the derivation of identities. These identities play an important role in calculus, the physical and life sciences, and economics, where they are used to simplify complicated expressions. The last two sections of this chapter deal with equations that contain trigonometric functions.

OUTLINE

6.1 The Inverse Sine, Cosine, and Tangent Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Inverse Functions (Section 4.2, pp. 256–267)
- Values of the Trigonometric Functions (Section 5.2, pp. 374–382)
- Properties of the Sine, Cosine, and Tangent Functions (Section 5.3, pp. 388–393)
- Graphs of the Sine, Cosine, and Tangent Functions (Section 5.4, pp. 402–407, and Section 5.5, pp. 419–422)

Now work the 'Are You Prepared?' problems on page 457.

OBJECTIVES
1 Find the Exact Value of the Inverse Sine, Cosine, and Tangent Functions
2 Find an Approximate Value of the Inverse Sine, Cosine, and Tangent Functions

In Section 4.2 we discussed inverse functions, and we noted that if a function is one-to-one it will have an inverse function. We also observed that if a function is not one-to-one it may be possible to restrict its domain in some suitable manner so that the restricted function is one-to-one.

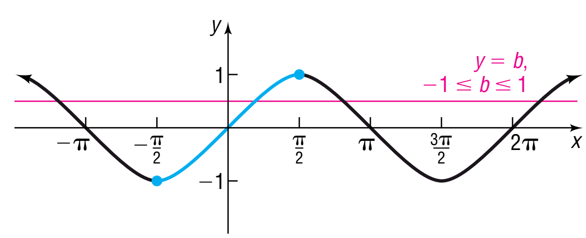
Next, we review some properties of a one-to-one function f and its inverse function f^{-1}.

1. $f^{-1}(f(x)) = x$ for every x in the domain of f and $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.
2. Domain of f = range of f^{-1}, and range of f = domain of f^{-1}.
3. The graph of f and the graph of f^{-1} are symmetric with respect to the line $y = x$.
4. If a function $y = f(x)$ has an inverse function, the equation of the inverse function is $x = f(y)$. The solution of this equation is $y = f^{-1}(x)$.

The Inverse Sine Function

In Figure 1, we reproduce the graph of $y = \sin x$. Because every horizontal line $y = b$, where b is between -1 and 1, intersects the graph of $y = \sin x$ infinitely many times, it follows from the horizontal-line test that the function $y = \sin x$ is not one-to-one.

Figure 1
$y = \sin x, -\infty < x < \infty, -1 \le y \le 1$

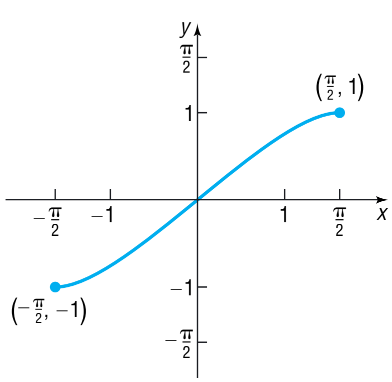


However, if we restrict the domain of $y = \sin x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the restricted function

$$y = \sin x, \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

is one-to-one and so will have an inverse function.* See Figure 2.

Figure 2
$y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}, -1 \le y \le 1$



*Although there are many other ways to restrict the domain and obtain a one-to-one function, mathematicians have agreed to use the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to define the inverse of $y = \sin x$.

An equation for the inverse of $y = f(x) = \sin x$ is obtained by interchanging x and y. The implicit form of the inverse function is $x = \sin y$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. The explicit form is called the **inverse sine** of x and is symbolized by $y = f^{-1}(x) = \sin^{-1} x$.

$$y = \sin^{-1} x \quad \text{means} \quad x = \sin y$$

$$\text{where} \quad -1 \le x \le 1 \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2} \tag{1}$$

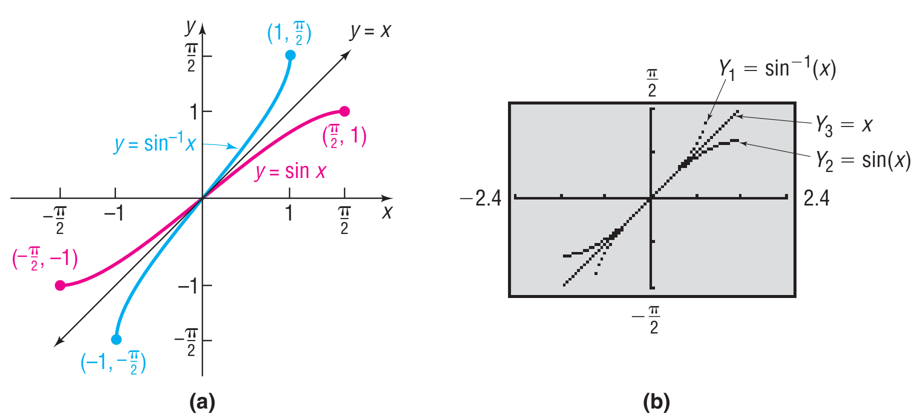
Because $y = \sin^{-1} x$ means $x = \sin y$, we read $y = \sin^{-1} x$ as "y is the angle or real number whose sine equals x." Alternatively, we can say that "y is the inverse sine of x." Be careful about the notation used. The superscript -1 that appears in $y = \sin^{-1} x$ is not an exponent, but is reminiscent of the symbolism f^{-1} used to denote the inverse function of f. (To avoid this notation, some books use the notation $y = \text{Arcsin}\, x$ instead of $y = \sin^{-1} x$.)

The inverse of a function f receives as input an element from the range of f and returns as output an element in the domain of f. The restricted sine function, $y = f(x) = \sin x$, receives as input an angle or real number x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and outputs a real number in the interval $[-1, 1]$. Therefore, the inverse sine function $y = \sin^{-1} x$ receives as input a real number in the interval $[-1, 1]$ or $-1 \le x \le 1$, its domain, and outputs an angle or real number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, its range.

The graph of the inverse sine function can be obtained by reflecting the restricted portion of the graph of $y = f(x) = \sin x$ about the line $y = x$, as shown in Figure 3(a). Figure 3(b) shows the graph using a graphing utility.

Figure 3

$y = \sin^{-1} x$, $-1 \le x \le 1$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



Find the Exact Value of the Inverse Sine Function

For some numbers x, it is possible to find the exact value of $y = \sin^{-1} x$.

EXAMPLE 1 **Finding the Exact Value of a Composite Function**

Find the exact value of: $\sin^{-1} 1$

Solution Let $\theta = \sin^{-1} 1$. We seek the angle, θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 1.

$$\theta = \sin^{-1} 1, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

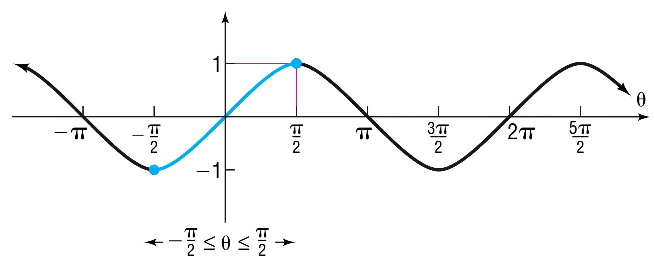
$$\sin\theta = 1, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \qquad \text{By definition of } y = \sin^{-1} x$$

Now look at Table 1 and Figure 4.

Table 1

θ	$\sin\theta$
$-\frac{\pi}{2}$	-1
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$-\frac{\pi}{6}$	$-\frac{1}{2}$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

Figure 4

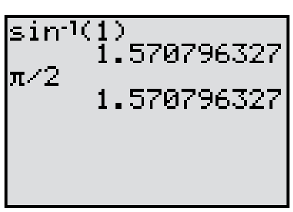


We see that the only angle θ within the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is 1 is $\frac{\pi}{2}$. (Note that $\sin\frac{5\pi}{2}$ also equals 1, but $\frac{5\pi}{2}$ lies outside the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence is not admissible.) So, since $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we conclude that

$$\sin^{-1} 1 = \frac{\pi}{2}$$

✔ **CHECK:** We can verify the solution by evaluating $\sin^{-1} 1$ with our graphing calculator in radian mode. See Figure 5. ◀

Figure 5



For the remainder of the section, the reader is encouraged to verify the solutions obtained using a graphing utility.

NOW WORK PROBLEM 13.

EXAMPLE 2 Finding the Exact Value of an Inverse Sine Function

Find the exact value of: $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution Let $\theta = \sin^{-1}\left(-\frac{1}{2}\right)$. We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\theta = \sin^{-1}\left(-\frac{1}{2}\right), \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\sin\theta = -\frac{1}{2}, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

(Refer to Table 1 and Figure 4, if necessary.) The only angle within the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{1}{2}$ is $-\frac{\pi}{6}$. So, since $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we conclude that

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

NOW WORK PROBLEM 19.

2 Find an Approximate Value of the Inverse Sine Function

For most numbers x, the value $y = \sin^{-1} x$ must be approximated.

EXAMPLE 3 **Finding an Approximate Value of an Inverse Sine Function**

Find an approximate value of:

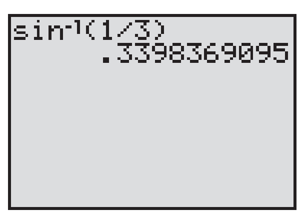
(a) $\sin^{-1}\frac{1}{3}$ (b) $\sin^{-1}\left(-\frac{1}{4}\right)$

Express the answer in radians rounded to two decimal places.

Solution Because we want the angle measured in radians, we first set the mode to radians.

(a) Figure 6(a) shows the solution using a TI-84 Plus graphing calculator.

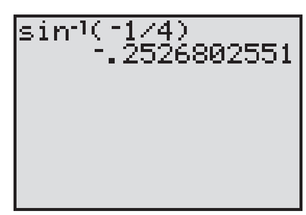
Figure 6a



We have $\sin^{-1}\frac{1}{3} = 0.34$, rounded to two decimal places.

(b) Figure 6(b) shows the solution using a TI-84 Plus graphing calculator.

Figure 6b



We have $\sin^{-1}\left(-\frac{1}{4}\right) = -0.25$, rounded to two decimal places.

NOW WORK PROBLEM 25.

When we discussed functions and their inverses in Section 4.2, we found that $f^{-1}(f(x)) = x$ for all x in the domain of f and $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. In terms of the sine function and its inverse, these properties are of the form

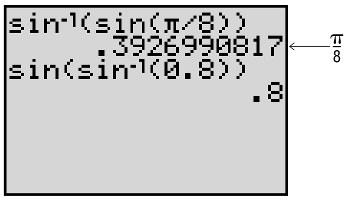
$$f^{-1}(f(x)) = \sin^{-1}(\sin x) = x, \quad \text{where } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \qquad \textbf{(2a)}$$

$$f(f^{-1}(x)) = \sin(\sin^{-1} x) = x, \quad \text{where } -1 \le x \le 1 \qquad \textbf{(2b)}$$

For example, because $\frac{\pi}{8}$ lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the restricted domain of the sine function, we can apply (2a) to get

$$\sin^{-1}\left[\sin\left(\frac{\pi}{8}\right)\right] = \frac{\pi}{8}$$

Figure 7

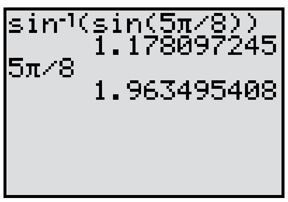


Also, because 0.8 lies in the interval $[-1, 1]$, the domain of the inverse sine function, we can apply (2b) to get

$$\sin[\sin^{-1}(0.8)] = 0.8$$

See Figure 7 for these calculations on a graphing calculator.

Figure 8



See Figure 8. Because $\frac{5\pi}{8}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\sin^{-1}\left[\sin\left(\frac{5\pi}{8}\right)\right] \neq \frac{5\pi}{8}$$

To find $\sin^{-1}\left(\sin\frac{5\pi}{8}\right)$, we use the fact that $\sin\frac{5\pi}{8} = \sin\frac{3\pi}{8}$. Since $\frac{3\pi}{8}$ lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can apply (2a) to get

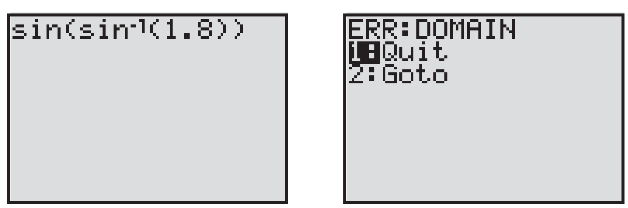
$$\sin^{-1}\left(\sin\frac{5\pi}{8}\right) = \sin^{-1}\left(\sin\frac{3\pi}{8}\right) = \frac{3\pi}{8} \approx 1.178097245$$

Also, because 1.8 is not in the interval $[-1, 1]$,

$$\sin[\sin^{-1}(1.8)] \neq 1.8$$

See Figure 9. Can you explain why the error appears?

Figure 9

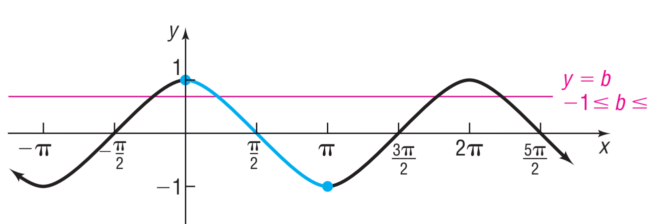


NOW WORK PROBLEM 37.

The Inverse Cosine Function

In Figure 10 we reproduce the graph of $y = \cos x$. Because every horizontal line $y = b$, where b is between -1 and 1, intersects the graph of $y = \cos x$ infinitely many times, it follows that the cosine function is not one-to-one.

Figure 10
$y = \cos x, -\infty < x < \infty,$
$-1 \leq y \leq 1$

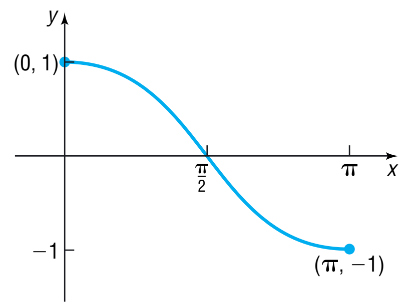


However, if we restrict the domain of $y = \cos x$ to the interval $[0, \pi]$, the restricted function

$$y = \cos x, \qquad 0 \le x \le \pi$$

is one-to-one and hence will have an inverse function.* See Figure 11.

Figure 11
$y = \cos x$, $0 \le x \le \pi$, $-1 \le y \le 1$



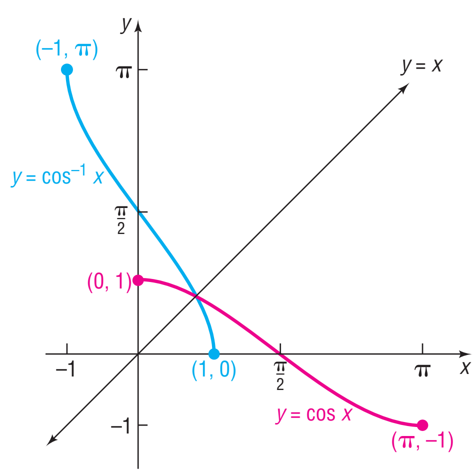
An equation for the inverse of $y = f(x) = \cos x$ is obtained by interchanging x and y. The implicit form of the inverse function is $x = \cos y$, $0 \le y \le \pi$. The explicit form is called the **inverse cosine** of x and is symbolized by $y = f^{-1}(x) = \cos^{-1} x$ (or by $y = \text{Arccos } x$).

$$y = \cos^{-1} x \quad \text{means} \quad x = \cos y$$
$$\text{where} \quad -1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \pi \qquad \textbf{(3)}$$

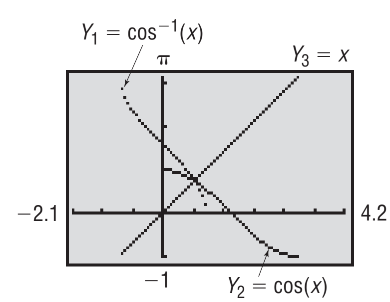
Here y is the angle whose cosine is x. Because the range of the cosine function, $y = \cos x$, is $-1 \le y \le 1$, the domain of the inverse function $y = \cos^{-1} x$ is $-1 \le x \le 1$. Because the restricted domain of the cosine function, $y = \cos x$, is $0 \le x \le \pi$, the range of the inverse function $y = \cos^{-1} x$ is $0 \le y \le \pi$.

The graph of $y = \cos^{-1} x$ can be obtained by reflecting the restricted portion of the graph of $y = \cos x$ about the line $y = x$, as shown in Figure 12.

Figure 12
$y = \cos^{-1} x$, $-1 \le x \le 1$, $0 \le y \le \pi$



(a)



(b)

*This is the generally accepted restriction to define the inverse cosine function.

EXAMPLE 4 Finding the Exact Value of an Inverse Cosine Function

Find the exact value of: $\cos^{-1} 0$

Solution Let $\theta = \cos^{-1} 0$. We seek the angle θ, $0 \leq \theta \leq \pi$, whose cosine equals 0.

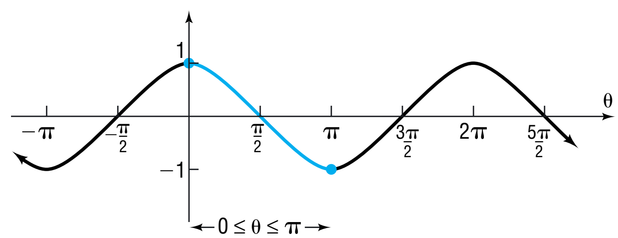
$$\theta = \cos^{-1} 0, \qquad 0 \leq \theta \leq \pi$$
$$\cos\theta = 0, \qquad 0 \leq \theta \leq \pi$$

Look at Table 2 and Figure 13.

Table 2

θ	$\cos\theta$
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$
π	-1

Figure 13



We see that the only angle θ within the interval $[0, \pi]$ whose cosine is 0 is $\frac{\pi}{2}$. (Note that $\cos\frac{3\pi}{2}$ also equals 0, but $\frac{3\pi}{2}$ lies outside the interval $[0, \pi]$ and hence is not admissible.) So, since $\cos\frac{\pi}{2} = 0$ and $\frac{\pi}{2}$ is in the interval $[0, \pi]$, we conclude that

$$\cos^{-1} 0 = \frac{\pi}{2}$$

◀

EXAMPLE 5 Finding the Exact Value of an Inverse Cosine Function

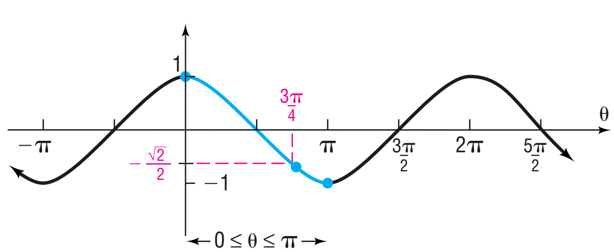
Find the exact value of: $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution Let $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. We seek the angle θ, $0 \leq \theta \leq \pi$, whose cosine equals $-\frac{\sqrt{2}}{2}$.

$$\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right), \qquad 0 \leq \theta \leq \pi$$
$$\cos\theta = -\frac{\sqrt{2}}{2}, \qquad 0 \leq \theta \leq \pi$$

Look at Table 2 and Figure 14.

Figure 14



We see that the only angle θ within the interval $[0, \pi]$, whose cosine is $-\dfrac{\sqrt{2}}{2}$ is $\dfrac{3\pi}{4}$. So, since $\cos\dfrac{3\pi}{4} = -\dfrac{\sqrt{2}}{2}$ and $\dfrac{3\pi}{4}$ is in the interval $[0, \pi]$, we conclude that

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

NOW WORK PROBLEM 23.

For the cosine function and its inverse, the following properties hold:

$$f^{-1}(f(x)) = \cos^{-1}(\cos x) = x, \qquad \text{where } 0 \le x \le \pi \qquad \textbf{(4a)}$$
$$f(f^{-1}(x)) = \cos(\cos^{-1} x) = x, \qquad \text{where } -1 \le x \le 1 \qquad \textbf{(4b)}$$

EXAMPLE 6 **Finding the Exact Value of an Inverse Cosine Function**

Find the exact value of: (a) $\cos^{-1}\left[\cos\left(\dfrac{\pi}{12}\right)\right]$ (b) $\cos[\cos^{-1}(-0.4)]$

Solution (a) $\cos^{-1}\left[\cos\left(\dfrac{\pi}{12}\right)\right] = \dfrac{\pi}{12}$ By Property (4a)

(b) $\cos[\cos^{-1}(-0.4)] = -0.4$ By Property (4b)

NOW WORK PROBLEM 39.

The Inverse Tangent Function

In Figure 15 we reproduce the graph of $y = \tan x$. Because every horizontal line intersects the graph infinitely many times, it follows that the tangent function is not one-to-one.

Figure 15

$y = \tan x$, $-\infty < x < \infty$, x not equal to odd multiples of $\dfrac{\pi}{2}$, $-\infty < y < \infty$

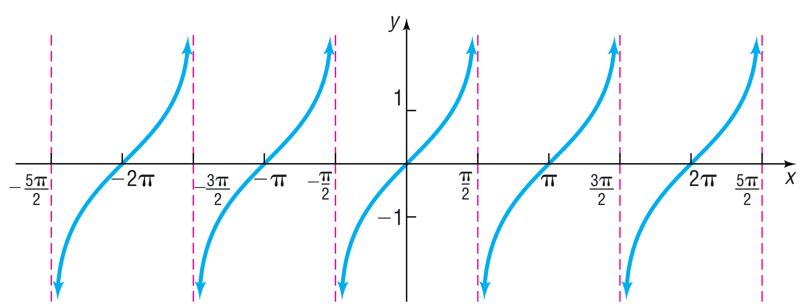
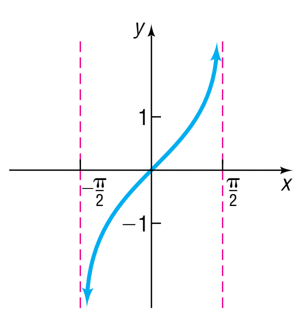


Figure 16

$y = \tan x$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, $-\infty < y < \infty$



However, if we restrict the domain of $y = \tan x$ to the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, the restricted function

$$y = \tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

is one-to-one and hence has an inverse function.* See Figure 16.

*This is the generally accepted restriction.

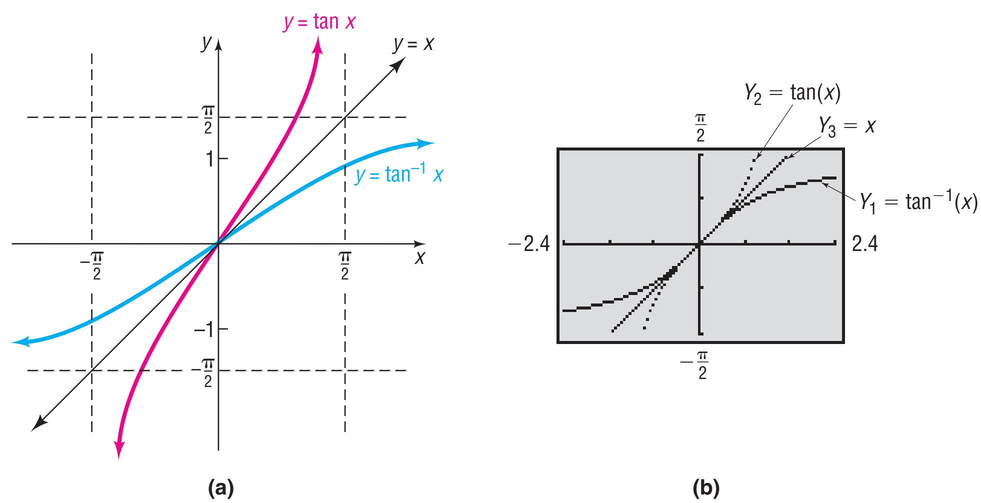
An equation for the inverse of $y = f(x) = \tan x$ is obtained by interchanging x and y. The implicit form of the inverse function is $x = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The explicit form is called the **inverse tangent** of x and is symbolized by $y = f^{-1}(x) = \tan^{-1} x$ (or by $y = \text{Arctan}\, x$).

$$y = \tan^{-1} x \quad \text{means} \quad x = \tan y$$
$$\text{where} \quad -\infty < x < \infty \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \qquad \textbf{(5)}$$

Here y is the angle whose tangent is x. The domain of the function $y = \tan^{-1} x$ is $-\infty < x < \infty$, and its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The graph of $y = \tan^{-1} x$ can be obtained by reflecting the restricted portion of the graph of $y = \tan x$ about the line $y = x$, as shown in Figure 17.

Figure 17

$y = \tan^{-1} x$, $-\infty < x < \infty$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$



EXAMPLE 7 Finding the Exact Value of an Inverse Tangent Function

Find the exact value of: $\tan^{-1} 1$

Solution Let $\theta = \tan^{-1} 1$. We seek the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 1.

$$\theta = \tan^{-1} 1, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\tan \theta = 1, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Look at Table 3. The only angle θ within the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is 1 is $\frac{\pi}{4}$. So, since $\tan \frac{\pi}{4} = 1$ and $\frac{\pi}{4}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we conclude that

$$\tan^{-1} 1 = \frac{\pi}{4}$$

Table 3

θ	$\tan \theta$
$-\frac{\pi}{2}$	Undefined
$-\frac{\pi}{3}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3}$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	Undefined

NOW WORK PROBLEM 17.

EXAMPLE 8 **Finding the Exact Value of an Inverse Tangent Function**

Find the exact value of: $\tan^{-1}(-\sqrt{3})$

Solution Let $\theta = \tan^{-1}(-\sqrt{3})$. We seek the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\sqrt{3}$.

$$\theta = \tan^{-1}(-\sqrt{3}), \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\tan\theta = -\sqrt{3}, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Look at Table 3 or Figure 16 if necessary. The only angle θ within the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $-\sqrt{3}$ is $-\frac{\pi}{3}$. So, since $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$ and $-\frac{\pi}{3}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we conclude that

$$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

For the tangent function and its inverse, the following properties hold:

$$f^{-1}(f(x)) = \tan^{-1}(\tan x) = x, \qquad \text{where } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(f^{-1}(x)) = \tan(\tan^{-1} x) = x, \qquad \text{where } -\infty < x < \infty$$

6.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. What is the domain and the range of $y = \sin x$? (pp. 388–389)
2. A suitable restriction on the domain of the function $f(x) = (x - 1)^2$ to make it one-to-one would be _____. (p. 266)
3. If the domain of a one-to-one function is $[3, \infty)$, then the range of its inverse is _____. (pp. 259–261)
4. *True or False:* The graph of $y = \cos x$ is decreasing on the interval $[0, \pi]$. (p. 406)
5. $\tan\frac{\pi}{4} =$ _____; $\sin\frac{\pi}{3} =$ _____ (p. 379)
6. $\sin\left(-\frac{\pi}{6}\right) =$ _____; $\cos\pi =$ _____ (pp. 374–381)

Concepts and Vocabulary

7. $y = \sin^{-1} x$ means _____, where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
8. The value of $\sin^{-1}\left[\sin\frac{\pi}{2}\right]$ is _____.
9. $\cos^{-1}\left[\cos\frac{\pi}{5}\right] =$ _____.
10. *True or False:* The domain of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
11. *True or False:* $\sin(\sin^{-1} 0) = 0$ and $\cos(\cos^{-1} 0) = 0$.
12. *True or False:* $y = \tan^{-1} x$ means $x = \tan y$, where $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Skill Building

In Problems 13–24, find the exact value of each expression. Verify your results using a graphing utility.

13. $\sin^{-1} 0$ **14.** $\cos^{-1} 1$ **15.** $\sin^{-1}(-1)$ **16.** $\cos^{-1}(-1)$

17. $\tan^{-1} 0$ **18.** $\tan^{-1}(-1)$ **19.** $\sin^{-1} \dfrac{\sqrt{2}}{2}$ **20.** $\tan^{-1} \dfrac{\sqrt{3}}{3}$

21. $\tan^{-1} \sqrt{3}$ **22.** $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ **23.** $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ **24.** $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

In Problems 25–36, use a calculator to find the value of each expression rounded to two decimal places.

25. $\sin^{-1} 0.1$ **26.** $\cos^{-1} 0.6$ **27.** $\tan^{-1} 5$ **28.** $\tan^{-1} 0.2$

29. $\cos^{-1} \dfrac{7}{8}$ **30.** $\sin^{-1} \dfrac{1}{8}$ **31.** $\tan^{-1}(-0.4)$ **32.** $\tan^{-1}(-3)$

33. $\sin^{-1}(-0.12)$ **34.** $\cos^{-1}(-0.44)$ **35.** $\cos^{-1} \dfrac{\sqrt{2}}{3}$ **36.** $\sin^{-1} \dfrac{\sqrt{3}}{5}$

In Problems 37–44, find the exact value of each expression. Do not use a calculator.

37. $\sin[\sin^{-1}(0.54)]$ **38.** $\tan[\tan^{-1}(7.4)]$ **39.** $\cos^{-1}\left[\cos\left(\dfrac{4\pi}{5}\right)\right]$ **40.** $\sin^{-1}\left[\sin\left(-\dfrac{\pi}{10}\right)\right]$

41. $\tan[\tan^{-1}(-3.5)]$ **42.** $\cos[\cos^{-1}(-0.05)]$ **43.** $\sin^{-1}\left[\sin\left(-\dfrac{3\pi}{7}\right)\right]$ **44.** $\tan^{-1}\left[\tan\left(\dfrac{2\pi}{5}\right)\right]$

In Problems 45–56, do not use a calculator. For your answer, also say why or why not.

45. Does $\sin^{-1}\left[\sin\left(-\dfrac{\pi}{6}\right)\right] = -\dfrac{\pi}{6}$? **46.** Does $\sin^{-1}\left[\sin\left(\dfrac{2\pi}{3}\right)\right] = \dfrac{2\pi}{3}$? **47.** Does $\sin[\sin^{-1}(2)] = 2$?

48. Does $\sin\left[\sin^{-1}\left(-\dfrac{1}{2}\right)\right] = -\dfrac{1}{2}$? **49.** Does $\cos^{-1}\left[\cos\left(-\dfrac{\pi}{6}\right)\right] = -\dfrac{\pi}{6}$? **50.** Does $\cos^{-1}\left[\cos\left(\dfrac{2\pi}{3}\right)\right] = \dfrac{2\pi}{3}$?

51. Does $\cos\left[\cos^{-1}\left(-\dfrac{1}{2}\right)\right] = -\dfrac{1}{2}$? **52.** Does $\cos[\cos^{-1}(2)] = 2$? **53.** Does $\tan^{-1}\left[\tan\left(-\dfrac{\pi}{3}\right)\right] = -\dfrac{\pi}{3}$?

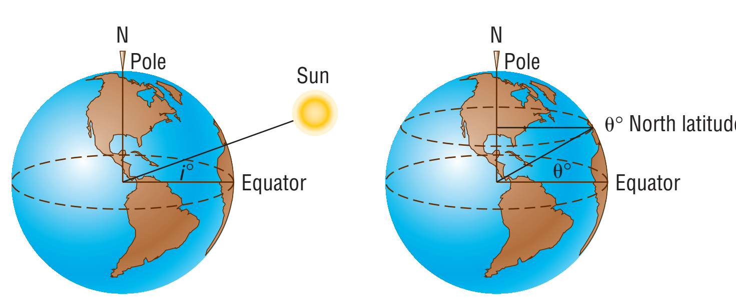
54. Does $\tan^{-1}\left[\tan\left(\dfrac{2\pi}{3}\right)\right] = \dfrac{2\pi}{3}$? **55.** Does $\tan[\tan^{-1}(2)] = 2$? **56.** Does $\tan\left[\tan^{-1}\left(-\dfrac{1}{2}\right)\right] = -\dfrac{1}{2}$?

Applications and Extensions

In Problems 57–62, use the following: The formula

$$D = 24\left[1 - \frac{\cos^{-1}(\tan i \tan\theta)}{\pi}\right]$$

can be used to approximate the number of hours of daylight when the declination of the Sun is $i°$ at a location $\theta°$ north latitude for any date between the vernal equinox and autumnal equinox. The declination of the Sun is defined as the angle i between the equatorial plane and any ray of light from the Sun. The latitude of a location is the angle θ between the Equator and the location on the surface of Earth, with the vertex of the angle located at the center of Earth. See the figure. To use the formula, $\cos^{-1}(\tan i \tan\theta)$ *must be expressed in radians.*



57. Approximate the number of hours of daylight in Houston, Texas (29°45′ north latitude), for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)

58. Approximate the number of hours of daylight in New York, New York (40°45′ north latitude), for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)

59. Approximate the number of hours of daylight in Honolulu, Hawaii (21°18′ north latitude), for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)

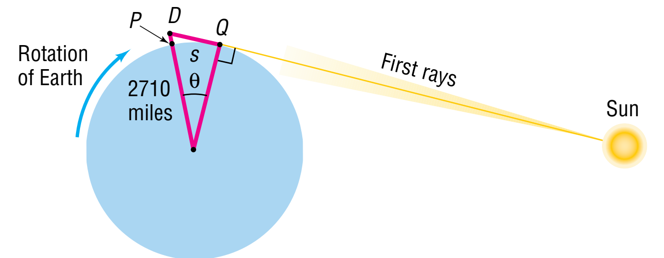
60. Approximate the number of hours of daylight in Anchorage, Alaska (61°10′ north latitude), for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)

61. Approximate the number of hours of daylight at the Equator (0° north latitude) for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)
(d) What do you conclude about the number of hours of daylight throughout the year for a location at the Equator?

62. Approximate the number of hours of daylight for any location that is 66°30′ north latitude for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)
(d) The number of hours of daylight on the winter solstice may be found by computing the number of hours of daylight on the summer solstice and subtracting this result from 24 hours, due to the symmetry of the orbital path of Earth around the Sun. Compute the number of hours of daylight for this location on the winter solstice. What do you conclude about daylight for a location at 66°30′ north latitude?

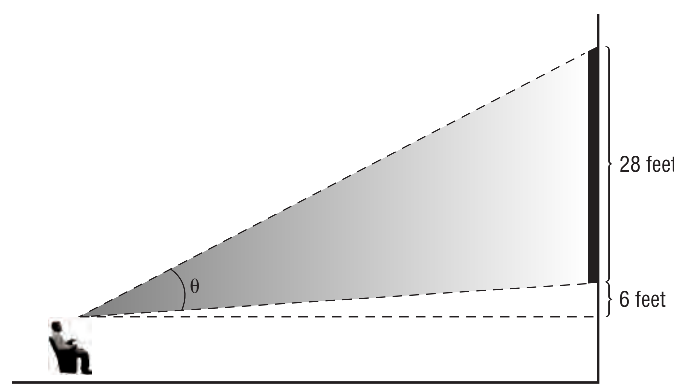
63. Being the First to See the Rising Sun Cadillac Mountain, elevation 1530 feet, is located in Acadia National Park, Maine, and is the highest peak on the east coast of the United States. It is said that a person standing on the summit will be the first person in the United States to see the rays of the rising Sun. How much sooner would a person atop Cadillac Mountain see the first rays than a person standing below, at sea level?

[**Hint:** Consult the figure. When the person at D sees the first rays of the Sun, the person at P does not. The person at P sees the first rays of the Sun only after Earth has rotated so that P is at location Q. Compute the length of the arc subtended by the central angle θ. Then use the fact that, at the latitude of Cadillac Mountain, in 24 hours a length of 2π (2710) miles is subtended, and find the time that it takes to subtend this length.]



64. The Movie Theater Suppose that a movie theater has a screen that is 28 feet tall. When you sit down, the bottom of the screen is 6 feet above your eye level. The angle formed by drawing a line from your eye to the bottom of the screen and your eye and the top of the screen is called the **viewing angle**. In the figure θ is the viewing angle.Suppose that you sit x feet from the screen. The viewing angle θ is given by the equation

$$\theta = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right).$$



(a) What is your viewing angle if you sit 10 feet from the screen? 15 feet? 20 feet?
(b) Using a graphing utility, graph

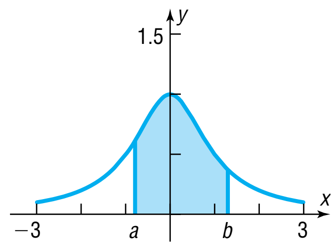
$$\theta = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right).$$

What value of x results in maximizing the viewing angle?
(c) If there is 5 feet between the screen and the first row of seats and there is 3 feet between each row, which row results in the optimal viewing angle?

65. Area under a Curve The area under the graph of $y = \dfrac{1}{1 + x^2}$ and above the x-axis between $x = a$ and $x = b$ is given by

$$\tan^{-1} b - \tan^{-1} a$$

See the figure.



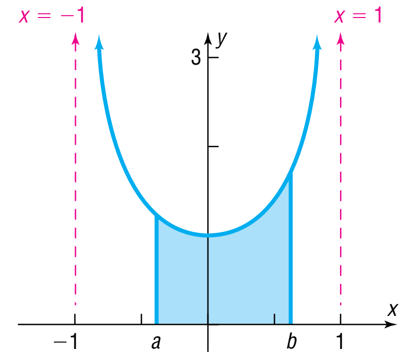
(a) Find the exact area under the graph of $y = \dfrac{1}{1 + x^2}$ and above the x-axis between $x = 0$ and $x = \sqrt{3}$.

(b) Find the exact area under the graph of $y = \dfrac{1}{1 + x^2}$ and above the x-axis between $x = -\dfrac{\sqrt{3}}{3}$ and $x = 1$.

66. Area under a Curve The area under the graph of $y = \dfrac{1}{\sqrt{1 - x^2}}$ and above the x-axis between $x = a$ and $x = b$ is given by

$$\sin^{-1} b - \sin^{-1} a$$

See the figure.



(a) Find the exact area under the graph of $y = \dfrac{1}{\sqrt{1 - x^2}}$ and above the x-axis between $x = 0$ and $x = \dfrac{\sqrt{3}}{2}$.

(b) Find the exact area under the graph of $y = \dfrac{1}{\sqrt{1 - x^2}}$ and above the x-axis between $x = -\dfrac{1}{2}$ and $x = \dfrac{1}{2}$.

'Are You Prepared?' Answers

1. domain: $-\infty < x < \infty$; range: $-1 \le y \le 1$ **2.** Two answers are possible: $x \le -1$ or $x \ge 1$ **3.** $[3, \infty)$ **4.** True **5.** $1; \dfrac{\sqrt{3}}{2}$ **6.** $-\dfrac{1}{2}; -1$

6.2 The Inverse Trigonometric Functions (Continued)

PREPARING FOR THIS SECTION *Before getting started, review the following concepts:*

- Finding Exact Values Given the Value of a Trigonometric Function and the Quadrant of the Angle (Section 5.3, pp. 395–398)
- Graphs of the Secant, Cosecant, and Cotangent Functions (Section 5.5, pp. 422–423)
- Domain and Range of the Secant, Cosecant, and Cotangent Functions (Section 5.3, p. 388–390)

Now work the 'Are You Prepared?' problems on page 464.

OBJECTIVES
1. Find the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions
2. Know the Definition of the Inverse Secant, Cosecant, and Cotangent Functions
3. Use a Calculator to Evaluate $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$

1 Find the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

EXAMPLE 1 Finding the Exact Value of Expressions Involving Inverse Trigonometric Functions

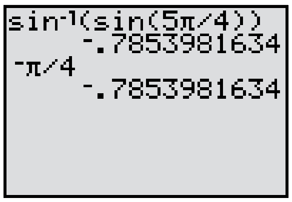
Find the exact value of: $\sin^{-1}\left(\sin \dfrac{5\pi}{4}\right)$

Solution

$$\sin^{-1}\left(\sin \frac{5\pi}{4}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

✔ **CHECK:** We can verify the solution by evaluating $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$ with our graphing calculator in radian mode. See Figure 18. ◀

Figure 18



Notice in the solution to Example 1 that we did not use Property (2a), page 451. This is because the argument of the sine function is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as required. If we use the fact that

$$\sin\frac{5\pi}{4} = \sin\left(-\frac{\pi}{4}\right)$$

then we can use Property (2a):

$$\sin^{-1}\left(\sin\frac{5\pi}{4}\right) = \sin^{-1}\left[\sin\left(-\frac{\pi}{4}\right)\right] \underset{\substack{\uparrow \\ \text{Property (2a)}}}{=} -\frac{\pi}{4}$$

For the remainder of the section, the reader is encouraged to verify the solutions obtained using a graphing utility.

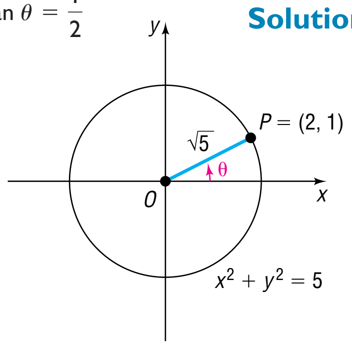
NOW WORK PROBLEM 23.

EXAMPLE 2 Finding the Exact Value of Expressions Involving Inverse Trigonometric Functions

Find the exact value of: $\sin\left(\tan^{-1}\frac{1}{2}\right)$

Figure 19

$\tan\theta = \frac{1}{2}$



Solution

Let $\theta = \tan^{-1}\frac{1}{2}$. Then $\tan\theta = \frac{1}{2}$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We seek $\sin\theta$. Because $\tan\theta > 0$, it follows that $0 < \theta < \frac{\pi}{2}$, so θ lies in quadrant I. Since $\tan\theta = \frac{1}{2} = \frac{y}{x}$, we let $x = 2$ and $y = 1$. Since $r = d(O, P) = \sqrt{2^2 + 1^2} = \sqrt{5}$, the point $P = (x, y) = (2, 1)$ is on the circle $x^2 + y^2 = 5$. See Figure 19. Then, with $x = 2$, $y = 1$, and $r = \sqrt{5}$, we have

$$\sin\left(\tan^{-1}\frac{1}{2}\right) = \sin\theta \underset{\substack{\uparrow \\ \sin\theta = \frac{y}{r}}}{=} \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

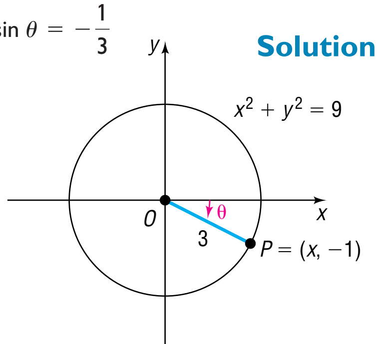
◀

EXAMPLE 3 Finding the Exact Value of Expressions Involving Inverse Trigonometric Functions

Find the exact value of: $\cos\left[\sin^{-1}\left(-\frac{1}{3}\right)\right]$

Figure 20

$\sin\theta = -\frac{1}{3}$



Solution

Let $\theta = \sin^{-1}\left(-\frac{1}{3}\right)$. Then $\sin\theta = -\frac{1}{3}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We seek $\cos\theta$. Because $\sin\theta < 0$, it follows that $-\frac{\pi}{2} \le \theta < 0$, so θ lies in quadrant IV. Since $\sin\theta = \frac{-1}{3} = \frac{y}{r}$, we let $y = -1$ and $r = 3$. The point $P = (x, y) = (x, -1)$, $x > 0$, is on a circle of radius 3, namely, $x^2 + y^2 = 9$. See Figure 20. Then,

$$\begin{aligned} x^2 + y^2 &= 9 \\ x^2 + (-1)^2 &= 9 && y = -1 \\ x^2 &= 8 \\ x &= 2\sqrt{2} && x > 0 \end{aligned}$$

Then we have $x = 2\sqrt{2}$, $y = -1$, $r = 3$, so that

$$\cos\left[\sin^{-1}\left(-\frac{1}{3}\right)\right] = \underset{\uparrow\ \cos\theta = \frac{x}{r}}{\cos\theta} = \frac{2\sqrt{2}}{3}$$

◀

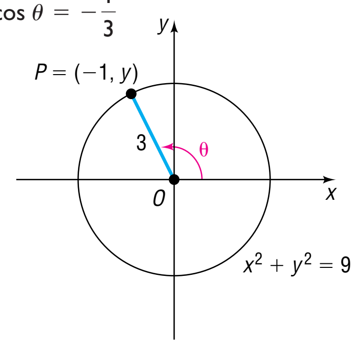
EXAMPLE 4 Finding the Exact Value of Expressions Involving Inverse Trigonometric Functions

Find the exact value of: $\tan\left[\cos^{-1}\left(-\frac{1}{3}\right)\right]$

Solution Let $\theta = \cos^{-1}\left(-\frac{1}{3}\right)$. Then $\cos\theta = -\frac{1}{3}$ and $0 \le \theta \le \pi$. We seek $\tan\theta$. Because $\cos\theta < 0$, it follows that $\frac{\pi}{2} < \theta \le \pi$, so θ lies in quadrant II. Since $\cos\theta = \frac{-1}{3} = \frac{x}{r}$, we let $x = -1$ and $r = 3$. The point $P = (x, y) = (-1, y)$, $y > 0$, is on a circle of radius $r = 3$, namely, $x^2 + y^2 = 9$. See Figure 21. Then,

Figure 21

$\cos\theta = -\frac{1}{3}$



$$\begin{aligned} x^2 + y^2 &= 9 \\ (-1)^2 + y^2 &= 9 \qquad x = -1 \\ y^2 &= 8 \\ y &= 2\sqrt{2} \qquad y > 0 \end{aligned}$$

Then, we have $x = -1$, $y = 2\sqrt{2}$, and $r = 3$, so that

$$\tan\left[\cos^{-1}\left(-\frac{1}{3}\right)\right] = \underset{\uparrow\ \tan\theta = \frac{y}{x}}{\tan\theta} = \frac{2\sqrt{2}}{-1} = -2\sqrt{2}$$

◀

NOW WORK PROBLEMS 9 AND 27.

Look at Example 1 and Examples 2–4. What differences do you see?

2 Know the Definition of the Inverse Secant, Cosecant, and Cotangent Functions

The inverse secant, inverse cosecant, and inverse cotangent functions are defined as follows:

$$y = \sec^{-1} x \quad \text{means} \quad x = \sec y$$
$$\text{where} \quad |x| \ge 1 \quad \text{and} \quad 0 \le y \le \pi, \quad y \ne \frac{\pi}{2}{}^{*} \qquad \textbf{(1)}$$

$$y = \csc^{-1} x \quad \text{means} \quad x = \csc y$$
$$\text{where} \quad |x| \ge 1 \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad y \ne 0^{\dagger} \qquad \textbf{(2)}$$

$$y = \cot^{-1} x \quad \text{means} \quad x = \cot y$$
$$\text{where} \quad -\infty < x < \infty \quad \text{and} \quad 0 < y < \pi \qquad \textbf{(3)}$$

*Most books use this definition. A few use the restriction $0 \le y < \frac{\pi}{2}, \pi \le y < \frac{3\pi}{2}$.

†Most books use this definition. A few use the restriction $-\pi < y \le -\frac{\pi}{2}, 0 < y \le \frac{\pi}{2}$.

You are encouraged to review the graphs of the cotangent, cosecant, and secant functions in Figures 77, 78, and 80 in Section 5.5 to help you to see the basis for these definitions.

EXAMPLE 5 Finding the Exact Value of an Inverse Cosecant Function

Find the exact value of: $\csc^{-1} 2$

Solution Let $\theta = \csc^{-1} 2$. We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose cosecant equals 2 $\left(\text{or, equivalently, whose sine equals } \frac{1}{2}\right)$.

$$\theta = \csc^{-1} 2, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad \theta \ne 0$$
$$\csc\theta = 2, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad \theta \ne 0$$

The only angle θ in the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose cosecant is 2, is $\frac{\pi}{6}$, so $\csc^{-1} 2 = \frac{\pi}{6}$.

NOW WORK PROBLEM 39.

3 Use a Calculator to Evaluate $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$

Most calculators do not have keys for evaluating the inverse cotangent, cosecant, and secant functions. The easiest way to evaluate them is to convert to an inverse trigonometric function whose range is the same as the one to be evaluated. In this regard, notice that $y = \cot^{-1} x$ and $y = \sec^{-1} x$, except where undefined, each have the same range as $y = \cos^{-1} x$; $y = \csc^{-1} x$, except where undefined, has the same range as $y = \sin^{-1} x$.

EXAMPLE 6 Approximating the Value of Inverse Trigonometric Functions

Use a calculator to approximate each expression in radians rounded to two decimal places.

(a) $\sec^{-1} 3$ (b) $\csc^{-1}(-4)$ (c) $\cot^{-1} \frac{1}{2}$ (d) $\cot^{-1}(-2)$

Solution First, set your calculator to radian mode.

(a) Let $\theta = \sec^{-1} 3$. Then $\sec\theta = 3$ and $0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$. We seek $\cos\theta$ because $y = \cos^{-1}x$ has the same range as $y = \sec^{-1}x$, except where undefined. Since $\sec\theta = \frac{1}{\cos\theta} = 3$, we have $\cos\theta = \frac{1}{3}$. Then $\theta = \cos^{-1}\frac{1}{3}$ and

$$\sec^{-1} 3 = \theta = \cos^{-1}\frac{1}{3} \approx 1.23$$

Use a calculator.

(b) Let $\theta = \csc^{-1}(-4)$. Then $\csc\theta = -4$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$. We seek $\sin\theta$ because $y = \sin^{-1}x$ has the same range as $y = \csc^{-1}x$, except where undefined. Since $\csc\theta = \frac{1}{\sin\theta} = -4$, we have $\sin\theta = -\frac{1}{4}$. Then $\theta = \sin^{-1}\left(-\frac{1}{4}\right)$, and

$$\csc^{-1}(-4) = \theta = \sin^{-1}\left(-\frac{1}{4}\right) \approx -0.25$$

Figure 22 $\cot\theta = \frac{1}{2}, 0 < \theta < \pi$

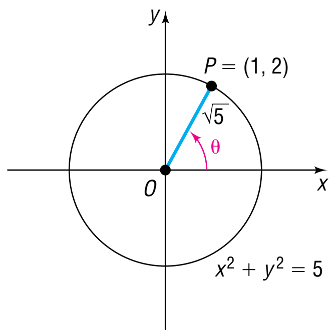
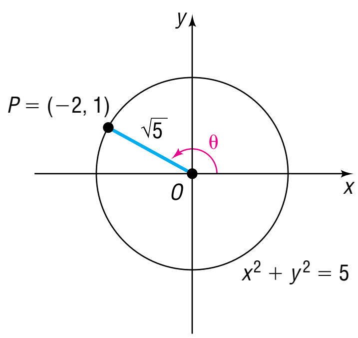


Figure 23 $\cot\theta = -2, 0 < \theta < \pi$



(c) There is no inverse cotangent key on the calculator. We proceed as before. Let $\theta = \cot^{-1}\frac{1}{2}$. Then $\cot\theta = \frac{1}{2}, 0 < \theta < \pi$. From these facts we know that θ lies in quadrant I. We seek $\cos\theta$ because $y = \cos^{-1}x$ has the same range as $y = \cot^{-1}x$, except where undefined. To find $\cos\theta$ we use Figure 22. Then

$$\cos\theta = \frac{1}{\sqrt{5}},\ 0 < \theta < \frac{\pi}{2},\ \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right), \text{ and}$$

$$\cot^{-1}\frac{1}{2} = \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 1.11$$

(d) Let $\theta = \cot^{-1}(-2)$. Then $\cot\theta = -2, 0 < \theta < \pi$. From these facts we know that θ lies in quadrant II. We seek $\cos\theta$. To find it we use Figure 23. Then

$$\cos\theta = -\frac{2}{\sqrt{5}},\ \frac{\pi}{2} < \theta < \pi,\ \theta = \cos^{-1}\left(-\frac{2}{\sqrt{5}}\right), \text{ and}$$

$$\cot^{-1}(-2) = \theta = \cos^{-1}\left(-\frac{2}{\sqrt{5}}\right) \approx 2.68$$

NOW WORK PROBLEM 45.

6.2 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. What is the domain and the range of $y = \sec x$? (pp. 388–390)

2. *True or False:* The graph of $y = \sec x$ is increasing on the interval $\left[0, \frac{\pi}{2}\right)$ and on the interval $\left(\frac{\pi}{2}, \pi\right]$. (p. 423)

3. If $\cot\theta = -2$ and $0 < \theta < \pi$, then $\cos\theta =$ _____. (pp. 395–398)

Concepts and Vocabulary

4. $y = \sec^{-1}x$ means _____, where $|x|$ _____ and _____ $\leq y \leq$ ____, $y \neq \frac{\pi}{2}$.

5. $\cos(\tan^{-1}1) =$ ____.

6. *True or False:* It is impossible to obtain exact values for the inverse secant function.

7. *True or False:* $\csc^{-1}0.5$ is not defined.

8. *True or False:* The domain of the inverse cotangent function is the set of real numbers.

Skill Building

In Problems 9–36, find the exact value of each expression. Verify your results using a graphing utility.

9. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

10. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

11. $\tan\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

12. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

13. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

14. $\cot\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

15. $\csc(\tan^{-1}1)$

16. $\sec(\tan^{-1}\sqrt{3})$

17. $\sin[\tan^{-1}(-1)]$

18. $\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

19. $\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

20. $\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

21. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$

22. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

23. $\sin^{-1}\left[\sin\left(-\frac{7\pi}{6}\right)\right]$

24. $\cos^{-1}\left[\cos\left(-\frac{\pi}{3}\right)\right]$

25. $\tan\left(\sin^{-1}\dfrac{1}{3}\right)$ **26.** $\tan\left(\cos^{-1}\dfrac{1}{3}\right)$ **27.** $\sec\left(\tan^{-1}\dfrac{1}{2}\right)$ **28.** $\cos\left(\sin^{-1}\dfrac{\sqrt{2}}{3}\right)$

29. $\cot\left[\sin^{-1}\left(-\dfrac{\sqrt{2}}{3}\right)\right]$ **30.** $\csc[\tan^{-1}(-2)]$ **31.** $\sin[\tan^{-1}(-3)]$ **32.** $\cot\left[\cos^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)\right]$

33. $\sec\left(\sin^{-1}\dfrac{2\sqrt{5}}{5}\right)$ **34.** $\csc\left(\tan^{-1}\dfrac{1}{2}\right)$ **35.** $\sin^{-1}\left(\cos\dfrac{3\pi}{4}\right)$ **36.** $\cos^{-1}\left(\sin\dfrac{7\pi}{6}\right)$

In Problems 37–44, find the exact value of each expression. Verify your results using a graphing utility.

37. $\cot^{-1}\sqrt{3}$ **38.** $\cot^{-1} 1$ **39.** $\csc^{-1}(-1)$ **40.** $\csc^{-1}\sqrt{2}$

41. $\sec^{-1}\dfrac{2\sqrt{3}}{3}$ **42.** $\sec^{-1}(-2)$ **43.** $\cot^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$ **44.** $\csc^{-1}\left(-\dfrac{2\sqrt{3}}{3}\right)$

In Problems 45–56, use a calculator to find the value of each expression rounded to two decimal places.

45. $\sec^{-1} 4$ **46.** $\csc^{-1} 5$ **47.** $\cot^{-1} 2$ **48.** $\sec^{-1}(-3)$

49. $\csc^{-1}(-3)$ **50.** $\cot^{-1}\left(-\dfrac{1}{2}\right)$ **51.** $\cot^{-1}(-\sqrt{5})$ **52.** $\cot^{-1}(-8.1)$

53. $\csc^{-1}\left(-\dfrac{3}{2}\right)$ **54.** $\sec^{-1}\left(-\dfrac{4}{3}\right)$ **55.** $\cot^{-1}\left(-\dfrac{3}{2}\right)$ **56.** $\cot^{-1}(-\sqrt{10})$

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57. Using a graphing utility, graph $y = \cot^{-1} x$.

58. Using a graphing utility, graph $y = \sec^{-1} x$.

59. Using a graphing utility, graph $y = \csc^{-1} x$.

Discussion and Writing

60. Explain in your own words how you would use your calculator to find the value of $\cot^{-1} 10$.

61. Consult three books on calculus and write down the definition in each of $y = \sec^{-1} x$ and $y = \csc^{-1} x$. Compare these with the definition given in this book.

'Are You Prepared?' Answers

1. Domain: $\left\{x \mid x \neq \text{odd multiples of } \dfrac{\pi}{2}\right\}$; Range: $\{y \leq -1 \text{ or } y \geq 1\}$ **2.** True **3.** $\dfrac{-2\sqrt{5}}{5}$

6.3 Trigonometric Identities

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Fundamental Identities (Section 5.3, p. 394)

Now work the 'Are You Prepared?' problems on page 471.

OBJECTIVES 1 Use Algebra to Simplify Trigonometric Expressions
2 Establish Identities

We saw in the previous chapter that the trigonometric functions lend themselves to a wide variety of identities. Before establishing some additional identities, let's review the definition of an *identity*.

Two functions f and g are said to be **identically equal** if

$$\boxed{f(x) = g(x)}$$

for every value of x for which both functions are defined. Such an equation is referred to as an **identity**. An equation that is not an identity is called a **conditional equation**.

For example, the following are identities:

$$(x+1)^2 = x^2 + 2x + 1 \qquad \sin^2 x + \cos^2 x = 1 \qquad \csc x = \frac{1}{\sin x}$$

The following are conditional equations:

$2x + 5 = 0$ True only if $x = -\frac{5}{2}$

$\sin x = 0$ True only if $x = k\pi$, k an integer

$\sin x = \cos x$ True only if $x = \frac{\pi}{4} + 2k\pi$ or $x = \frac{5\pi}{4} + 2k\pi$, k an integer

The following boxes summarize the trigonometric identities that we have established thus far.

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

This list of identities comprises what we shall refer to as the **basic trigonometric identities**. These identities should not merely be memorized, but should be *known* (just as you know your name rather than have it memorized). In fact, minor variations of a basic identity are often used. For example, we might want to use

$$\sin^2\theta = 1 - \cos^2\theta \quad \text{or} \quad \cos^2\theta = 1 - \sin^2\theta$$

instead of $\sin^2\theta + \cos^2\theta = 1$. For this reason, among others, you need to know these relationships and be quite comfortable with variations of them.

1 Use Algebra to Simplify Trigonometric Expressions

The ability to use algebra to manipulate trigonometric expressions is a key skill that one must have to establish identities. Some of the techniques that are used in establishing identities are multiplying by a "well-chosen 1," writing a trigonometric expression over a common denominator, rewriting a trigonometric expression in terms of sine and cosine only, and factoring.

EXAMPLE 1 Using Algebraic Techniques to Simplify Trigonometric Expressions

(a) Simplify $\dfrac{\cot\theta}{\csc\theta}$ by rewriting each trigonometric function in terms of sine and cosine.

(b) Show that $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$ by multiplying the numerator and denominator by $1-\sin\theta$.

(c) Simplify $\dfrac{1+\sin\theta}{\sin\theta} + \dfrac{\cot\theta - \cos\theta}{\cos\theta}$ by rewriting the expression over a common denominator.

(d) Simplify $\dfrac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta}$ by factoring.

Solution

(a) $$\frac{\cot\theta}{\csc\theta} = \frac{\dfrac{\cos\theta}{\sin\theta}}{\dfrac{1}{\sin\theta}} = \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1} = \cos\theta$$

(b) $$\frac{\cos\theta}{1+\sin\theta} \underset{\uparrow}{=} \frac{\cos\theta}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta} = \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} = \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} = \frac{1-\sin\theta}{\cos\theta}$$

Well-chosen 1: $\dfrac{1-\sin\theta}{1-\sin\theta}$

(c) $$\frac{1+\sin\theta}{\sin\theta} + \frac{\cot\theta-\cos\theta}{\cos\theta} = \frac{1+\sin\theta}{\sin\theta}\cdot\frac{\cos\theta}{\cos\theta} + \frac{\cot\theta-\cos\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}$$

$$= \frac{\cos\theta + \sin\theta\cos\theta + \cot\theta\sin\theta - \cos\theta\sin\theta}{\sin\theta\cos\theta} \underset{\uparrow}{=} \frac{\cos\theta + \dfrac{\cos\theta}{\sin\theta}\cdot\sin\theta}{\sin\theta\cos\theta}$$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$$= \frac{\cos\theta + \cos\theta}{\sin\theta\cos\theta} = \frac{2\cos\theta}{\sin\theta\cos\theta} = \frac{2}{\sin\theta}$$

(d) $$\frac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta} = \frac{(\sin\theta+1)(\sin\theta-1)}{\tan\theta(\sin\theta-1)} = \frac{\sin\theta+1}{\tan\theta}$$

◀

NOW WORK PROBLEMS 9, 11, AND 13.

2 Establish Identities

In the examples that follow, the directions will read "Establish the identity. . . ." As you will see, this is accomplished by starting with one side of the given equation (usually the one containing the more complicated expression) and, using appropriate basic identities and algebraic manipulations, arriving at the other side. The selection of appropriate basic identities to obtain the desired result is learned only through experience and lots of practice.

EXAMPLE 2 Establishing an Identity

Establish the identity: $\csc\theta\cdot\tan\theta = \sec\theta$

Solution We start with the left side, because it contains the more complicated expression, and apply a reciprocal identity and a quotient identity.

$$\csc\theta\cdot\tan\theta = \frac{1}{\cancel{\sin\theta}}\cdot\frac{\cancel{\sin\theta}}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$$

Having arrived at the right side, the identity is established. ◀

NOTE A graphing utility can be used to provide evidence of an identity. For example, if we graph $Y_1 = \csc\theta \cdot \tan\theta$ and $Y_2 = \sec\theta$, the graphs appear to be the same. This provides evidence that $Y_1 = Y_2$. However, it does not prove their equality. A graphing utility *cannot be used to establish an identity*—identities must be established algebraically.

NOW WORK PROBLEM 19.

EXAMPLE 3 Establishing an Identity

Establish the identity: $\sin^2(-\theta) + \cos^2(-\theta) = 1$

Solution We begin with the left side and apply Even–Odd Identities.

$$\begin{aligned}
\sin^2(-\theta) + \cos^2(-\theta) &= [\sin(-\theta)]^2 + [\cos(-\theta)]^2 \\
&= (-\sin\theta)^2 + (\cos\theta)^2 && \text{Even–Odd Identities} \\
&= (\sin\theta)^2 + (\cos\theta)^2 \\
&= 1 && \text{Pythagorean Identity}
\end{aligned}$$

EXAMPLE 4 Establishing an Identity

Establish the identity: $\dfrac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$

Solution We begin with two observations: The left side contains the more complicated expression. Also, the left side contains expressions with the argument $-\theta$, whereas the right side contains expressions with the argument θ. We decide, therefore, to start with the left side and apply Even–Odd Identities.

$$\begin{aligned}
\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} &= \frac{[\sin(-\theta)]^2 - [\cos(-\theta)]^2}{\sin(-\theta) - \cos(-\theta)} \\
&= \frac{(-\sin\theta)^2 - (\cos\theta)^2}{-\sin\theta - \cos\theta} && \text{Even–Odd Identities} \\
&= \frac{(\sin\theta)^2 - (\cos\theta)^2}{-\sin\theta - \cos\theta} && \text{Simplify.} \\
&= \frac{(\sin\theta - \cos\theta)\cancel{(\sin\theta + \cos\theta)}}{-\cancel{(\sin\theta + \cos\theta)}} && \text{Factor.} \\
&= \cos\theta - \sin\theta && \text{Cancel and simplify.}
\end{aligned}$$

EXAMPLE 5 Establishing an Identity

Establish the identity: $\dfrac{1 + \tan\theta}{1 + \cot\theta} = \tan\theta$

Solution

$$\frac{1 + \tan\theta}{1 + \cot\theta} = \frac{1 + \tan\theta}{1 + \dfrac{1}{\tan\theta}} = \frac{1 + \tan\theta}{\dfrac{\tan\theta + 1}{\tan\theta}} = \frac{\tan\theta\cancel{(1 + \tan\theta)}}{\cancel{\tan\theta + 1}} = \tan\theta$$

NOW WORK PROBLEM 27.

When sums or differences of quotients appear, it is usually best to rewrite them as a single quotient, especially if the other side of the identity consists of only one term.

EXAMPLE 6 Establishing an Identity

Establish the identity: $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\csc\theta$

Solution The left side is more complicated, so we start with it and proceed to add.

$$\begin{aligned}
\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} &= \frac{\sin^2\theta + (1+\cos\theta)^2}{(1+\cos\theta)(\sin\theta)} && \text{Add the quotients.}\\
&= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{(1+\cos\theta)(\sin\theta)} && \text{Remove parentheses in the numerator.}\\
&= \frac{(\sin^2\theta + \cos^2\theta) + 1 + 2\cos\theta}{(1+\cos\theta)(\sin\theta)} && \text{Regroup.}\\
&= \frac{2 + 2\cos\theta}{(1+\cos\theta)(\sin\theta)} && \text{Pythagorean identity}\\
&= \frac{2\cancel{(1+\cos\theta)}}{\cancel{(1+\cos\theta)}(\sin\theta)} && \text{Factor and cancel.}\\
&= \frac{2}{\sin\theta}\\
&= 2\csc\theta && \text{Reciprocal Identity}
\end{aligned}$$

◀

Sometimes it helps to write one side in terms of sines and cosines only.

EXAMPLE 7 Establishing an Identity

Establish the identity: $\dfrac{\tan\theta + \cot\theta}{\sec\theta\csc\theta} = 1$

Solution

$$\frac{\tan\theta + \cot\theta}{\sec\theta\csc\theta} \underset{\substack{\uparrow\\ \text{Change to sines}\\ \text{and cosines.}}}{=} \frac{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}}{\dfrac{1}{\cos\theta}\cdot\dfrac{1}{\sin\theta}} \underset{\substack{\uparrow\\ \text{Add the quotients}\\ \text{in the numerator.}}}{=} \frac{\dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}}{\dfrac{1}{\cos\theta\sin\theta}}$$

$$\underset{\substack{\uparrow\\ \text{Divide the quotients;}\\ \sin^2\theta + \cos^2\theta = 1.}}{=} \frac{1}{\cos\theta\sin\theta}\cdot\frac{\cos\theta\sin\theta}{1} = 1$$

◀

NOW WORK PROBLEM 69.

Sometimes, multiplying the numerator and denominator by an appropriate factor will result in a simplification.

EXAMPLE 8 Establishing an Identity

Establish the identity: $\dfrac{1 - \sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1 + \sin\theta}$

Solution We start with the left side and multiply the numerator and the denominator by $1 + \sin\theta$. (Alternatively, we could multiply the numerator and denominator of the right side by $1 - \sin\theta$.)

$$\begin{aligned}
\frac{1 - \sin\theta}{\cos\theta} &= \frac{1 - \sin\theta}{\cos\theta} \cdot \frac{1 + \sin\theta}{1 + \sin\theta} && \text{Multiply the numerator and denominator by } 1 + \sin\theta. \\
&= \frac{1 - \sin^2\theta}{\cos\theta(1 + \sin\theta)} \\
&= \frac{\cos^2\theta}{\cos\theta(1 + \sin\theta)} && 1 - \sin^2\theta = \cos^2\theta \\
&= \frac{\cos\theta}{1 + \sin\theta} && \text{Cancel.}
\end{aligned}$$

◀

NOW WORK PROBLEM 53.

EXAMPLE 9 Establishing an Identity Involving Inverse Trigonometric Functions

Show that $\sin(\tan^{-1} v) = \dfrac{v}{\sqrt{1 + v^2}}$.

Solution Let $\theta = \tan^{-1} v$ so that $\tan\theta = v$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. As a result, we know that $\sec\theta > 0$.

$$\sin(\tan^{-1} v) = \sin\theta \underset{\uparrow}{=} \sin\theta \cdot \frac{\cos\theta}{\cos\theta} \underset{\uparrow}{=} \tan\theta\cos\theta = \frac{\tan\theta}{\sec\theta} \underset{\uparrow}{=} \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} = \frac{v}{\sqrt{1 + v^2}}$$

Multiply by 1: $\dfrac{\cos\theta}{\cos\theta}$. $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ $\sec^2\theta = 1 + \tan^2\theta$, $\sec\theta > 0$

◀

NOW WORK PROBLEM 99.

WARNING Be careful not to handle identities to be established as if they were conditional equations. You *cannot* establish an identity by such methods as adding the same expression to each side and obtaining a true statement. This practice is not allowed, because the original statement is precisely the one that you are trying to establish. You do not know until it has been established that it is, in fact, true. ■

Although a lot of practice is the only real way to learn how to establish identities, the following guidelines should prove helpful.

Guidelines for Establishing Identities

1. It is almost always preferable to start with the side containing the more complicated expression.
2. Rewrite sums or differences of quotients as a single quotient.
3. Sometimes rewriting one side in terms of sines and cosines only will help.
4. Always keep your goal in mind. As you manipulate one side of the expression, you must keep in mind the form of the expression on the other side.

6.3 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. *True or False:* $\sin^2\theta = 1 - \cos^2\theta$. (p. 394)

2. *True or False:* $\sin(-\theta) + \cos(-\theta) = \cos\theta - \sin\theta$. (p. 398)

Concepts and Vocabulary

3. Suppose that f and g are two functions with the same domain. If $f(x) = g(x)$ for every x in the domain, the equation is called a(n) _____. Otherwise, it is called a(n) _____ equation.

4. $\tan^2\theta - \sec^2\theta =$ ____.

5. $\cos(-\theta) - \cos\theta =$ ____.

6. *True or False:* $\sin(-\theta) + \sin\theta = 0$ for any value of θ.

7. *True or False:* In establishing an identity, it is often easiest to just multiply both sides by a well-chosen nonzero expression involving the variable.

8. *True or False:* $\tan\theta \cdot \cos\theta = \sin\theta$ for any $\theta \neq (2k+1)\frac{\pi}{2}$.

Skill Building

In Problems 9–18, simplify each trigonometric expression by following the indicated direction.

9. Rewrite in terms of sine and cosine: $\tan\theta \cdot \csc\theta$.

10. Rewrite in terms of sine and cosine: $\cot\theta \cdot \sec\theta$.

11. Multiply $\dfrac{\cos\theta}{1 - \sin\theta}$ by $\dfrac{1 + \sin\theta}{1 + \sin\theta}$.

12. Multiply $\dfrac{\sin\theta}{1 + \cos\theta}$ by $\dfrac{1 - \cos\theta}{1 - \cos\theta}$.

13. Rewrite over a common denominator:

$$\frac{\sin\theta + \cos\theta}{\cos\theta} + \frac{\cos\theta - \sin\theta}{\sin\theta}$$

14. Rewrite over a common denominator:

$$\frac{1}{1 - \cos\theta} + \frac{1}{1 + \cos\theta}$$

15. Multiply and simplify: $\dfrac{(\sin\theta + \cos\theta)(\sin\theta + \cos\theta) - 1}{\sin\theta\cos\theta}$

16. Multiply and simplify: $\dfrac{(\tan\theta + 1)(\tan\theta + 1) - \sec^2\theta}{\tan\theta}$

17. Factor and simplify: $\dfrac{3\sin^2\theta + 4\sin\theta + 1}{\sin^2\theta + 2\sin\theta + 1}$

18. Factor and simplify: $\dfrac{\cos^2\theta - 1}{\cos^2\theta - \cos\theta}$

In Problems 19–98, establish each identity.

19. $\csc\theta \cdot \cos\theta = \cot\theta$

20. $\sec\theta \cdot \sin\theta = \tan\theta$

21. $1 + \tan^2(-\theta) = \sec^2\theta$

22. $1 + \cot^2(-\theta) = \csc^2\theta$

23. $\cos\theta(\tan\theta + \cot\theta) = \csc\theta$

24. $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$

25. $\tan\theta\cot\theta - \cos^2\theta = \sin^2\theta$

26. $\sin\theta\csc\theta - \cos^2\theta = \sin^2\theta$

27. $(\sec\theta - 1)(\sec\theta + 1) = \tan^2\theta$

28. $(\csc\theta - 1)(\csc\theta + 1) = \cot^2\theta$

29. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

30. $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$

31. $\cos^2\theta(1 + \tan^2\theta) = 1$

32. $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$

33. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

34. $\tan^2\theta\cos^2\theta + \cot^2\theta\sin^2\theta = 1$

35. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

36. $\csc^4\theta - \csc^2\theta = \cot^4\theta + \cot^2\theta$

37. $\sec\theta - \tan\theta = \dfrac{\cos\theta}{1 + \sin\theta}$

38. $\csc\theta - \cot\theta = \dfrac{\sin\theta}{1 + \cos\theta}$

39. $3\sin^2\theta + 4\cos^2\theta = 3 + \cos^2\theta$

40. $9\sec^2\theta - 5\tan^2\theta = 5 + 4\sec^2\theta$

41. $1 - \dfrac{\cos^2\theta}{1 + \sin\theta} = \sin\theta$

42. $1 - \dfrac{\sin^2\theta}{1 - \cos\theta} = -\cos\theta$

43. $\dfrac{1 + \tan\theta}{1 - \tan\theta} = \dfrac{\cot\theta + 1}{\cot\theta - 1}$

44. $\dfrac{\csc\theta - 1}{\csc\theta + 1} = \dfrac{1 - \sin\theta}{1 + \sin\theta}$

45. $\dfrac{\sec\theta}{\csc\theta} + \dfrac{\sin\theta}{\cos\theta} = 2\tan\theta$

46. $\dfrac{\csc\theta - 1}{\cot\theta} = \dfrac{\cot\theta}{\csc\theta + 1}$

47. $\dfrac{1 + \sin\theta}{1 - \sin\theta} = \dfrac{\csc\theta + 1}{\csc\theta - 1}$

48. $\dfrac{\cos\theta + 1}{\cos\theta - 1} = \dfrac{1 + \sec\theta}{1 - \sec\theta}$

49. $\dfrac{1-\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{1-\sin\theta}=2\sec\theta$

50. $\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{1+\sin\theta}{\cos\theta}=2\sec\theta$

51. $\dfrac{\sin\theta}{\sin\theta-\cos\theta}=\dfrac{1}{1-\cot\theta}$

52. $1-\dfrac{\sin^2\theta}{1+\cos\theta}=\cos\theta$

53. $\dfrac{1-\sin\theta}{1+\sin\theta}=(\sec\theta-\tan\theta)^2$

54. $\dfrac{1-\cos\theta}{1+\cos\theta}=(\csc\theta-\cot\theta)^2$

55. $\dfrac{\cos\theta}{1-\tan\theta}+\dfrac{\sin\theta}{1-\cot\theta}=\sin\theta+\cos\theta$

56. $\dfrac{\cot\theta}{1-\tan\theta}+\dfrac{\tan\theta}{1-\cot\theta}=1+\tan\theta+\cot\theta$

57. $\tan\theta+\dfrac{\cos\theta}{1+\sin\theta}=\sec\theta$

58. $\dfrac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\dfrac{\tan\theta}{1-\tan^2\theta}$

59. $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\tan\theta+\sec\theta$

60. $\dfrac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}=\dfrac{\sin\theta+1}{\cos\theta}$

61. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=\sin^2\theta-\cos^2\theta$

62. $\dfrac{\sec\theta-\cos\theta}{\sec\theta+\cos\theta}=\dfrac{\sin^2\theta}{1+\cos^2\theta}$

63. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}+1=2\sin^2\theta$

64. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}+2\cos^2\theta=1$

65. $\dfrac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta}=\tan\theta\sec\theta$

66. $\dfrac{\sec\theta}{1+\sec\theta}=\dfrac{1-\cos\theta}{\sin^2\theta}$

67. $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}+1=2\cos^2\theta$

68. $\dfrac{1-\cot^2\theta}{1+\cot^2\theta}+2\cos^2\theta=1$

69. $\dfrac{\sec\theta-\csc\theta}{\sec\theta\csc\theta}=\sin\theta-\cos\theta$

70. $\dfrac{\sin^2\theta-\tan\theta}{\cos^2\theta-\cot\theta}=\tan^2\theta$

71. $\sec\theta-\cos\theta=\sin\theta\tan\theta$

72. $\tan\theta+\cot\theta=\sec\theta\csc\theta$

73. $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta$

74. $\dfrac{1+\sin\theta}{1-\sin\theta}-\dfrac{1-\sin\theta}{1+\sin\theta}=4\tan\theta\sec\theta$

75. $\dfrac{\sec\theta}{1-\sin\theta}=\dfrac{1+\sin\theta}{\cos^3\theta}$

76. $\dfrac{1-\sin\theta}{1+\sin\theta}=(\sec\theta-\tan\theta)^2$

77. $\dfrac{(\sec\theta-\tan\theta)^2+1}{\csc\theta(\sec\theta-\tan\theta)}=2\tan\theta$

78. $\dfrac{\sec^2\theta-\tan^2\theta+\tan\theta}{\sec\theta}=\sin\theta+\cos\theta$

79. $\dfrac{\sin\theta+\cos\theta}{\cos\theta}-\dfrac{\sin\theta-\cos\theta}{\sin\theta}=\sec\theta\csc\theta$

80. $\dfrac{\sin\theta+\cos\theta}{\sin\theta}-\dfrac{\cos\theta-\sin\theta}{\cos\theta}=\sec\theta\csc\theta$

81. $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=1-\sin\theta\cos\theta$

82. $\dfrac{\sin^3\theta+\cos^3\theta}{1-2\cos^2\theta}=\dfrac{\sec\theta-\sin\theta}{\tan\theta-1}$

83. $\dfrac{\cos^2\theta-\sin^2\theta}{1-\tan^2\theta}=\cos^2\theta$

84. $\dfrac{\cos\theta+\sin\theta-\sin^3\theta}{\sin\theta}=\cot\theta+\cos^2\theta$

85. $\dfrac{(2\cos^2\theta-1)^2}{\cos^4\theta-\sin^4\theta}=1-2\sin^2\theta$

86. $\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta}=\tan\theta-\cot\theta$

87. $\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}=\dfrac{1+\cos\theta}{\sin\theta}$

88. $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}=\sec\theta+\tan\theta$

89. $(a\sin\theta+b\cos\theta)^2+(a\cos\theta-b\sin\theta)^2=a^2+b^2$

90. $(2a\sin\theta\cos\theta)^2+a^2(\cos^2\theta-\sin^2\theta)^2=a^2$

91. $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}=\tan\alpha\tan\beta$

92. $(\tan\alpha+\tan\beta)(1-\cot\alpha\cot\beta)+(\cot\alpha+\cot\beta)(1-\tan\alpha\tan\beta)=0$

93. $(\sin\alpha+\cos\beta)^2+(\cos\beta+\sin\alpha)(\cos\beta-\sin\alpha)=2\cos\beta(\sin\alpha+\cos\beta)$

94. $(\sin\alpha-\cos\beta)^2+(\cos\beta+\sin\alpha)(\cos\beta-\sin\alpha)=-2\cos\beta(\sin\alpha-\cos\beta)$

95. $\ln|\sec\theta|=-\ln|\cos\theta|$

96. $\ln|\tan\theta|=\ln|\sin\theta|-\ln|\cos\theta|$

97. $\ln|1+\cos\theta|+\ln|1-\cos\theta|=2\ln|\sin\theta|$

98. $\ln|\sec\theta+\tan\theta|+\ln|\sec\theta-\tan\theta|=0$

99. Show that $\sec(\tan^{-1}v)=\sqrt{1+v^2}$.

100. Show that $\tan(\sin^{-1}v)=\dfrac{v}{\sqrt{1-v^2}}$.

101. Show that $\tan(\cos^{-1}v)=\dfrac{\sqrt{1-v^2}}{v}$.

102. Show that $\sin(\cos^{-1}v)=\sqrt{1-v^2}$.

103. Show that $\cos(\sin^{-1}v)=\sqrt{1-v^2}$.

104. Show that $\cos(\tan^{-1}v)=\dfrac{1}{\sqrt{1+v^2}}$.

Discussion and Writing

105. Write a few paragraphs outlining your strategy for establishing identities.

106. Write down the three Pythagorean Identities.

107. Why do you think it is usually preferable to start with the side containing the more complicated expression when establishing an identity?

108. Make up an identity that is not a Fundamental Identity.

'Are You Prepared?' Answers

1. True **2.** True

6.4 Sum and Difference Formulas

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Distance Formula (Section 1.1, p. 5)
- Values of the Trigonometric Functions (Section 5.2, pp. 374–382)
- Finding Exact Values Given the Value of a Trigonometric Function and the Quadrant of the Angle (Section 5.3, pp. 395–398)

Now work the 'Are You Prepared?' problems on page 481.

OBJECTIVES
1 Use Sum and Difference Formulas to Find Exact Values
2 Use Sum and Difference Formulas to Establish Identities
3 Use Sum and Difference Formulas Involving Inverse Trigonometric Functions

In this section, we continue our derivation of trigonometric identities by obtaining formulas that involve the sum or difference of two angles, such as $\cos(\alpha + \beta)$, $\cos(\alpha - \beta)$, or $\sin(\alpha + \beta)$. These formulas are referred to as the **sum and difference formulas**. We begin with the formulas for $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$.

Theorem

Sum and Difference Formulas for Cosines

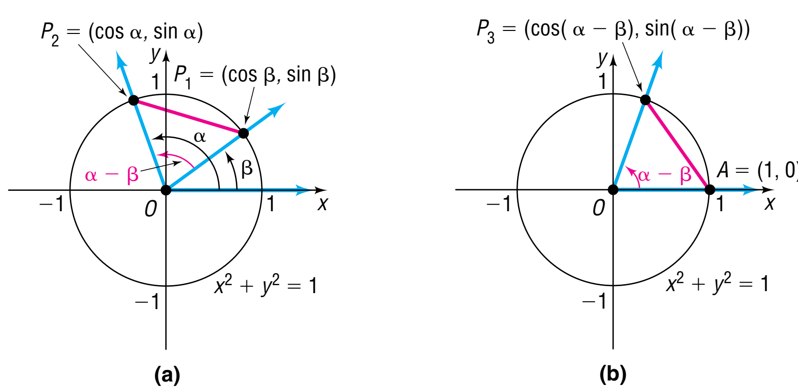
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (1)$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (2)$$

In Words
Formula (1) states that the cosine of the sum of two angles equals the cosine of the first angle times the cosine of the second angle minus the sine of the first angle times the sine of the second angle.

Proof We will prove formula (2) first. Although this formula is true for all numbers α and β, we shall assume in our proof that $0 < \beta < \alpha < 2\pi$. We begin with the unit circle and place the angles α and β in standard position, as shown in Figure 24(a). The point P_1 lies on the terminal side of β, so its coordinates are $(\cos\beta, \sin\beta)$; and the point P_2 lies on the terminal side of α, so its coordinates are $(\cos\alpha, \sin\alpha)$.

Figure 24



Now place the angle $\alpha - \beta$ in standard position, as shown in Figure 24(b). The point A has coordinates $(1, 0)$, and the point P_3 is on the terminal side of the angle $\alpha - \beta$, so its coordinates are $(\cos(\alpha - \beta), \sin(\alpha - \beta))$.

Looking at triangle OP_1P_2 in Figure 24(a) and triangle OAP_3 in Figure 24(b), we see that these triangles are congruent. (Do you see why? Two sides and the included angle, $\alpha - \beta$, are equal.) As a result, the unknown side of each triangle must be equal; that is,

$$d(A, P_3) = d(P_1, P_2)$$

Using the distance formula, we find that

$$\begin{aligned}
\sqrt{[\cos(\alpha-\beta)-1]^2+[\sin(\alpha-\beta)-0]^2} &= \sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2} && d(A, P_3) = d(P_1, P_2)\\
[\cos(\alpha-\beta)-1]^2+\sin^2(\alpha-\beta) &= (\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2 && \text{Square both sides.}\\
\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta) &= \cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta && \text{Multiply out the squared terms.}\\
&\quad+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta\\
2-2\cos(\alpha-\beta) &= 2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta && \text{Apply a Pythagorean Identity (3 times).}\\
-2\cos(\alpha-\beta) &= -2\cos\alpha\cos\beta-2\sin\alpha\sin\beta && \text{Subtract 2 from each side.}\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta && \text{Divide each side by } -2.
\end{aligned}$$

This is formula (2). ■

The proof of formula (1) follows from formula (2) and the Even–Odd Identities. We use the fact that $\alpha + \beta = \alpha - (-\beta)$. Then

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos[\alpha-(-\beta)]\\
&= \cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta) && \text{Use formula (2).}\\
&= \cos\alpha\cos\beta-\sin\alpha\sin\beta && \text{Even–Odd Identities}
\end{aligned}$$

■

1 Use Sum and Difference Formulas to Find Exact Values

One use of formulas (1) and (2) is to obtain the exact value of the cosine of an angle that can be expressed as the sum or difference of angles whose sine and cosine are known exactly.

EXAMPLE 1 Using the Sum Formula to Find Exact Values

Find the exact value of $\cos 75°$.

Solution Since $75° = 45° + 30°$, we use formula (1) to obtain

$$\cos 75° = \cos(45° + 30°) \underset{\text{Formula (1)}}{\uparrow}{=} \cos 45°\cos 30° - \sin 45°\sin 30°$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)$$

◀

EXAMPLE 2 Using the Difference Formula to Find Exact Values

Find the exact value of $\cos \dfrac{\pi}{12}$.

Solution

$$\cos \frac{\pi}{12} = \cos\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} \qquad \text{Use formula (2).}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

◀

NOW WORK PROBLEM 11.

2 Use Sum and Difference Formula to Establish Identities

Another use of formulas (1) and (2) is to establish other identities. One important pair of identities is given next.

— Seeing the Concept —

Graph $Y_1 = \cos\left(\dfrac{\pi}{2} - \theta\right)$ and $Y_2 = \sin \theta$ on the same screen. Does this demonstrate the result 3(a)? How would you demonstrate the result 3(b)?

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \qquad \textbf{(3a)}$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \qquad \textbf{(3b)}$$

Proof To prove formula (3a), we use the formula for $\cos(\alpha - \beta)$ with $\alpha = \dfrac{\pi}{2}$ and $\beta = \theta$.

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos \frac{\pi}{2} \cos \theta + \sin \frac{\pi}{2} \sin \theta$$

$$= 0 \cdot \cos \theta + 1 \cdot \sin \theta$$

$$= \sin \theta$$

■

To prove formula (3b), we make use of the identity (3a) just established.

$$\sin\left(\frac{\pi}{2} - \theta\right) \underset{\uparrow \text{ Use (3a).}}{=} \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right] = \cos \theta$$

■

Also, since

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\left[-\left(\theta - \frac{\pi}{2}\right)\right] \underset{\uparrow \text{ Even Property of Cosine}}{=} \cos\left(\theta - \frac{\pi}{2}\right)$$

and since

$$\cos\left(\frac{\pi}{2} - \theta\right) \underset{\uparrow \text{ 3(a)}}{=} \sin \theta$$

it follows that $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$. The graphs of $y = \cos\left(\theta - \frac{\pi}{2}\right)$ and $y = \sin\theta$ are identical, a fact that we conjectured earlier in Section 5.4.

Having established the identities in formulas (3a) and (3b), we now can derive the sum and difference formulas for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$.

Proof

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right] \qquad \text{Formula (3a)}$$

$$= \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$$

$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \qquad \text{Formula (2)}$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \text{Formulas (3a) and (3b)}$$

■

$$\sin(\alpha - \beta) = \sin[\alpha + (-\beta)]$$

$$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \qquad \text{Use the sum formula for sine just obtained.}$$

$$= \sin\alpha\cos\beta + \cos\alpha(-\sin\beta) \qquad \text{Even–Odd Identities.}$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

■

Theorem

Sum and Difference Formulas for Sines

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \textbf{(4)}$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad \textbf{(5)}$$

In Words

Formula (4) states that the sine of the sum of two angles equals the sine of the first angle times the cosine of the second angle plus the cosine of the first angle times the sine of the second angle.

EXAMPLE 3 Using the Sum Formula to Find Exact Values

Find the exact value of $\sin\frac{7\pi}{12}$.

Solution

$$\sin\frac{7\pi}{12} = \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3} \qquad \text{Formula (4)}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{1}{4}\left(\sqrt{2} + \sqrt{6}\right)$$

◀

NOW WORK PROBLEM 17.

EXAMPLE 4 Using the Difference Formula to Find Exact Values

Find the exact value of $\sin 80° \cos 20° - \cos 80° \sin 20°$.

Solution The form of the expression $\sin 80° \cos 20° - \cos 80° \sin 20°$ is that of the right side of the formula (5) for $\sin(\alpha - \beta)$ with $\alpha = 80°$ and $\beta = 20°$. That is,

$$\sin 80° \cos 20° - \cos 80° \sin 20° = \sin(80° - 20°) = \sin 60° = \frac{\sqrt{3}}{2}$$

NOW WORK PROBLEMS 23 AND 27.

EXAMPLE 5 Finding Exact Values

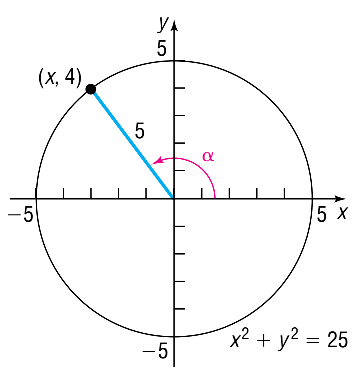
If it is known that $\sin \alpha = \frac{4}{5}$, $\frac{\pi}{2} < \alpha < \pi$, and that $\sin \beta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$, $\pi < \beta < \frac{3\pi}{2}$, find the exact value of

(a) $\cos \alpha$ (b) $\cos \beta$ (c) $\cos(\alpha + \beta)$ (d) $\sin(\alpha + \beta)$

Solution

Figure 25

$\sin \alpha = \frac{4}{5}, \frac{\pi}{2} < \alpha < \pi$



(a) Since $\sin \alpha = \frac{4}{5} = \frac{y}{r}$ and $\frac{\pi}{2} < \alpha < \pi$, we let $y = 4$ and $r = 5$ and place α in quadrant II. The point $P = (x, y) = (x, 4)$, $x < 0$, is on a circle of radius 5, namely, $x^2 + y^2 = 25$. See Figure 25. Then,

$$\begin{aligned} x^2 + y^2 &= 25, \\ x^2 + 16 &= 25 && y = 4 \\ x^2 &= 25 - 16 = 9 \\ x &= -3 && x < 0 \end{aligned}$$

Then,

$$\cos \alpha = \frac{x}{r} = -\frac{3}{5}$$

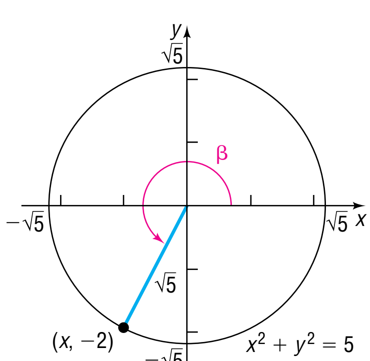
Alternatively, we can find $\cos \alpha$ using identities, as follows:

$$\cos \alpha = -\sqrt{1 - \sin^2 \alpha} = -\sqrt{1 - \frac{16}{25}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

α in quadrant II, $\cos \alpha < 0$

Figure 26

Given $\sin \beta = \frac{-2}{\sqrt{5}}, \pi < \beta < \frac{3\pi}{2}$



(b) Since $\sin \beta = \frac{-2}{\sqrt{5}} = \frac{y}{r}$ and $\pi < \beta < \frac{3\pi}{2}$, we let $y = -2$ and $r = \sqrt{5}$ and place β in quadrant III. The point $P = (x, y) = (x, -2)$, $x < 0$, is on a circle of radius $\sqrt{5}$, namely, $x^2 + y^2 = 5$. See Figure 26. Then,

$$\begin{aligned} x^2 + y^2 &= 5, \\ x^2 + 4 &= 5 && y = -2 \\ x^2 &= 1 \\ x &= -1 && x < 0 \end{aligned}$$

Then,

$$\cos\beta = \frac{x}{r} = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

Alternatively, we can find $\cos\beta$ using identities, as follows:

$$\cos\beta = -\sqrt{1-\sin^2\beta} = -\sqrt{1-\frac{4}{5}} = -\sqrt{\frac{1}{5}} = -\frac{\sqrt{5}}{5}$$

(c) Using the results found in parts (a) and (b) and formula (1), we have

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= -\frac{3}{5}\left(-\frac{\sqrt{5}}{5}\right) - \frac{4}{5}\left(-\frac{2\sqrt{5}}{5}\right) = \frac{11\sqrt{5}}{25}\end{aligned}$$

(d) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{4}{5}\left(-\frac{\sqrt{5}}{5}\right) + \left(-\frac{3}{5}\right)\left(-\frac{2\sqrt{5}}{5}\right) = \frac{2\sqrt{5}}{25}$$

◀

NOW WORK PROBLEMS 31(a), (b), AND (c).

EXAMPLE 6 Establishing an Identity

Establish the identity: $\dfrac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1$

Solution

$$\begin{aligned}\frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} &= \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\sin\beta} \\ &= \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} + \frac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta} \\ &= \frac{\cos\alpha}{\sin\alpha}\cdot\frac{\cos\beta}{\sin\beta} + 1 \\ &= \cot\alpha\cot\beta + 1\end{aligned}$$

◀

NOW WORK PROBLEMS 39 AND 51.

We use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and the sum formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ to derive a formula for $\tan(\alpha+\beta)$.

Proof $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$

Now we divide the numerator and denominator by $\cos\alpha\cos\beta$.

$$\tan(\alpha+\beta) = \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}}+\dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\cancel{\cos\alpha}\,\cancel{\cos\beta}}{\cancel{\cos\alpha}\,\cancel{\cos\beta}}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$ ■

Proof We use the sum formula for $\tan(\alpha+\beta)$ and even–odd properties to get the difference formula.

$$\tan(\alpha-\beta) = \tan[\alpha+(-\beta)] = \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)} = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$ ■

We have proved the following results:

Theorem

Sum and Difference Formulas for Tangents

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \quad (6)$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \quad (7)$$

In Words

Formula (6) states that the tangent of the sum of two angles equals the tangent of the first angle plus the tangent of the second angle, all divided by 1 minus their product.

NOW WORK PROBLEM 31(d).

EXAMPLE 7 Establishing an Identity

Prove the identity: $\tan(\theta+\pi) = \tan\theta$

Solution

$$\tan(\theta+\pi) = \frac{\tan\theta+\tan\pi}{1-\tan\theta\tan\pi} = \frac{\tan\theta+0}{1-\tan\theta\cdot 0} = \tan\theta$$ ◀

The result obtained in Example 7 verifies that the tangent function is periodic with period π, a fact that we mentioned earlier.

EXAMPLE 8 Establishing an Identity

Prove the identity: $\tan\left(\theta+\dfrac{\pi}{2}\right) = -\cot\theta$

Solution

We cannot use formula (6), since $\tan\dfrac{\pi}{2}$ is not defined. Instead, we proceed as follows:

$$\tan\left(\theta+\frac{\pi}{2}\right) = \frac{\sin\left(\theta+\dfrac{\pi}{2}\right)}{\cos\left(\theta+\dfrac{\pi}{2}\right)} = \frac{\sin\theta\cos\dfrac{\pi}{2}+\cos\theta\sin\dfrac{\pi}{2}}{\cos\theta\cos\dfrac{\pi}{2}-\sin\theta\sin\dfrac{\pi}{2}}$$

$$= \frac{(\sin\theta)(0)+(\cos\theta)(1)}{(\cos\theta)(0)-(\sin\theta)(1)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$ ◀

WARNING Be careful when using formulas (6) and (7). These formulas can be used only for angles α and β for which $\tan\alpha$ and $\tan\beta$ are defined, that is, all angles except odd multiples of $\dfrac{\pi}{2}$. ■

3 Use Sum and Difference Formulas Involving Inverse Trigonometric Functions

EXAMPLE 9 Finding the Exact Value of an Expression Involving Inverse Trigonometric Functions

Find the exact value of: $\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right)$

Solution We seek the sine of the sum of two angles, $\alpha = \cos^{-1}\frac{1}{2}$ and $\beta = \sin^{-1}\frac{3}{5}$. Then

$$\cos\alpha = \frac{1}{2}, \quad 0 \le \alpha \le \pi, \quad \text{and} \quad \sin\beta = \frac{3}{5}, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$$

We use Pythagorean Identities to obtain $\sin\alpha$ and $\cos\beta$. Since $\sin\alpha \ge 0$ and $\cos\beta \ge 0$ (do you know why?), we find

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\cos\beta = \sqrt{1 - \sin^2\beta} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

As a result,

$$\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right) = \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{3}{5} = \frac{4\sqrt{3} + 3}{10}$$

NOTE
In Example 9, we could also find $\sin\alpha$ by using $\cos\alpha = \frac{1}{2} = \frac{x}{r}$, so $x = 1$ and $r = 2$. Then, $y = \sqrt{3}$ and $\sin\alpha = \frac{y}{r} = \frac{\sqrt{3}}{2}$. We could find $\cos\beta$ in a similar fashion.

NOW WORK PROBLEM 67.

EXAMPLE 10 Writing a Trigonometric Expression as an Algebraic Expression

Write $\sin(\sin^{-1}u + \cos^{-1}v)$ as an algebraic expression containing u and v (that is, without any trigonometric functions).

Solution Let $\alpha = \sin^{-1}u$ and $\beta = \cos^{-1}v$. Then

$$\sin\alpha = u, \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}, \quad \text{and} \quad \cos\beta = v, \quad 0 \le \beta \le \pi$$

Since $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, we know that $\cos\alpha \ge 0$. As a result,

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - u^2}$$

Similarly, since $0 \le \beta \le \pi$, we know that $\sin\beta \ge 0$. Then

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \sqrt{1 - v^2}$$

Now

$$\sin(\sin^{-1}u + \cos^{-1}v) = \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= uv + \sqrt{1 - u^2}\sqrt{1 - v^2}$$

NOW WORK PROBLEM 77.

Summary

Sum and Difference Formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

6.4 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The distance d from the point $(2, -3)$ to the point $(5, 1)$ is _____. (p. 5)

2. If $\sin\theta = \frac{4}{5}$ and θ is in quadrant II, then $\cos\theta =$ _____. (pp. 395–398)

3. (a) $\sin\frac{\pi}{4}\cdot\cos\frac{\pi}{3} =$ _____. (p. 379)

(b) $\tan\frac{\pi}{4} - \sin\frac{\pi}{6} =$ _____. (p. 379)

Concepts and Vocabulary

4. $\cos(\alpha+\beta) = \cos\alpha\cos\beta$ ____ $\sin\alpha\sin\beta$.

5. $\sin(\alpha-\beta) = \sin\alpha\cos\beta$ ____ $\cos\alpha\sin\beta$.

6. *True or False:* $\sin(\alpha+\beta) = \sin\alpha + \sin\beta + 2\sin\alpha\sin\beta$

7. *True or False:* $\tan 75° = \tan 30° + \tan 45°$

8. *True or False:* $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Skill Building

In Problems 9–20, find the exact value of each trigonometric function.

9. $\sin\frac{5\pi}{12}$ **10.** $\sin\frac{\pi}{12}$ **11.** $\cos\frac{7\pi}{12}$ **12.** $\tan\frac{7\pi}{12}$ **13.** $\cos 165°$ **14.** $\sin 105°$

15. $\tan 15°$ **16.** $\tan 195°$ **17.** $\sin\frac{17\pi}{12}$ **18.** $\tan\frac{19\pi}{12}$ **19.** $\sec\left(-\frac{\pi}{12}\right)$ **20.** $\cot\left(-\frac{5\pi}{12}\right)$

In Problems 21–30, find the exact value of each expression.

21. $\sin 20°\cos 10° + \cos 20°\sin 10°$

22. $\sin 20°\cos 80° - \cos 20°\sin 80°$

23. $\cos 70°\cos 20° - \sin 70°\sin 20°$

24. $\cos 40°\cos 10° + \sin 40°\sin 10°$

25. $\frac{\tan 20° + \tan 25°}{1 - \tan 20°\tan 25°}$

26. $\frac{\tan 40° - \tan 10°}{1 + \tan 40°\tan 10°}$

27. $\sin\frac{\pi}{12}\cos\frac{7\pi}{12} - \cos\frac{\pi}{12}\sin\frac{7\pi}{12}$

28. $\cos\frac{5\pi}{12}\cos\frac{7\pi}{12} - \sin\frac{5\pi}{12}\sin\frac{7\pi}{12}$

29. $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}\sin\frac{\pi}{12}$

30. $\sin\frac{\pi}{18}\cos\frac{5\pi}{18} + \cos\frac{\pi}{18}\sin\frac{5\pi}{18}$

In Problems 31–36, find the exact value of each of the following under the given conditions:

(a) $\sin(\alpha+\beta)$ (b) $\cos(\alpha+\beta)$ (c) $\sin(\alpha-\beta)$ (d) $\tan(\alpha-\beta)$

31. $\sin\alpha = \frac{3}{5}, 0 < \alpha < \frac{\pi}{2};\quad \cos\beta = \frac{2\sqrt{5}}{5}, -\frac{\pi}{2} < \beta < 0$

32. $\cos\alpha = \frac{\sqrt{5}}{5}, 0 < \alpha < \frac{\pi}{2};\quad \sin\beta = -\frac{4}{5}, -\frac{\pi}{2} < \beta < 0$

33. $\tan\alpha = -\frac{4}{3}, \frac{\pi}{2} < \alpha < \pi;\quad \cos\beta = \frac{1}{2}, 0 < \beta < \frac{\pi}{2}$

34. $\tan\alpha = \frac{5}{12}, \pi < \alpha < \frac{3\pi}{2};\quad \sin\beta = -\frac{1}{2}, \pi < \beta < \frac{3\pi}{2}$

35. $\sin\alpha = \frac{5}{13}, -\frac{3\pi}{2} < \alpha < -\pi;\quad \tan\beta = -\sqrt{3}, \frac{\pi}{2} < \beta < \pi$

36. $\cos\alpha = \frac{1}{2}, -\frac{\pi}{2} < \alpha < 0;\quad \sin\beta = \frac{1}{3}, 0 < \beta < \frac{\pi}{2}$

37. If $\sin\theta = \frac{1}{3}$, θ in quadrant II, find the exact value of:

(a) $\cos\theta$ (b) $\sin\left(\theta + \frac{\pi}{6}\right)$ (c) $\cos\left(\theta - \frac{\pi}{3}\right)$ (d) $\tan\left(\theta + \frac{\pi}{4}\right)$

38. If $\cos\theta = \frac{1}{4}$, θ in quadrant IV, find the exact value of:

(a) $\sin\theta$ (b) $\sin\left(\theta - \frac{\pi}{6}\right)$ (c) $\cos\left(\theta + \frac{\pi}{3}\right)$ (d) $\tan\left(\theta - \frac{\pi}{4}\right)$

In Problems 39–64, establish each identity.

39. $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

40. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

41. $\sin(\pi - \theta) = \sin\theta$

42. $\cos(\pi - \theta) = -\cos\theta$

43. $\sin(\pi + \theta) = -\sin\theta$

44. $\cos(\pi + \theta) = -\cos\theta$

45. $\tan(\pi - \theta) = -\tan\theta$

46. $\tan(2\pi - \theta) = -\tan\theta$

47. $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$

48. $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

49. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$

50. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$

51. $\frac{\sin(\alpha + \beta)}{\sin\alpha\cos\beta} = 1 + \cot\alpha\tan\beta$

52. $\frac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

53. $\frac{\cos(\alpha + \beta)}{\cos\alpha\cos\beta} = 1 - \tan\alpha\tan\beta$

54. $\frac{\cos(\alpha - \beta)}{\sin\alpha\cos\beta} = \cot\alpha + \tan\beta$

55. $\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta}$

56. $\frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan\alpha\tan\beta}{1 + \tan\alpha\tan\beta}$

57. $\cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}$

58. $\cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$

59. $\sec(\alpha + \beta) = \frac{\csc\alpha\csc\beta}{\cot\alpha\cot\beta - 1}$

60. $\sec(\alpha - \beta) = \frac{\sec\alpha\sec\beta}{1 + \tan\alpha\tan\beta}$

61. $\sin(\alpha - \beta)\sin(\alpha + \beta) = \sin^2\alpha - \sin^2\beta$

62. $\cos(\alpha - \beta)\cos(\alpha + \beta) = \cos^2\alpha - \sin^2\beta$

63. $\sin(\theta + k\pi) = (-1)^k\sin\theta$, k any integer

64. $\cos(\theta + k\pi) = (-1)^k\cos\theta$, k any integer

In Problems 65–76, find the exact value of each expression.

65. $\sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}0\right)$

66. $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}1\right)$

67. $\sin\left[\sin^{-1}\frac{3}{5} - \cos^{-1}\left(-\frac{4}{5}\right)\right]$

68. $\sin\left[\sin^{-1}\left(-\frac{4}{5}\right) - \tan^{-1}\frac{3}{4}\right]$

69. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{5}{13}\right)$

70. $\cos\left[\tan^{-1}\frac{5}{12} - \sin^{-1}\left(-\frac{3}{5}\right)\right]$

71. $\cos\left(\sin^{-1}\frac{5}{13} - \tan^{-1}\frac{3}{4}\right)$

72. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{12}{13}\right)$

73. $\tan\left(\sin^{-1}\frac{3}{5} + \frac{\pi}{6}\right)$

74. $\tan\left(\frac{\pi}{4} - \cos^{-1}\frac{3}{5}\right)$

75. $\tan\left(\sin^{-1}\frac{4}{5} + \cos^{-1}1\right)$

76. $\tan\left(\cos^{-1}\frac{4}{5} + \sin^{-1}1\right)$

In Problems 77–82, write each trigonometric expression as an algebraic expression containing u and v.

77. $\cos(\cos^{-1}u + \sin^{-1}v)$

78. $\sin(\sin^{-1}u - \cos^{-1}v)$

79. $\sin(\tan^{-1}u - \sin^{-1}v)$

80. $\cos(\tan^{-1}u + \tan^{-1}v)$

81. $\tan(\sin^{-1}u - \cos^{-1}v)$

82. $\sec(\tan^{-1}u + \cos^{-1}v)$

Applications and Extensions

83. Show that $\sin^{-1}v + \cos^{-1}v = \frac{\pi}{2}$.

84. Show that $\tan^{-1}v + \cot^{-1}v = \frac{\pi}{2}$.

85. Show that $\tan^{-1}\left(\frac{1}{v}\right) = \frac{\pi}{2} - \tan^{-1}v$, if $v > 0$.

86. Show that $\cot^{-1}e^v = \tan^{-1}e^{-v}$.

87. Show that $\sin(\sin^{-1}v + \cos^{-1}v) = 1$.

88. Show that $\cos(\sin^{-1}v + \cos^{-1}v) = 0$.

89. Calculus Show that the difference quotient for $f(x) = \sin x$ is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$$
$$= \cos x \cdot \frac{\sin h}{h} - \sin x \cdot \frac{1 - \cos h}{h}$$

90. Calculus Show that the difference quotient for $f(x) = \cos x$ is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos x}{h}$$
$$= -\sin x \cdot \frac{\sin h}{h} - \cos x \cdot \frac{1 - \cos h}{h}$$

91. Explain why formula (7) cannot be used to show that

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

Establish this identity by using formulas (3a) and (3b).

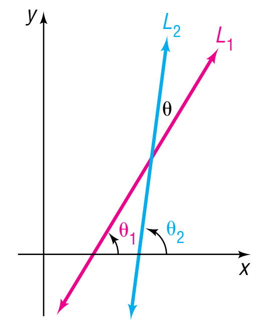
92. If $\tan\alpha = x + 1$ and $\tan\beta = x - 1$, show that

$$2\cot(\alpha - \beta) = x^2$$

93. Geometry: Angle Between Two Lines Let L_1 and L_2 denote two nonvertical intersecting lines, and let θ denote the acute angle between L_1 and L_2 (see the figure). Show that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2, respectively. [**Hint:** Use the facts that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.]



94. If $\alpha + \beta + \gamma = 180°$ and

$$\cot\theta = \cot\alpha + \cot\beta + \cot\gamma, \quad 0 < \theta < 90°$$

show that

$$\sin^3\theta = \sin(\alpha - \theta)\sin(\beta - \theta)\sin(\gamma - \theta)$$

Discussion and Writing

95. Discuss the following derivation:

$$\tan\left(\theta + \frac{\pi}{2}\right) = \frac{\tan\theta + \tan\frac{\pi}{2}}{1 - \tan\theta\tan\frac{\pi}{2}} = \frac{\frac{\tan\theta}{\tan\frac{\pi}{2}} + 1}{\frac{1}{\tan\frac{\pi}{2}} - \tan\theta} = \frac{0 + 1}{0 - \tan\theta} = \frac{1}{-\tan\theta} = -\cot\theta$$

Can you justify each step?

'Are You Prepared?' Answers

1. 5 **2.** $-\frac{3}{5}$ **3.** (a) $\frac{\sqrt{2}}{4}$ (b) $\frac{1}{2}$

6.5 Double-angle and Half-angle Formulas

OBJECTIVES
1 Use Double-angle Formulas to Find Exact Values
2 Use Double-angle Formulas to Establish Identities
3 Use Half-angle Formulas to Find Exact Values

In this section we derive formulas for $\sin(2\theta)$, $\cos(2\theta)$, $\sin\left(\frac{1}{2}\theta\right)$, and $\cos\left(\frac{1}{2}\theta\right)$ in terms of $\sin\theta$ and $\cos\theta$. They are derived using the sum formulas.

In the sum formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, let $\alpha = \beta = \theta$. Then

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\theta + \theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta \end{aligned}$$

and

$$\begin{aligned} \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\theta + \theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \end{aligned}$$

An application of the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$ results in two other ways to express $\cos(2\theta)$.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

and

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

We have established the following **Double-angle Formulas**:

Theorem

Double-angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \textbf{(1)}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \qquad \textbf{(2)}$$

$$\cos(2\theta) = 1 - 2\sin^2\theta \qquad \textbf{(3)}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \qquad \textbf{(4)}$$

1 Use Double-angle Formulas to Find Exact Values

EXAMPLE 1 **Finding Exact Values Using the Double-angle Formula**

If $\sin\theta = \frac{3}{5}$, $\frac{\pi}{2} < \theta < \pi$, find the exact value of:

(a) $\sin(2\theta)$ (b) $\cos(2\theta)$

Solution

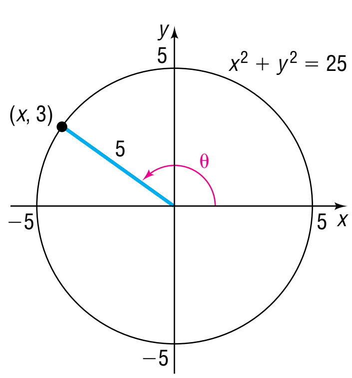
(a) Because $\sin(2\theta) = 2\sin\theta\cos\theta$ and we already know that $\sin\theta = \frac{3}{5}$, we only need to find $\cos\theta$. Since $\sin\theta = \frac{3}{5} = \frac{y}{r}$, $\frac{\pi}{2} < \theta < \pi$, we let $y = 3$ and $r = 5$ and place θ in quadrant II. The point $P = (x, y) = (x, 3)$, $x < 0$, is on a circle of radius 5, namely, $x^2 + y^2 = 25$. See Figure 27. Then

$$\begin{aligned} x^2 + y^2 &= 25, \\ x^2 + 9 &= 25 && y = 3 \\ x^2 &= 25 - 9 = 16 \\ x &= -4 && x < 0 \end{aligned}$$

We find that $\cos\theta = \frac{x}{r} = \frac{-4}{5}$. Now we use formula (1) to obtain

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$$

Figure 27



(b) Because we are given $\sin\theta = \frac{3}{5}$, it is easiest to use formula (4b) to get $\cos(2\theta)$.

$$\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\left(\frac{9}{25}\right) = 1 - \frac{18}{25} = \frac{7}{25}$$

WARNING In finding $\cos(2\theta)$ in Example 1(b), we chose to use a version of the Double-angle Formula, formula (3). Note that we are unable to use the Pythagorean Identity $\cos(2\theta) = \pm\sqrt{1 - \sin^2(2\theta)}$, with $\sin(2\theta) = -\frac{24}{25}$, because we have no way of knowing which sign to choose.

NOW WORK PROBLEMS 7(a) AND (b).

2 Use Double-angle Formulas to Establish Identities

EXAMPLE 2 **Establishing Identities**

(a) Develop a formula for $\tan(2\theta)$ in terms of $\tan\theta$.
(b) Develop a formula for $\sin(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

Solution (a) In the sum formula for $\tan(\alpha + \beta)$, let $\alpha = \beta = \theta$. Then

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \qquad \textbf{(5)}$$

(b) To get a formula for $\sin(3\theta)$, we use the sum formula and write 3θ as $2\theta + \theta$.

$$\sin(3\theta) = \sin(2\theta + \theta) = \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta$$

Now use the Double-angle Formulas to get

$$\begin{aligned}\sin(3\theta) &= (2\sin\theta\cos\theta)(\cos\theta) + (\cos^2\theta - \sin^2\theta)(\sin\theta)\\ &= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\end{aligned}$$

The formula obtained in Example 2(b) can also be written as

$$\begin{aligned}\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta &= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

That is, $\sin(3\theta)$ is a third-degree polynomial in the variable $\sin\theta$. In fact, $\sin(n\theta)$, n a positive odd integer, can always be written as a polynomial of degree n in the variable $\sin\theta$.*

NOW WORK PROBLEM 53.

By rearranging the Double-angle Formulas (3) and (4), we obtain other formulas that we will use later in this section.

*Due to the work done by P.L. Chebyshëv, these polynomials are sometimes called *Chebyshëv polynomials*.

We begin with formula (3) and proceed to solve for $\sin^2\theta$.

$$\cos(2\theta) = 1 - 2\sin^2\theta$$
$$2\sin^2\theta = 1 - \cos(2\theta)$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \textbf{(6)}$$

Similarly, using formula (4), we proceed to solve for $\cos^2\theta$.

$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$2\cos^2\theta = 1 + \cos(2\theta)$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \textbf{(7)}$$

Formulas (6) and (7) can be used to develop a formula for $\tan^2\theta$.

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\dfrac{1-\cos(2\theta)}{2}}{\dfrac{1+\cos(2\theta)}{2}}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \qquad \textbf{(8)}$$

Formulas (6) through (8) do not have to be memorized since their derivations are so straightforward.

Formulas (6) and (7) are important in calculus. The next example illustrates a problem that arises in calculus requiring the use of formula (7).

EXAMPLE 3 Establishing an Identity

Write an equivalent expression for $\cos^4\theta$ that does not involve any powers of sine or cosine greater than 1.

Solution The idea here is to apply formula (7) twice.

$$\begin{aligned}
\cos^4\theta = (\cos^2\theta)^2 &= \left(\frac{1+\cos(2\theta)}{2}\right)^2 && \text{Formula (7)}\\
&= \frac{1}{4}[1 + 2\cos(2\theta) + \cos^2(2\theta)]\\
&= \frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta)\\
&= \frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\left\{\frac{1+\cos[2(2\theta)]}{2}\right\} && \text{Formula (7)}\\
&= \frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}[1+\cos(4\theta)]\\
&= \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)
\end{aligned}$$

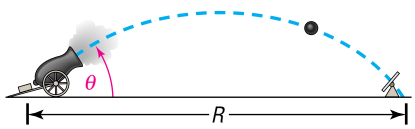
◀

NOW WORK PROBLEM 29.

Identities, such as the Double-angle Formulas, can sometimes be used to rewrite expressions in a more suitable form. Let's look at an example.

EXAMPLE 4 Projectile Motion

Figure 28



An object is propelled upward at an angle θ to the horizontal with an initial velocity of v_0 feet per second. See Figure 28. If air resistance is ignored, the **range** R, the horizontal distance that the object travels, is given by

$$R = \frac{1}{16}v_0^2 \sin\theta\cos\theta$$

(a) Show that $R = \dfrac{1}{32}v_0^2\sin(2\theta)$.

(b) Find the angle θ for which R is a maximum.

Solution (a) We rewrite the given expression for the range using the Double-angle Formula $\sin(2\theta) = 2\sin\theta\cos\theta$. Then

$$R = \frac{1}{16}v_0^2\sin\theta\cos\theta = \frac{1}{16}v_0^2\frac{2\sin\theta\cos\theta}{2} = \frac{1}{32}v_0^2\sin(2\theta)$$

(b) In this form, the largest value for the range R can be found. For a fixed initial speed v_0, the angle θ of inclination to the horizontal determines the value of R. Since the largest value of a sine function is 1, occurring when the argument 2θ is $90°$, it follows that for maximum R we must have

$$2\theta = 90°$$
$$\theta = 45°$$

An inclination to the horizontal of $45°$ results in the maximum range. ◀

3 Use Half-angle Formulas to Find Exact Values

Another important use of formulas (6) through (8) is to prove the *Half-angle Formulas*. In formulas (6) through (8), let $\theta = \dfrac{\alpha}{2}$. Then

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \qquad \tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha} \qquad \textbf{(9)}$$

The identities in box (9) will prove useful in integral calculus.

If we solve for the trigonometric functions on the left sides of equations (9), we obtain the Half-angle Formulas.

Theorem

Half-angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \textbf{(10a)}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \textbf{(10b)}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad \textbf{(10c)}$$

where the + or − sign is determined by the quadrant of the angle $\dfrac{\alpha}{2}$.

EXAMPLE 5 Finding Exact Values Using Half-angle Formulas

Use a Half-angle Formula find the exact value of:

(a) $\cos 15°$ (b) $\sin(-15°)$

Solution (a) Because $15° = \frac{30°}{2}$, we can use the Half-angle Formula for $\cos\frac{\alpha}{2}$ with $\alpha = 30°$. Also, because $15°$ is in quadrant I, $\cos 15° > 0$, we choose the $+$ sign in using formula (10b):

$$\cos 15° = \cos\frac{30°}{2} = \sqrt{\frac{1 + \cos 30°}{2}}$$

$$= \sqrt{\frac{1 + \sqrt{3}/2}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

(b) We use the fact that $\sin(-15°) = -\sin 15°$ and then apply formula (10a).

$$\sin(-15°) = -\sin\frac{30°}{2} = -\sqrt{\frac{1 - \cos 30°}{2}}$$

$$= -\sqrt{\frac{1 - \sqrt{3}/2}{2}} = -\sqrt{\frac{2 - \sqrt{3}}{4}} = -\frac{\sqrt{2 - \sqrt{3}}}{2}$$

It is interesting to compare the answer found in Example 5(a) with the answer to Example 2 of Section 6.4. There we calculated

$$\cos\frac{\pi}{12} = \cos 15° = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

Based on this and the result of Example 5(a), we conclude that

$$\frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right) \quad \text{and} \quad \frac{\sqrt{2 + \sqrt{3}}}{2}$$

are equal. (Since each expression is positive, you can verify this equality by squaring each expression.) Two very different looking, yet correct, answers can be obtained, depending on the approach taken to solve a problem.

NOW WORK PROBLEM 19.

EXAMPLE 6 Finding Exact Values Using Half-angle Formulas

If $\cos\alpha = -\frac{3}{5}$, $\pi < \alpha < \frac{3\pi}{2}$, find the exact value of:

(a) $\sin\frac{\alpha}{2}$ (b) $\cos\frac{\alpha}{2}$ (c) $\tan\frac{\alpha}{2}$

Solution First, we observe that if $\pi < \alpha < \frac{3\pi}{2}$ then $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$. As a result, $\frac{\alpha}{2}$ lies in quadrant II.

(a) Because $\frac{\alpha}{2}$ lies in quadrant II, $\sin\frac{\alpha}{2} > 0$, so we use the + sign in formula (10a) to get

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}$$

$$= \sqrt{\frac{\frac{8}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

(b) Because $\frac{\alpha}{2}$ lies in quadrant II, $\cos\frac{\alpha}{2} < 0$, so we use the − sign in formula (10b) to get

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}}$$

$$= -\sqrt{\frac{\frac{2}{5}}{2}} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

(c) Because $\frac{\alpha}{2}$ lies in quadrant II, $\tan\frac{\alpha}{2} < 0$, so we use the − sign in formula (10c) to get

$$\tan\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{1+\left(-\frac{3}{5}\right)}} = -\sqrt{\frac{\frac{8}{5}}{\frac{2}{5}}} = -2$$

Another way to solve Example 6(c) is to use the solutions found in parts (a) and (b).

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\frac{2\sqrt{5}}{5}}{-\frac{\sqrt{5}}{5}} = -2$$

NOW WORK PROBLEMS 7(c) AND (d).

There is a formula for $\tan\frac{\alpha}{2}$ that does not contain + and − signs, making it more useful than Formula 10(c). To derive it, we use the formulas

$$1 - \cos\alpha = 2\sin^2\frac{\alpha}{2} \qquad \text{Formula (9)}$$

and

$$\sin\alpha = \sin\left[2\left(\frac{\alpha}{2}\right)\right] = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \qquad \text{Double-angle Formula}$$

Then

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2}$$

Since it also can be shown that

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

we have the following two Half-angle Formulas:

Half-angle Formulas for tan $\frac{\alpha}{2}$

$$\boxed{\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} \qquad \textbf{(11)}}$$

With this formula, the solution to Example 6(c) can be obtained as follows

$$\cos\alpha = -\frac{3}{5}, \pi < \alpha < \frac{3\pi}{2}$$

$$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Then, by equation (11),

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}} = \frac{\frac{8}{5}}{-\frac{4}{5}} = -2$$

6.5 Assess Your Understanding

Concepts and Vocabulary

1. $\cos(2\theta) = \cos^2\theta - $ ____ $=$ ____ $- 1 = 1 -$ ____.

2. $\sin^2\frac{\theta}{2} = \frac{\_\_}{2}$.

3. $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\_\_\_\_}$.

4. *True or False:* $\cos(2\theta)$ has three equivalent forms:

$$\cos^2\theta - \sin^2\theta, \quad 1 - 2\sin^2\theta, \quad 2\cos^2\theta - 1$$

5. *True or False:* $\sin(2\theta)$ has two equivalent forms:

$$2\sin\theta\cos\theta \quad \text{and} \quad \sin^2\theta - \cos^2\theta$$

6. *True or False:* $\tan(2\theta) + \tan(2\theta) = \tan(4\theta)$

Skill Building

In Problems 7–18, use the information given about the angle θ, $0 \le \theta \le 2\pi$, to find the exact value of

(a) $\sin(2\theta)$ (b) $\cos(2\theta)$ (c) $\sin\frac{\theta}{2}$ (d) $\cos\frac{\theta}{2}$

7. $\sin\theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$

8. $\cos\theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$

9. $\tan\theta = \frac{4}{3}$, $\pi < \theta < \frac{3\pi}{2}$

10. $\tan\theta = \frac{1}{2}$, $\pi < \theta < \frac{3\pi}{2}$

11. $\cos\theta = -\frac{\sqrt{6}}{3}$, $\frac{\pi}{2} < \theta < \pi$

12. $\sin\theta = -\frac{\sqrt{3}}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$

13. $\sec\theta = 3, \quad \sin\theta > 0$

14. $\csc\theta = -\sqrt{5}, \quad \cos\theta < 0$

15. $\cot\theta = -2, \quad \sec\theta < 0$

16. $\sec\theta = 2, \quad \csc\theta < 0$

17. $\tan\theta = -3, \quad \sin\theta < 0$

18. $\cot\theta = 3, \quad \cos\theta < 0$

In Problems 19–28, use the Half-angle Formulas to find the exact value of each trigonometric function.

19. $\sin 22.5°$

20. $\cos 22.5°$

21. $\tan \frac{7\pi}{8}$

22. $\tan \frac{9\pi}{8}$

23. $\cos 165°$

24. $\sin 195°$

25. $\sec \frac{15\pi}{8}$

26. $\csc \frac{7\pi}{8}$

27. $\sin\left(-\frac{\pi}{8}\right)$

28. $\cos\left(-\frac{3\pi}{8}\right)$

29. Show that $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$.

30. Show that $\sin(4\theta) = (\cos\theta)(4\sin\theta - 8\sin^3\theta)$.

31. Develop a formula for $\cos(3\theta)$ as a third-degree polynomial in the variable $\cos\theta$.

32. Develop a formula for $\cos(4\theta)$ as a fourth-degree polynomial in the variable $\cos\theta$.

33. Find an expression for $\sin(5\theta)$ as a fifth-degree polynomial in the variable $\sin\theta$.

34. Find an expression for $\cos(5\theta)$ as a fifth-degree polynomial in the variable $\cos\theta$.

In Problems 35–56, establish each identity.

35. $\cos^4\theta - \sin^4\theta = \cos(2\theta)$

36. $\dfrac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta} = \cos(2\theta)$

37. $\cot(2\theta) = \dfrac{\cot^2\theta - 1}{2\cot\theta}$

38. $\cot(2\theta) = \frac{1}{2}(\cot\theta - \tan\theta)$

39. $\sec(2\theta) = \dfrac{\sec^2\theta}{2 - \sec^2\theta}$

40. $\csc(2\theta) = \frac{1}{2}\sec\theta\csc\theta$

41. $\cos^2(2\theta) - \sin^2(2\theta) = \cos(4\theta)$

42. $(4\sin\theta\cos\theta)(1 - 2\sin^2\theta) = \sin(4\theta)$

43. $\dfrac{\cos(2\theta)}{1 + \sin(2\theta)} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$

44. $\sin^2\theta\cos^2\theta = \frac{1}{8}[1 - \cos(4\theta)]$

45. $\sec^2\frac{\theta}{2} = \dfrac{2}{1 + \cos\theta}$

46. $\csc^2\frac{\theta}{2} = \dfrac{2}{1 - \cos\theta}$

47. $\cot^2\frac{\theta}{2} = \dfrac{\sec\theta + 1}{\sec\theta - 1}$

48. $\tan\frac{\theta}{2} = \csc\theta - \cot\theta$

49. $\cos\theta = \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

50. $1 - \frac{1}{2}\sin(2\theta) = \dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$

51. $\dfrac{\sin(3\theta)}{\sin\theta} - \dfrac{\cos(3\theta)}{\cos\theta} = 2$

52. $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} - \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\tan(2\theta)$

53. $\tan(3\theta) = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

54. $\tan\theta + \tan(\theta + 120°) + \tan(\theta + 240°) = 3\tan(3\theta)$

55. $\ln|\sin\theta| = \frac{1}{2}(\ln|1 - \cos(2\theta)| - \ln 2)$

56. $\ln|\cos\theta| = \frac{1}{2}(\ln|1 + \cos(2\theta)| - \ln 2)$

In Problems 57–68, find the exact value of each expression.

57. $\sin\left(2\sin^{-1}\frac{1}{2}\right)$

58. $\sin\left[2\sin^{-1}\frac{\sqrt{3}}{2}\right]$

59. $\cos\left(2\sin^{-1}\frac{3}{5}\right)$

60. $\cos\left(2\cos^{-1}\frac{4}{5}\right)$

61. $\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$

62. $\tan\left(2\tan^{-1}\frac{3}{4}\right)$

63. $\sin\left(2\cos^{-1}\frac{4}{5}\right)$

64. $\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right]$

65. $\sin^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$

66. $\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$

67. $\sec\left(2\tan^{-1}\frac{3}{4}\right)$

68. $\csc\left[2\sin^{-1}\left(-\frac{3}{5}\right)\right]$

Applications and Extensions

69. If $x = 2\tan\theta$, express $\sin(2\theta)$ as a function of x.

70. If $x = 2\tan\theta$, express $\cos(2\theta)$ as a function of x.

71. Find the value of the number C:

$$\frac{1}{2}\sin^2 x + C = -\frac{1}{4}\cos(2x)$$

72. Find the value of the number C:

$$\frac{1}{2}\cos^2 x + C = \frac{1}{4}\cos(2x)$$

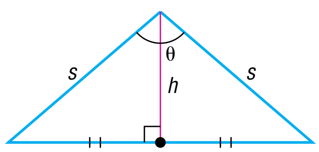
73. If $z = \tan\frac{\alpha}{2}$, show that $\sin\alpha = \frac{2z}{1+z^2}$.

74. If $z = \tan\frac{\alpha}{2}$, show that $\cos\alpha = \frac{1-z^2}{1+z^2}$.

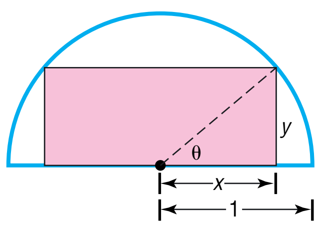
75. Area of an Isosceles Triangle Show that the area A of an isosceles triangle whose equal sides are of length s and θ is the angle between them is

$$\frac{1}{2}s^2\sin\theta$$

[**Hint:** See the illustration. The height h bisects the angle θ and is the perpendicular bisector of the base.]



76. Geometry A rectangle is inscribed in a semicircle of radius 1. See the illustration.



(a) Express the area A of the rectangle as a function of the angle θ shown in the illustration.
(b) Show that $A = \sin(2\theta)$.
(c) Find the angle θ that results in the largest area A.
(d) Find the dimensions of this largest rectangle.

77. Graph $f(x) = \sin^2 x = \frac{1-\cos(2x)}{2}$ for $0 \le x \le 2\pi$ by using transformations.

78. Repeat Problem 77 for $g(x) = \cos^2 x$.

79. Use the fact that

$$\cos\frac{\pi}{12} = \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

to find $\sin\frac{\pi}{24}$ and $\cos\frac{\pi}{24}$.

80. Show that

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

and use it to find $\sin\frac{\pi}{16}$ and $\cos\frac{\pi}{16}$.

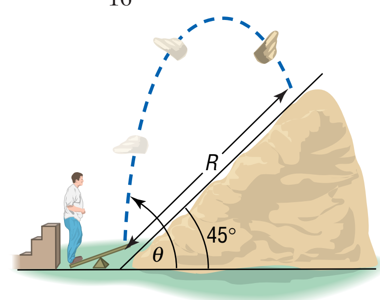
81. Show that

$$\sin^3\theta + \sin^3(\theta+120°) + \sin^3(\theta+240°) = -\frac{3}{4}\sin(3\theta)$$

82. If $\tan\theta = a\tan\frac{\theta}{3}$, express $\tan\frac{\theta}{3}$ in terms of a.

83. Projectile Motion An object is propelled upward at an angle θ, $45° < \theta < 90°$, to the horizontal with an initial velocity of v_0 feet per second from the base of a plane that makes an angle of 45° with the horizontal. See the illustration. If air resistance is ignored, the distance R that it travels up the inclined plane is given by

$$R = \frac{v_0^2\sqrt{2}}{16}\cos\theta(\sin\theta - \cos\theta)$$



(a) Show that

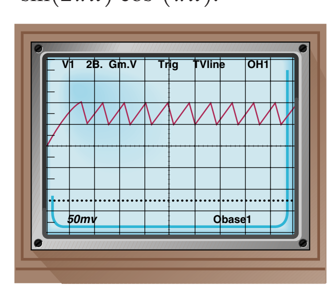
$$R = \frac{v_0^2\sqrt{2}}{32}[\sin(2\theta) - \cos(2\theta) - 1]$$

(b) Graph $R = R(\theta)$. (Use $v_0 = 32$ feet per second.)
(c) What value of θ makes R the largest? (Use $v_0 = 32$ feet per second.)

84. Sawtooth Curve An oscilloscope often displays a sawtooth curve. This curve can be approximated by sinusoidal curves of varying periods and amplitudes. A first approximation to the sawtooth curve is given by

$$y = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x)$$

Show that $y = \sin(2\pi x)\cos^2(\pi x)$.



Discussion and Writing

85. Go to the library and research Chebyshëv polynomials. Write a report on your findings.

6.6 Product-to-Sum and Sum-to-Product Formulas

OBJECTIVES 1 Express Products as Sums
2 Express Sums as Products

1 Express Products as Sums

Sum and difference formulas can be used to derive formulas for writing the products of sines and/or cosines as sums or differences. These identities are usually called the **Product-to-Sum Formulas**.

Theorem

Product-to-Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \quad \textbf{(1)}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \quad \textbf{(2)}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \quad \textbf{(3)}$$

These formulas do not have to be memorized. Instead, you should remember how they are derived. Then, when you want to use them, either look them up or derive them, as needed.

To derive formulas (1) and (2), write down the sum and difference formulas for the cosine:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \textbf{(4)}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \textbf{(5)}$$

Subtract equation (5) from equation (4) to get

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

from which

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

Now add equations (4) and (5) to get

$$\cos(\alpha-\beta) + \cos(\alpha+\beta) = 2\cos\alpha\cos\beta$$

from which

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

To derive Product-to-Sum Formula (3), use the sum and difference formulas for sine in a similar way. (You are asked to do this in Problem 41.)

EXAMPLE 1 Expressing Products as Sums

Express each of the following products as a sum containing only sines or cosines.

(a) $\sin(6\theta)\sin(4\theta)$ (b) $\cos(3\theta)\cos\theta$ (c) $\sin(3\theta)\cos(5\theta)$

Solution (a) We use formula (1) to get

$$\begin{aligned}\sin(6\theta)\sin(4\theta) &= \frac{1}{2}[\cos(6\theta - 4\theta) - \cos(6\theta + 4\theta)]\\ &= \frac{1}{2}[\cos(2\theta) - \cos(10\theta)]\end{aligned}$$

(b) We use formula (2) to get

$$\begin{aligned}\cos(3\theta)\cos\theta &= \frac{1}{2}[\cos(3\theta - \theta) + \cos(3\theta + \theta)]\\ &= \frac{1}{2}[\cos(2\theta) + \cos(4\theta)]\end{aligned}$$

(c) We use formula (3) to get

$$\begin{aligned}\sin(3\theta)\cos(5\theta) &= \frac{1}{2}[\sin(3\theta + 5\theta) + \sin(3\theta - 5\theta)]\\ &= \frac{1}{2}[\sin(8\theta) + \sin(-2\theta)] = \frac{1}{2}[\sin(8\theta) - \sin(2\theta)]\end{aligned}$$

◀

NOW WORK PROBLEM 1.

2 Express Sums as Products

The **Sum-to-Product Formulas** are given next.

Theorem

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \quad \textbf{(6)}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} \quad \textbf{(7)}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \quad \textbf{(8)}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \quad \textbf{(9)}$$

We will derive formula (6) and leave the derivations of formulas (7) through (9) as exercises (see Problems 42 through 44).

Proof

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \underset{\substack{\uparrow\\ \text{Product-to-Sum Formula (3)}}}{=} 2\cdot\frac{1}{2}\left[\sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)\right]$$

$$= \sin\frac{2\alpha}{2} + \sin\frac{2\beta}{2} = \sin\alpha + \sin\beta$$

■

EXAMPLE 2 **Expressing Sums (or Differences) as a Product**

Express each sum or difference as a product of sines and/or cosines.

(a) $\sin(5\theta) - \sin(3\theta)$ (b) $\cos(3\theta) + \cos(2\theta)$

Solution (a) We use formula (7) to get

$$\sin(5\theta) - \sin(3\theta) = 2\sin\frac{5\theta - 3\theta}{2}\cos\frac{5\theta + 3\theta}{2}$$
$$= 2\sin\theta\cos(4\theta)$$

(b) $$\cos(3\theta) + \cos(2\theta) = 2\cos\frac{3\theta + 2\theta}{2}\cos\frac{3\theta - 2\theta}{2} \quad \text{Formula (8)}$$
$$= 2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}$$

NOW WORK PROBLEM 11.

6.6 Assess Your Understanding

Skill Building

In Problems 1–10, express each product as a sum containing only sines or cosines.

1. $\sin(4\theta)\sin(2\theta)$
2. $\cos(4\theta)\cos(2\theta)$
3. $\sin(4\theta)\cos(2\theta)$
4. $\sin(3\theta)\sin(5\theta)$
5. $\cos(3\theta)\cos(5\theta)$
6. $\sin(4\theta)\cos(6\theta)$
7. $\sin\theta\sin(2\theta)$
8. $\cos(3\theta)\cos(4\theta)$
9. $\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$
10. $\sin\frac{\theta}{2}\cos\frac{5\theta}{2}$

In Problems 11–18, express each sum or difference as a product of sines and/or cosines.

11. $\sin(4\theta) - \sin(2\theta)$
12. $\sin(4\theta) + \sin(2\theta)$
13. $\cos(2\theta) + \cos(4\theta)$
14. $\cos(5\theta) - \cos(3\theta)$
15. $\sin\theta + \sin(3\theta)$
16. $\cos\theta + \cos(3\theta)$
17. $\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}$
18. $\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}$

In Problems 19–36, establish each identity.

19. $\dfrac{\sin\theta + \sin(3\theta)}{2\sin(2\theta)} = \cos\theta$
20. $\dfrac{\cos\theta + \cos(3\theta)}{2\cos(2\theta)} = \cos\theta$
21. $\dfrac{\sin(4\theta) + \sin(2\theta)}{\cos(4\theta) + \cos(2\theta)} = \tan(3\theta)$
22. $\dfrac{\cos\theta - \cos(3\theta)}{\sin(3\theta) - \sin\theta} = \tan(2\theta)$
23. $\dfrac{\cos\theta - \cos(3\theta)}{\sin\theta + \sin(3\theta)} = \tan\theta$
24. $\dfrac{\cos\theta - \cos(5\theta)}{\sin\theta + \sin(5\theta)} = \tan(2\theta)$
25. $\sin\theta[\sin\theta + \sin(3\theta)] = \cos\theta[\cos\theta - \cos(3\theta)]$
26. $\sin\theta[\sin(3\theta) + \sin(5\theta)] = \cos\theta[\cos(3\theta) - \cos(5\theta)]$
27. $\dfrac{\sin(4\theta) + \sin(8\theta)}{\cos(4\theta) + \cos(8\theta)} = \tan(6\theta)$
28. $\dfrac{\sin(4\theta) - \sin(8\theta)}{\cos(4\theta) - \cos(8\theta)} = -\cot(6\theta)$
29. $\dfrac{\sin(4\theta) + \sin(8\theta)}{\sin(4\theta) - \sin(8\theta)} = -\dfrac{\tan(6\theta)}{\tan(2\theta)}$
30. $\dfrac{\cos(4\theta) - \cos(8\theta)}{\cos(4\theta) + \cos(8\theta)} = \tan(2\theta)\tan(6\theta)$
31. $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \tan\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$
32. $\dfrac{\cos\alpha + \cos\beta}{\cos\alpha - \cos\beta} = -\cot\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$
33. $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \tan\dfrac{\alpha+\beta}{2}$
34. $\dfrac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta} = -\cot\dfrac{\alpha+\beta}{2}$
35. $1 + \cos(2\theta) + \cos(4\theta) + \cos(6\theta) = 4\cos\theta\cos(2\theta)\cos(3\theta)$
36. $1 - \cos(2\theta) + \cos(4\theta) - \cos(6\theta) = 4\sin\theta\cos(2\theta)\sin(3\theta)$

Applications and Extensions

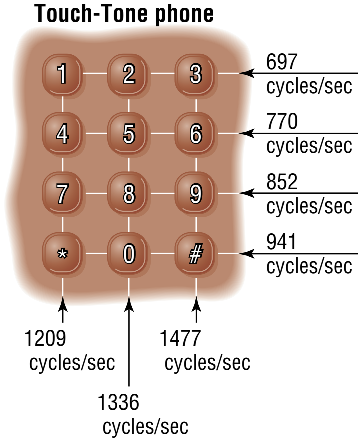
37. Touch-Tone Phones On a Touch-Tone phone, each button produces a unique sound. The sound produced is the sum of two tones, given by

$$y = \sin(2\pi l t) \quad \text{and} \quad y = \sin(2\pi h t)$$

where l and h are the low and high frequencies (cycles per second) shown on the illustration. For example, if you touch 7, the low frequency is $l = 852$ cycles per second and the high frequency is $h = 1209$ cycles per second. The sound emitted by touching 7 is

$$y = \sin[2\pi(852)t] + \sin[2\pi(1209)t]$$

Touch-Tone phone



(a) Write this sound as a product of sines and/or cosines.
(b) Determine the maximum value of y.
(c) Graph the sound emitted by touching 7.

38. Touch-Tone Phones
(a) Write the sound emitted by touching the # key as a product of sines and/or cosines.
(b) Determine the maximum value of y.
(c) Graph the sound emitted by touching the # key.

39. If $\alpha + \beta + \gamma = \pi$, show that

$$\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 4 \sin\alpha \sin\beta \sin\gamma$$

40. If $\alpha + \beta + \gamma = \pi$, show that

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \tan\beta \tan\gamma$$

41. Derive formula (3).

42. Derive formula (7).

43. Derive formula (8).

44. Derive formula (9).

6.7 Trigonometric Equations (I)

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Equations (Appendix, Section A.5, pp. 984–987)
- Values of the Trigonometric Functions of Angles (Section 5.2, pp. 374–382)

Now work the 'Are You Prepared?' problems on page 500.

OBJECTIVE 1 Solve Equations Involving a Single Trigonometric Function

1 Solve Equations Involving a Single Trigonometric Function

The previous four sections of this chapter were devoted to trigonometric identities, that is, equations involving trigonometric functions that are satisfied by every value in the domain of the variable. In the remaining two sections, we discuss **trigonometric equations**, that is, equations involving trigonometric functions that are satisfied only by some values of the variable (or, possibly, are not satisfied by any values of the variable). The values that satisfy the equation are called **solutions** of the equation.

EXAMPLE 1 Checking Whether a Given Number Is a Solution of a Trigonometric Equation

Determine whether $\theta = \frac{\pi}{4}$ is a solution of the equation $\sin\theta = \frac{1}{2}$. Is $\theta = \frac{\pi}{6}$ a solution?

Solution Replace θ by $\frac{\pi}{4}$ in the given equation. The result is

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \neq \frac{1}{2}$$

We conclude that $\frac{\pi}{4}$ is not a solution.

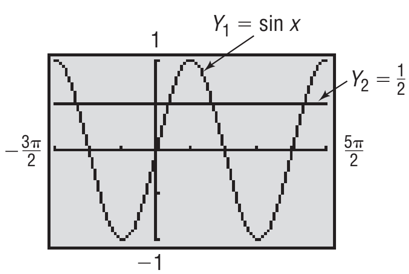
Next replace θ by $\frac{\pi}{6}$ in the equation. The result is

$$\sin\frac{\pi}{6} = \frac{1}{2}$$

We conclude that $\frac{\pi}{6}$ is a solution of the given equation. ◀

The equation given in Example 1 has other solutions besides $\theta = \frac{\pi}{6}$. For example, $\theta = \frac{5\pi}{6}$ is also a solution, as is $\theta = \frac{13\pi}{6}$. (You should check this for yourself.) In fact, the equation has an infinite number of solutions due to the periodicity of the sine function. See Figure 29.

Figure 29



As before, our practice will be to solve equations, whenever possible, by finding exact solutions. In such cases, we will also verify the solution obtained by using a graphing utility. When traditional methods cannot be used, approximate solutions will be obtained using a graphing utility. The reader is encouraged to pay particular attention to the form of equations for which exact solutions are possible.

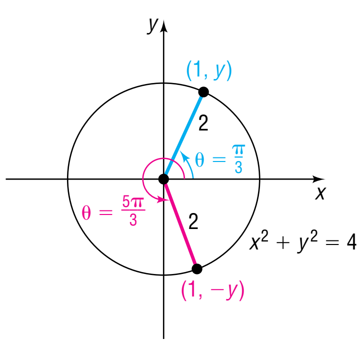
Unless the domain of the variable is restricted, we need to find *all* the solutions of a trigonometric equation. As the next example illustrates, finding all the solutions can be accomplished by first finding solutions over an interval whose length equals the period of the function and then adding multiples of that period to the solutions found. Let's look at some examples.

EXAMPLE 2 Finding All the Solutions of a Trigonometric Equation

Solve the equation: $\cos\theta = \frac{1}{2}$

Give a general formula for all the solutions. List eight of the solutions.

Figure 30



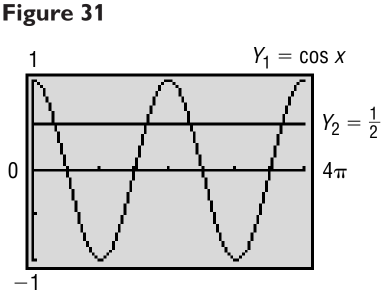
Solution The period of the cosine function is 2π. In the interval $[0, 2\pi)$, there are two angles θ for which $\cos\theta = \frac{1}{2}$: $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. See Figure 30. Because the cosine function has period 2π, all the solutions of $\cos\theta = \frac{1}{2}$ may be given by the general formula

$$\theta = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2k\pi \qquad k \text{ any integer}$$

Eight of the solutions are

$$\underbrace{-\frac{5\pi}{3}, \ -\frac{\pi}{3}}_{k=-1}, \ \underbrace{\frac{\pi}{3}, \ \frac{5\pi}{3}}_{k=0}, \ \underbrace{\frac{7\pi}{3}, \ \frac{11\pi}{3}}_{k=1}, \ \underbrace{\frac{13\pi}{3}, \ \frac{17\pi}{3}}_{k=2}$$

Figure 31



✔ **CHECK:** We can verify the solutions by graphing $Y_1 = \cos x$ and $Y_2 = \frac{1}{2}$ to determine where the graphs intersect. (Be sure to graph in radian mode.) See Figure 31. The graph of Y_1 intersects the graph of Y_2 at $x = 1.05\left(\approx\frac{\pi}{3}\right)$, $5.24\left(\approx\frac{5\pi}{3}\right)$, $7.33\left(\approx\frac{7\pi}{3}\right)$, and $11.52\left(\approx\frac{11\pi}{3}\right)$, rounded to two decimal places. ◀

NOW WORK PROBLEM 31.

In most of our work, we shall be interested only in finding solutions of trigonometric equations for $0 \le \theta < 2\pi$.

EXAMPLE 3 Solving a Linear Trigonometric Equation

Solve the equation: $2 \sin\theta + \sqrt{3} = 0, \quad 0 \le \theta < 2\pi$

Solution We solve the equation for $\sin\theta$.

$$2 \sin\theta + \sqrt{3} = 0$$

$$2 \sin\theta = -\sqrt{3} \qquad \text{Subtract } \sqrt{3} \text{ from both sides.}$$

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \text{Divide both sides by 2.}$$

The period of the sine function is 2π. In the interval $[0, 2\pi)$, there are two angles θ for which $\sin\theta = -\frac{\sqrt{3}}{2}$: $\theta = \frac{4\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

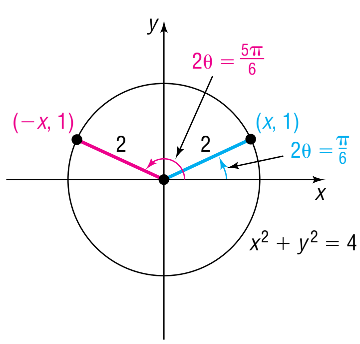
NOW WORK PROBLEM 7.

EXAMPLE 4 Solving a Trigonometric Equation

Solve the equation: $\sin(2\theta) = \frac{1}{2}, \quad 0 \le \theta < 2\pi$

Solution The period of the sine function is 2π. In the interval $[0, 2\pi)$, the sine function has a value $\frac{1}{2}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. See Figure 32. Because the argument is 2θ in the equation $\sin(2\theta) = \frac{1}{2}$, we have

Figure 32



$$2\theta = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad 2\theta = \frac{5\pi}{6} + 2k\pi \qquad k \text{ any integer}$$

$$\theta = \frac{\pi}{12} + k\pi \qquad \theta = \frac{5\pi}{12} + k\pi \qquad \text{Divide by 2.}$$

Then

$$\theta = \frac{\pi}{12} + (-1)\pi = \frac{-11\pi}{12} \qquad k = -1 \qquad \theta = \frac{5\pi}{12} + (-1)\pi = \frac{-7\pi}{12}$$

$$\theta = \frac{\pi}{12} + (0)\pi = \frac{\pi}{12} \qquad k = 0 \qquad \theta = \frac{5\pi}{12} + (0)\pi = \frac{5\pi}{12}$$

$$\theta = \frac{\pi}{12} + (1)\pi = \frac{13\pi}{12} \qquad k = 1 \qquad \theta = \frac{5\pi}{12} + (1)\pi = \frac{17\pi}{12}$$

$$\theta = \frac{\pi}{12} + (2)\pi = \frac{25\pi}{12} \qquad k = 2 \qquad \theta = \frac{5\pi}{12} + (2)\pi = \frac{29\pi}{12}$$

In the interval $[0, 2\pi)$, the solutions of $\sin(2\theta) = \frac{1}{2}$ are $\theta = \frac{\pi}{12}, \theta = \frac{5\pi}{12}, \theta = \frac{13\pi}{12}$, and $\theta = \frac{17\pi}{12}$.

✔ **CHECK:** Verify these solutions by graphing $Y_1 = \sin(2x)$ and $Y_2 = \frac{1}{2}$ for $0 \le x \le 2\pi$.

WARNING In solving a trigonometric equation for $\theta, 0 \le \theta < 2\pi$, in which the argument is not θ (as in Example 4), you must write down all the solutions first and then list those that are in the interval $[0, 2\pi)$. Otherwise, solutions may be lost. For example, in solving $\sin(2\theta) = \frac{1}{2}$, if you merely write the solutions $2\theta = \frac{\pi}{6}$ and $2\theta = \frac{5\pi}{6}$, you will find only $\theta = \frac{\pi}{12}$ and $\theta = \frac{5\pi}{12}$ and miss the other solutions.

NOW WORK PROBLEM 13.

EXAMPLE 5 Solving a Trigonometric Equation

Solve the equation: $\tan\left(\theta - \frac{\pi}{2}\right) = 1, \quad 0 \le \theta < 2\pi$

Solution The period of the tangent function is π. In the interval $[0, \pi)$, the tangent function has the value 1 when the argument is $\frac{\pi}{4}$. Because the argument is $\theta - \frac{\pi}{2}$ in the given equation, we have

$$\theta - \frac{\pi}{2} = \frac{\pi}{4} + k\pi \qquad k \text{ any integer}$$

$$\theta = \frac{3\pi}{4} + k\pi$$

In the interval $[0, 2\pi)$, $\theta = \frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$ are the only solutions.

✔ **CHECK:** Verify these solutions using a graphing utility.

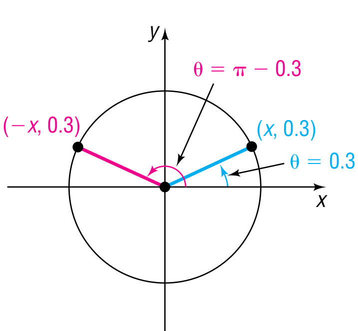
The next example illustrates how to solve trigonometric equations using a calculator. Remember that the function keys on a calculator will only give values consistent with the definition of the function.

EXAMPLE 6 Solving a Trigonometric Equation with a Calculator

Use a calculator to solve the equation: $\sin\theta = 0.3, 0 \le \theta < 2\pi$ Express any solutions in radians, rounded to two decimal places.

Solution To solve $\sin\theta = 0.3$ on a calculator, first set the mode to radians. Then use the $\boxed{\sin^{-1}}$ key to obtain

Figure 33



$$\theta = \sin^{-1}(0.3) \approx 0.3046927$$

Rounded to two decimal places, $\theta = \sin^{-1}(0.3) = 0.30$ radian. Because of the definition of $y = \sin^{-1} x$, the angle θ that we obtain is the angle $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ for which $\sin\theta = 0.3$. Another angle for which $\sin\theta = 0.3$ is $\pi - 0.30$. See Figure 33. The angle $\pi - 0.30$ is the angle in quadrant II, where $\sin\theta = 0.3$. The solutions for $\sin\theta = 0.3, 0 \le \theta < 2\pi$, are

$$\theta = 0.30 \text{ radian} \quad \text{and} \quad \theta = \pi - 0.30 \approx 2.84 \text{ radians}$$

A second method for solving $\sin\theta = 0.3, 0 \le \theta < 2\pi$, would be to graph $Y_1 = \sin x$ and $Y_2 = 0.3$ for $0 \le x < 2\pi$ and find their point(s) of intersection. Try this method for yourself to verify the results obtained in Example 6.

WARNING Example 6 illustrates that caution must be exercised when solving trigonometric equations on a calculator. Remember that the calculator supplies an angle only within the restrictions of the definition of the inverse trigonometric function. To find the remaining solutions, you must identify other quadrants, if any, in which a solution may be located.

NOW WORK PROBLEM 41.

6.7 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Solve: $3x - 5 = -x + 1$. (p. 986)

2. $\sin\left(\frac{\pi}{4}\right) =$ ____; $\cos\left(\frac{8\pi}{3}\right) =$ ____. (p. 379 and pp. 380–381)

Concepts and Vocabulary

3. Two solutions of the equation $\sin\theta = \frac{1}{2}$ are _____ and _____.

4. All the solutions of the equation $\sin\theta = \frac{1}{2}$ are _____.

5. *True or False:* Most trigonometric equations have unique solutions.

6. *True or False:* The equation $\sin\theta = 2$ has a real solution that can be found using a graphing calculator.

Skill Building

In Problems 7–30, solve each equation on the interval $0 \le \theta < 2\pi$.

7. $2\sin\theta + 3 = 2$
8. $1 - \cos\theta = \frac{1}{2}$
9. $4\cos^2\theta = 1$
10. $\tan^2\theta = \frac{1}{3}$
11. $2\sin^2\theta - 1 = 0$
12. $4\cos^2\theta - 3 = 0$
13. $\sin(3\theta) = -1$
14. $\tan\frac{\theta}{2} = \sqrt{3}$
15. $\cos(2\theta) = -\frac{1}{2}$
16. $\tan(2\theta) = -1$
17. $\sec\frac{3\theta}{2} = -2$
18. $\cot\frac{2\theta}{3} = -\sqrt{3}$
19. $2\sin\theta + 1 = 0$
20. $\cos\theta + 1 = 0$
21. $\tan\theta + 1 = 0$
22. $\sqrt{3}\cot\theta + 1 = 0$
23. $4\sec\theta + 6 = -2$
24. $5\csc\theta - 3 = 2$
25. $3\sqrt{2}\cos\theta + 2 = -1$
26. $4\sin\theta + 3\sqrt{3} = \sqrt{3}$
27. $\cos\left(2\theta - \frac{\pi}{2}\right) = -1$
28. $\sin\left(3\theta + \frac{\pi}{18}\right) = 1$
29. $\tan\left(\frac{\theta}{2} + \frac{\pi}{3}\right) = 1$
30. $\cos\left(\frac{\theta}{3} - \frac{\pi}{4}\right) = \frac{1}{2}$

In Problems 31–40, solve each equation. Give a general formula for all the solutions. List six solutions.

31. $\sin\theta = \frac{1}{2}$
32. $\tan\theta = 1$
33. $\tan\theta = -\frac{\sqrt{3}}{3}$
34. $\cos\theta = -\frac{\sqrt{3}}{2}$
35. $\cos\theta = 0$

36. $\sin\theta = \frac{\sqrt{2}}{2}$ **37.** $\cos(2\theta) = -\frac{1}{2}$ **38.** $\sin(2\theta) = -1$ **39.** $\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{2}$ **40.** $\tan\frac{\theta}{2} = -1$

In Problems 41–52, use a calculator to solve each equation on the interval $0 \le \theta < 2\pi$. *Round answers to two decimal places.*

41. $\sin\theta = 0.4$ **42.** $\cos\theta = 0.6$ **43.** $\tan\theta = 5$ **44.** $\cot\theta = 2$

45. $\cos\theta = -0.9$ **46.** $\sin\theta = -0.2$ **47.** $\sec\theta = -4$ **48.** $\csc\theta = -3$

49. $5\tan\theta + 9 = 0$ **50.** $4\cot\theta = -5$ **51.** $3\sin\theta - 2 = 0$ **52.** $4\cos\theta + 3 = 0$

Applications and Extensions

53. Suppose that $f(x) = 3\sin x$.
(a) Solve $f(x) = \frac{3}{2}$.
(b) For what values of x is $f(x) > \frac{3}{2}$ on the interval $[0, 2\pi)$?

54. Suppose that $f(x) = 2\cos x$.
(a) Solve $f(x) = -\sqrt{3}$.
(b) For what values of x is $f(x) < -\sqrt{3}$ on the interval $[0, 2\pi)$?

55. Suppose that $f(x) = 4\tan x$.
(a) Solve $f(x) = -4$.
(b) For what values of x is $f(x) < -4$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

56. Suppose that $f(x) = \cot x$.
(a) Solve $f(x) = -\sqrt{3}$.
(b) For what values of x is $f(x) > -\sqrt{3}$ on the interval $(0, \pi)$?

57. The Ferris Wheel In 1893, George Ferris engineered the Ferris Wheel. It was 250 feet in diameter. If the wheel makes 1 revolution every 40 seconds, then

$$h(t) = 125\sin\left(0.157t - \frac{\pi}{2}\right) + 125$$

represents the height h, in feet, of a seat on the wheel as a function of time t, where t is measured in seconds. The ride begins when $t = 0$.
(a) During the first 40 seconds of the ride, at what time t is an individual on the Ferris Wheel exactly 125 feet above the ground?
(b) During the first 80 seconds of the ride, at what time t is an individual on the Ferris Wheel exactly 250 feet above the ground?
(c) During the first 40 seconds of the ride, over what interval of time t is an individual on the Ferris Wheel more than 125 feet above the ground?

58. Tire Rotation The P215/65R15 Cobra Radial G/T tire has a diameter of exactly 26 inches. Suppose that a car's wheel is making 2 revolutions per second (the car is traveling a little less than 10 miles per hour). Then $h(t) = 13\sin\left(4\pi t - \frac{\pi}{2}\right) + 13$ represents the height h (in inches) of a point on the tire as a function of time t (in seconds). The car starts to move when $t = 0$.
(a) During the first second that the car is moving, at what time t is the point on the tire exactly 13 inches above the ground?
(b) During the first second that the car is moving, at what time t is the point on the tire exactly 6.5 inches above the ground?
(c) During the first second that the car is moving, at what time t is the point on the tire more than 13 inches above the ground?
SOURCE: Cobra Tire

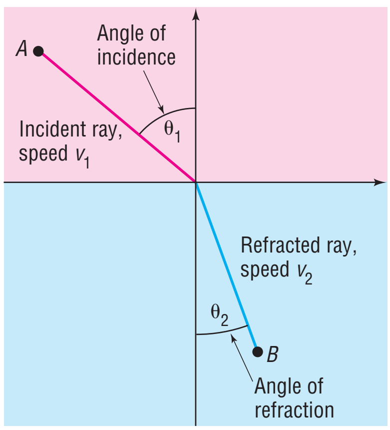
59. Holding Pattern Suppose that an airplane is asked to stay within a holding pattern near Chicago's O'Hare International Airport. The function $d(x) = 70\sin(0.65x) + 150$ represents the distance d, in miles, that the airplane is from the airport at time x, in minutes.
(a) When the plane enters the holding pattern, $x = 0$, how far is it from O'Hare?
(b) During the first 20 minutes after the plane enters the holding pattern, at what time x is the plane exactly 100 miles from the airport?
(c) During the first 20 minutes after the plane enters the holding pattern, at what time x is the plane more than 100 miles from the airport?
(d) While the plane is in the holding pattern, will it ever be within 70 miles of the airport? Why?

60. Projectile Motion A golfer hits a golf ball with an initial velocity of 100 miles per hour. The range R of the ball as a function of the angle θ to the horizontal is given by $R(\theta) = 672\sin(2\theta)$, where R is measured in feet.
(a) At what angle θ should the ball be hit if the golfer wants the ball to travel 450 feet (150 yards)?
(b) At what angle θ should the ball be hit if the golfer wants the ball to travel 540 feet (180 yards)?
(c) At what angle θ should the ball be hit if the golfer wants the ball to travel at least 480 feet (160 yards)?
(d) Can the golfer hit the ball 720 feet (240 yards)?

The following discussion of Snell's Law of Refraction (named after Willebrord Snell, 1580–1626) is needed for Problems 61–67. Light, sound, and other waves travel at different speeds, depending on the media (air, water, wood, and so on) through which they pass. Suppose that light travels from a point A in one medium, where its speed is v_1, to a point B in another medium, where its speed is v_2. Refer to the figure, where the angle θ_1 is called the ***angle of incidence*** *and the angle θ_2 is the* ***angle of refraction****. Snell's Law,* which can be proved using calculus, states that*

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

The ratio $\frac{v_1}{v_2}$ is called the ***index of refraction****. Some values are given in the following table.*



SOME INDEXES OF REFRACTION

Medium	Index of Refraction†
Water	1.33
Ethyl alcohol	1.36
Carbon disulfide	1.63
Air (1 atm and 20°C)	1.0003
Methylene iodide	1.74
Fused quartz	1.46
Glass, crown	1.52
Glass, dense flint	1.66
Sodium chloride	1.54

†For light of wavelength 589 nanometers, measured with respect to a vacuum. The index with respect to air is negligibly different in most cases.

61. The index of refraction of light in passing from a vacuum into water is 1.33. If the angle of incidence is 40°, determine the angle of refraction.

62. The index of refraction of light in passing from a vacuum into dense glass is 1.66. If the angle of incidence is 50°, determine the angle of refraction.

63. Ptolemy, who lived in the city of Alexandria in Egypt during the second century AD, gave the measured values in the table below for the angle of incidence θ_1 and the angle of refraction θ_2 for a light beam passing from air into water. Do these values agree with Snell's Law? If so, what index of refraction results? (These data are interesting as the oldest recorded physical measurements.)†

θ_1	θ_2	θ_1	θ_2
10°	7°45′	50°	35°0′
20°	15°30′	60°	40°30′
30°	22°30′	70°	45°30′
40°	29°0′	80°	50°0′

64. The speed of yellow sodium light (wavelength of 589 nanometers) in a certain liquid is measured to be 1.92×10^8 meters per second. What is the index of refraction of this liquid, with respect to air, for sodium light?†

[**Hint:** The speed of light in air is approximately 2.997×10^8 meters per second.]

65. A beam of light with a wavelength of 589 nanometers traveling in air makes an angle of incidence of 40° on a slab of transparent material, and the refracted beam makes an angle of refraction of 26°. Find the index of refraction of the material.†

66. A light ray with a wavelength of 589 nanometers (produced by a sodium lamp) traveling through air makes an angle of incidence of 30° on a smooth, flat slab of crown glass. Find the angle of refraction.†

67. A light beam passes through a thick slab of material whose index of refraction is n_2. Show that the emerging beam is parallel to the incident beam.‡

*Because this law was also deduced by René Descartes in France, it is also known as Descartes's Law.
†Adapted from Halliday and Resnick, *Physics*, Part 1 & 2, 3rd ed. New York: Wiley.
‡*Physics for Scientists & Engineers* 3/E by Serway. © 1990. Reprinted with permission of Brooks/Cole, a division of Thomson Learning.

Discussion and Writing

68. Explain in your own words how you would use your calculator to solve the equation $\sin x = 0.3, 0 \le x < 2\pi$. How would you modify your approach in order to solve the equation $\cot x = 5, 0 < x < 2\pi$?

'Are You Prepared?' Answers

1. $\left\{\frac{3}{2}\right\}$ **2.** $\frac{\sqrt{2}}{2}; -\frac{1}{2}$

6.8 Trigonometric Equations (II)

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Quadratic Equations by Factoring (Appendix, Section A.5, p. 989)
- Solving Equations in One Variable Using a Graphing Utility (Section 1.3, pp. 24–26)
- The Quadratic Formula (Appendix, Section A.5, pp. 992–994)
- Solve Equations Quadratic in Form (Appendix, Section A.5, pp. 995–996)

Now work the 'Are You Prepared?' problems on page 508.

OBJECTIVES
1 Solve Trigonometric Equations Quadratic in Form
2 Solve Trigonometric Equations Using Identities
3 Solve Trigonometric Equations Linear in Sine and Cosine
4 Solve Trigonometric Equations Using a Graphing Utility

1 Solve Trigonometric Equations Quadratic in From

In this section we continue our study of trigonometric equations. Many trigonometric equations can be solved by applying techniques that we already know, such as applying the quadratic formula (if the equation is a second-degree polynomial) or factoring.

EXAMPLE 1 Solving a Trigonometric Equation Quadratic in Form

Solve the equation: $2\sin^2\theta - 3\sin\theta + 1 = 0, \quad 0 \le \theta < 2\pi$

Solution The equation that we wish to solve is a quadratic equation (in $\sin\theta$) that can be factored.

$$2\sin^2\theta - 3\sin\theta + 1 = 0 \qquad 2x^2 - 3x + 1 = 0, \quad x = \sin\theta$$
$$(2\sin\theta - 1)(\sin\theta - 1) = 0 \qquad (2x - 1)(x - 1) = 0$$
$$2\sin\theta - 1 = 0 \quad \text{or} \quad \sin\theta - 1 = 0 \qquad \text{Zero-product Property.}$$
$$\sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = 1$$

Solving each equation in the interval $[0, 2\pi)$, we obtain

$$\theta = \frac{\pi}{6}, \quad \theta = \frac{5\pi}{6}, \quad \theta = \frac{\pi}{2}$$

NOW WORK PROBLEM 7.

2 Solve Trigonometric Equations Using Identities

When a trigonometric equation contains more than one trigonometric function, identities sometimes can be used to obtain an equivalent equation that contains only one trigonometric function.

EXAMPLE 2 Solving a Trigonometric Equation Using Identities

Solve the equation: $3\cos\theta + 3 = 2\sin^2\theta, \quad 0 \le \theta < 2\pi$

Solution The equation in its present form contains sines and cosines. However, a form of the Pythagorean Identity can be used to transform the equation into an equivalent expression containing only cosines.

$$\begin{aligned}
3\cos\theta + 3 &= 2\sin^2\theta \\
3\cos\theta + 3 &= 2(1 - \cos^2\theta) && \sin^2\theta = 1 - \cos^2\theta \\
3\cos\theta + 3 &= 2 - 2\cos^2\theta \\
2\cos^2\theta + 3\cos\theta + 1 &= 0 && \text{Quadratic in } \cos\theta \\
(2\cos\theta + 1)(\cos\theta + 1) &= 0 && \text{Factor.} \\
2\cos\theta + 1 = 0 \quad &\text{or} \quad \cos\theta + 1 = 0 && \text{Zero-product Property.} \\
\cos\theta = -\frac{1}{2} \quad &\text{or} \quad \cos\theta = -1
\end{aligned}$$

Solving each equation in the interval $[0, 2\pi)$, we obtain

$$\theta = \frac{2\pi}{3}, \qquad \theta = \frac{4\pi}{3}, \qquad \theta = \pi$$

✔ **CHECK:** Graph $Y_1 = 3\cos x + 3$ and $Y_2 = 2\sin^2 x, 0 \le x \le 2\pi$, and find the points of intersection. How close are your approximate solutions to the exact ones found in this example? ◀

EXAMPLE 3 Solving a Trigonometric Equation Using Identities

Solve the equation: $\cos(2\theta) + 3 = 5\cos\theta, \quad 0 \le \theta < 2\pi$

Solution First, we observe that the given equation contains two cosine functions, but with different arguments, θ and 2θ. We use the Double-angle Formula $\cos(2\theta) = 2\cos^2\theta - 1$ to obtain an equivalent equation containing only $\cos\theta$.

$$\begin{aligned}
\cos(2\theta) + 3 &= 5\cos\theta \\
(2\cos^2\theta - 1) + 3 &= 5\cos\theta && \cos(2\theta) = 2\cos^2\theta - 1 \\
2\cos^2\theta - 5\cos\theta + 2 &= 0 && \text{Place in standard form.} \\
(\cos\theta - 2)(2\cos\theta - 1) &= 0 && \text{Factor.} \\
\cos\theta = 2 \quad \text{or} \quad \cos\theta &= \frac{1}{2}
\end{aligned}$$

For any angle θ, $-1 \le \cos\theta \le 1$; therefore, the equation $\cos\theta = 2$ has no solution. The solutions of $\cos\theta = \frac{1}{2}$, $0 \le \theta < 2\pi$, are

$$\theta = \frac{\pi}{3}, \qquad \theta = \frac{5\pi}{3}$$

✔ **CHECK:** Graph $Y_1 = \cos(2x) + 3$ and $Y_2 = 5\cos x$, $0 \le x \le 2\pi$, and find the points of intersection. Compare your results with those of Example 3. ◀

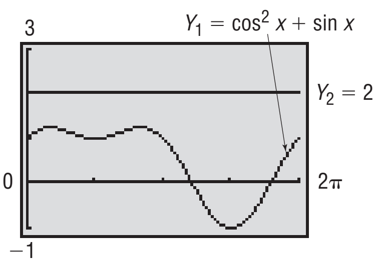
NOW WORK PROBLEM 23.

EXAMPLE 4 Solving a Trigonometric Equation Using Identities

Solve the equation: $\cos^2\theta + \sin\theta = 2, \quad 0 \le \theta < 2\pi$

Solution This equation involves two trigonometric functions, sine and cosine. We use a form of the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$ to rewrite the equation in terms of $\sin\theta$.

Figure 34



$$\begin{aligned} \cos^2\theta + \sin\theta &= 2 \\ (1 - \sin^2\theta) + \sin\theta &= 2 \qquad \cos^2\theta = 1 - \sin^2\theta \\ \sin^2\theta - \sin\theta + 1 &= 0 \end{aligned}$$

This is a quadratic equation in $\sin\theta$. The discriminant is $b^2 - 4ac = 1 - 4 = -3 < 0$. Therefore, the equation has no real solution.

✔ **CHECK:** Graph $Y_1 = \cos^2 x + \sin x$ and $Y_2 = 2$. See Figure 34. The two graphs do not intersect, so the equation $Y_1 = Y_2$ has no real solution. ◀

EXAMPLE 5 Solving a Trigonometric Equation Using Identities

Solve the equation: $\sin\theta\cos\theta = -\frac{1}{2}, \quad 0 \le \theta < 2\pi$

Solution The left side of the given equation is in the form of the Double-angle Formula $2\sin\theta\cos\theta = \sin(2\theta)$, except for a factor of 2. We multiply each side by 2.

$$\begin{aligned} \sin\theta\cos\theta &= -\frac{1}{2} \\ 2\sin\theta\cos\theta &= -1 \qquad \text{Multiply each side by 2.} \\ \sin(2\theta) &= -1 \qquad \text{Double-angle Formula} \end{aligned}$$

The argument here is 2θ. So we need to write all the solutions of this equation and then list those that are in the interval $[0, 2\pi)$. Because $\sin\left(\frac{3\pi}{2} + 2\pi k\right) = -1$, for any integer k we have

$$\begin{aligned} 2\theta &= \frac{3\pi}{2} + 2k\pi \qquad k \text{ any integer} \\ \theta &= \frac{3\pi}{4} + k\pi \end{aligned}$$

$$\underset{k=-1}{\theta = \frac{3\pi}{4} + (-1)\pi = -\frac{\pi}{4}}, \quad \underset{k=0}{\theta = \frac{3\pi}{4} + (0)\pi = \frac{3\pi}{4}}, \quad \underset{k=1}{\theta = \frac{3\pi}{4} + (1)\pi = \frac{7\pi}{4}}, \quad \underset{k=2}{\theta = \frac{3\pi}{4} + (2)\pi = \frac{11\pi}{4}}$$

The solutions in the interval $[0, 2\pi)$ are

$$\theta = \frac{3\pi}{4}, \qquad \theta = \frac{7\pi}{4}$$

3 Solve Trigonometric Equations Linear in Sine and Cosine

Sometimes it is necessary to square both sides of an equation to obtain expressions that allow the use of identities. Remember, squaring both sides of an equation may introduce extraneous solutions. As a result, apparent solutions must be checked.

EXAMPLE 6 Solving a Trigonometric Equation Linear in Sine and Cosine

Solve the equation: $\sin\theta + \cos\theta = 1, \quad 0 \le \theta < 2\pi$

Solution A Attempts to use available identities do not lead to equations that are easy to solve. (Try it yourself.) Given the form of this equation, we decide to square each side.

$$\begin{aligned} \sin\theta + \cos\theta &= 1 & \\ (\sin\theta + \cos\theta)^2 &= 1 & \text{Square each side.} \\ \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta &= 1 & \text{Remove parentheses.} \\ 2\sin\theta\cos\theta &= 0 & \sin^2\theta + \cos^2\theta = 1 \\ \sin\theta\cos\theta &= 0 & \end{aligned}$$

Setting each factor equal to zero, we obtain

$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = 0$$

The apparent solutions are

$$\theta = 0, \qquad \theta = \pi, \qquad \theta = \frac{\pi}{2}, \qquad \theta = \frac{3\pi}{2}$$

Because we squared both sides of the original equation, we must check these apparent solutions to see if any are extraneous.

$$\begin{aligned} &\theta = 0: && \sin 0 + \cos 0 = 0 + 1 = 1 && \text{A solution} \\ &\theta = \pi: && \sin\pi + \cos\pi = 0 + (-1) = -1 && \text{Not a solution} \\ &\theta = \frac{\pi}{2}: && \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1 && \text{A solution} \\ &\theta = \frac{3\pi}{2}: && \sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1 + 0 = -1 && \text{Not a solution} \end{aligned}$$

The values $\theta = \pi$ and $\theta = \dfrac{3\pi}{2}$ are extraneous. The solution set is $\left\{0, \dfrac{\pi}{2}\right\}$.

Solution B We start with the equation

$$\sin\theta + \cos\theta = 1$$

and divide each side by $\sqrt{2}$. (The reason for this choice will become apparent shortly.) Then

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$$

The left side now resembles the formula for the sine of the sum of two angles, one of which is θ. The other angle is unknown (call it ϕ.) Then

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{1}$$

where

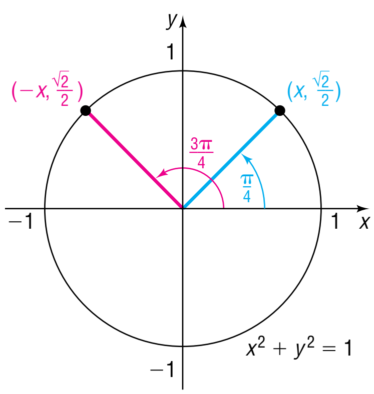
$$\cos\phi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \sin\phi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad 0 \le \phi < 2\pi$$

The angle ϕ is therefore $\frac{\pi}{4}$. As a result, equation (1) becomes

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

In the interval $[0, 2\pi)$, there are two angles whose sine is $\frac{\sqrt{2}}{2}$: $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. See Figure 35. As a result,

Figure 35



$$\theta + \frac{\pi}{4} = \frac{\pi}{4} \quad \text{or} \quad \theta + \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\theta = 0 \quad \text{or} \quad \theta = \frac{\pi}{2}$$

The solution set is $\left\{0, \frac{\pi}{2}\right\}$. ◀

This second method of solution can be used to solve any linear equation in the variables $\sin\theta$ and $\cos\theta$.

EXAMPLE 7 Solving a Trigonometric Equation Linear in Sin θ and Cos θ

Solve:

$$a\sin\theta + b\cos\theta = c, \qquad 0 \le \theta < 2\pi \tag{2}$$

where a, b, and c are constants and either $a \ne 0$ or $b \ne 0$.

Solution We divide each side of equation (2) by $\sqrt{a^2 + b^2}$. Then

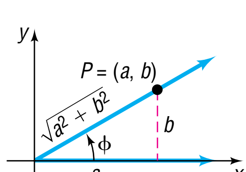
$$\frac{a}{\sqrt{a^2 + b^2}}\sin\theta + \frac{b}{\sqrt{a^2 + b^2}}\cos\theta = \frac{c}{\sqrt{a^2 + b^2}} \tag{3}$$

There is a unique angle ϕ, $0 \le \phi < 2\pi$, for which

$$\cos\phi = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \sin\phi = \frac{b}{\sqrt{a^2 + b^2}} \tag{4}$$

See Figure 36. Equation (3) may be written as

Figure 36



$$\sin\theta\cos\phi + \cos\theta\sin\phi = \frac{c}{\sqrt{a^2 + b^2}}$$

or, equivalently,

$$\sin(\theta + \phi) = \frac{c}{\sqrt{a^2 + b^2}} \tag{5}$$

where ϕ satisfies equation (4).

If $|c| > \sqrt{a^2 + b^2}$, then $\sin(\theta + \phi) > 1$ or $\sin(\theta + \phi) < -1$, and equation (5) has no solution.

If $|c| \leq \sqrt{a^2 + b^2}$, then the solutions of equation (5) are

$$\theta + \phi = \sin^{-1}\frac{c}{\sqrt{a^2 + b^2}} \quad \text{or} \quad \theta + \phi = \pi - \sin^{-1}\frac{c}{\sqrt{a^2 + b^2}}$$

Because the angle ϕ is determined by equations (4), these are the solutions to equation (2).

NOW WORK PROBLEM 41.

4 Solve Trigonometric Equations Using a Graphing Utility

The techniques introduced in this section apply only to certain types of trigonometric equations. Solutions for other types are usually studied in calculus, using numerical methods. In the next example, we show how a graphing utility may be used to obtain solutions.

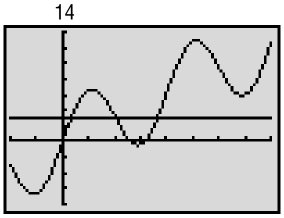
EXAMPLE 8 Solving Trigonometric Equations Using a Graphing Utility

Solve: $5 \sin x + x = 3$

Express the solution(s) rounded to two decimal places.

Solution This type of trigonometric equation cannot be solved by previous methods. A graphing utility, though, can be used here. The solution(s) of this equation is the same as the points of intersection of the graphs of $Y_1 = 5 \sin x + x$ and $Y_2 = 3$. See Figure 37.

Figure 37



There are three points of intersection; the x-coordinates are the solutions that we seek. Using INTERSECT, we find

$$x = 0.52, \qquad x = 3.18, \qquad x = 5.71$$

NOW WORK PROBLEM 53.

6.8 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find the real solutions of $4x^2 - x - 5 = 0$. (p. 989)
2. Find the real solutions of $x^2 - x - 1 = 0$. (pp. 992–994)
3. Find the real solutions of $(2x - 1)^2 - 3(2x - 1) - 4 = 0$ (pp. 995–996)
4. Use a graphing utility to solve $5x^3 - 2 = x - x^2$. Round answers to two decimal places. (pp. 24–26)

Skill Building

In Problems 5–46, solve each equation on the interval $0 \le \theta < 2\pi$.

5. $2\cos^2\theta + \cos\theta = 0$
6. $\sin^2\theta - 1 = 0$
7. $2\sin^2\theta - \sin\theta - 1 = 0$
8. $2\cos^2\theta + \cos\theta - 1 = 0$
9. $(\tan\theta - 1)(\sec\theta - 1) = 0$
10. $(\cot\theta + 1)\left(\csc\theta - \frac{1}{2}\right) = 0$
11. $\sin^2\theta - \cos^2\theta = 1 + \cos\theta$
12. $\cos^2\theta - \sin^2\theta + \sin\theta = 0$
13. $\sin^2\theta = 6(\cos\theta + 1)$
14. $2\sin^2\theta = 3(1 - \cos\theta)$
15. $\cos(2\theta) + 6\sin^2\theta = 4$
16. $\cos(2\theta) = 2 - 2\sin^2\theta$
17. $\cos\theta = \sin\theta$
18. $\cos\theta + \sin\theta = 0$
19. $\tan\theta = 2\sin\theta$
20. $\sin(2\theta) = \cos\theta$
21. $\sin\theta = \csc\theta$
22. $\tan\theta = \cot\theta$
23. $\cos(2\theta) = \cos\theta$
24. $\sin(2\theta)\sin\theta = \cos\theta$
25. $\sin(2\theta) + \sin(4\theta) = 0$
26. $\cos(2\theta) + \cos(4\theta) = 0$
27. $\cos(4\theta) - \cos(6\theta) = 0$
28. $\sin(4\theta) - \sin(6\theta) = 0$
29. $1 + \sin\theta = 2\cos^2\theta$
30. $\sin^2\theta = 2\cos\theta + 2$
31. $2\sin^2\theta - 5\sin\theta + 3 = 0$
32. $2\cos^2\theta - 7\cos\theta - 4 = 0$
33. $3(1 - \cos\theta) = \sin^2\theta$
34. $4(1 + \sin\theta) = \cos^2\theta$
35. $\tan^2\theta = \frac{3}{2}\sec\theta$
36. $\csc^2\theta = \cot\theta + 1$
37. $3 - \sin\theta = \cos(2\theta)$
38. $\cos(2\theta) + 5\cos\theta + 3 = 0$
39. $\sec^2\theta + \tan\theta = 0$
40. $\sec\theta = \tan\theta + \cot\theta$
41. $\sin\theta - \sqrt{3}\cos\theta = 1$
42. $\sqrt{3}\sin\theta + \cos\theta = 1$
43. $\tan(2\theta) + 2\sin\theta = 0$
44. $\tan(2\theta) + 2\cos\theta = 0$
45. $\sin\theta + \cos\theta = \sqrt{2}$
46. $\sin\theta + \cos\theta = -\sqrt{2}$

In Problems 47–52, solve each equation for x, $-\pi \le x \le \pi$. *Express the solution(s) rounded to two decimal places.*

47. Solve the equation $\cos x = e^x$ by graphing $Y_1 = \cos x$ and $Y_2 = e^x$ and finding their point(s) of intersection.

48. Solve the equation $\cos x = e^x$ by graphing $Y_1 = \cos x - e^x$ and finding the x-intercept(s).

49. Solve the equation $2\sin x = 0.7x$ by graphing $Y_1 = 2\sin x$ and $Y_2 = 0.7x$ and finding their point(s) of intersection.

50. Solve the equation $2\sin x = 0.7x$ by graphing $Y_1 = 2\sin x - 0.7x$ and finding the x-intercept(s).

51. Solve the equation $\cos x = x^2$ by graphing $Y_1 = \cos x$ and $Y_2 = x^2$ and finding their point(s) of intersection.

52. Solve the equation $\cos x = x^2$ by graphing $Y_1 = \cos x - x^2$ and finding the x-intercept(s).

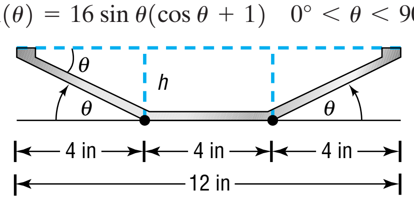
In Problems 53–64, use a graphing utility to solve each equation. Express the solution(s) rounded to two decimal places.

53. $x + 5\cos x = 0$
54. $x - 4\sin x = 0$
55. $22x - 17\sin x = 3$
56. $19x + 8\cos x = 2$
57. $\sin x + \cos x = x$
58. $\sin x - \cos x = x$
59. $x^2 - 2\cos x = 0$
60. $x^2 + 3\sin x = 0$
61. $x^2 - 2\sin(2x) = 3x$
62. $x^2 = x + 3\cos(2x)$
63. $6\sin x - e^x = 2, \quad x > 0$
64. $4\cos(3x) - e^x = 1, \quad x > 0$

Applications and Extensions

65. Constructing a Rain Gutter A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, this length is bent up at an angle θ. See the illustration. The area A of the opening as a function of θ is given by

$$A(\theta) = 16\sin\theta(\cos\theta + 1) \quad 0° < \theta < 90°$$



(a) In calculus, you will be asked to find the angle θ that maximizes A by solving the equation

$$\cos(2\theta) + \cos\theta = 0, \quad 0° < \theta < 90°$$

Solve this equation for θ by using the Double-angle Formula.

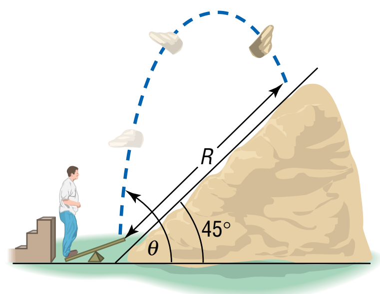
(b) Solve the equation for θ by writing the sum of the two cosines as a product.

(c) What is the maximum area A of the opening?

(d) Graph $A = A(\theta)$, $0° \le \theta \le 90°$, and find the angle θ that maximizes the area A. Also find the maximum area. Compare the results to the answers found earlier.

66. Projectile Motion An object is propelled upward at an angle θ, $45^\circ < \theta < 90^\circ$, to the horizontal with an initial velocity of v_0 feet per second from the base of a plane that makes an angle of 45° with the horizontal. See the illustration. If air resistance is ignored, the distance R that it travels up the inclined plane is given by

$$R(\theta) = \frac{v_0^2\sqrt{2}}{32}[\sin(2\theta) - \cos(2\theta) - 1]$$



(a) In calculus, you will be asked to find the angle θ that maximizes R by solving the equation

$$\sin(2\theta) + \cos(2\theta) = 0$$

Solve this equation for θ using the method of Example 7.

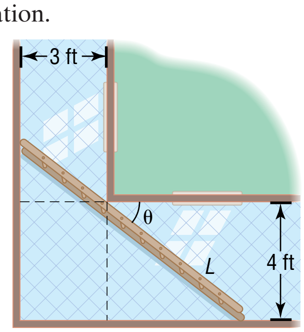
(b) Solve this equation for θ by dividing each side by $\cos(2\theta)$.

(c) What is the maximum distance R if $v_0 = 32$ feet per second?

(d) Graph $R = R(\theta)$, $45^\circ \le \theta \le 90^\circ$, and find the angle θ that maximizes the distance R. Also find the maximum distance. Use $v_0 = 32$ feet per second. Compare the results with the answers found earlier.

67. Heat Transfer In the study of heat transfer, the equation $x + \tan x = 0$ occurs. Graph $Y_1 = -x$ and $Y_2 = \tan x$ for $x \ge 0$. Conclude that there are an infinite number of points of intersection of these two graphs. Now find the first two positive solutions of $x + \tan x = 0$ rounded to two decimal places.

68. Carrying a Ladder Around a Corner Two hallways, one of width 3 feet, the other of width 4 feet, meet at a right angle. See the illustration.



(a) Express the length L of the line segment shown as a function of θ.

(b) In calculus, you will be asked to find the length of the longest ladder that can turn the corner by solving the equation

$$3 \sec\theta \tan\theta - 4 \csc\theta \cot\theta = 0, \quad 0^\circ < \theta < 90^\circ$$

Solve this equation for θ.

(c) What is the length of the longest ladder that can be carried around the corner?

(d) Graph $L = L(\theta)$, $0^\circ \le \theta \le 90^\circ$, and find the angle θ that minimizes the length L.

(e) Compare the result with the one found in part (b). Explain why the two answers are the same.

69. Projectile Motion The horizontal distance that a projectile will travel in the air (ignoring air resistance) is given by the equation

$$R(\theta) = \frac{v_0^2 \sin(2\theta)}{g}$$

where v_0 is the initial velocity of the projectile, θ is the angle of elevation, and g is acceleration due to gravity (9.8 meters per second squared).

(a) If you can throw a baseball with an initial speed of 34.8 meters per second, at what angle of elevation θ should you direct the throw so that the ball travels a distance of 107 meters before striking the ground?

(b) Determine the maximum distance that you can throw the ball.

(c) Graph $R = R(\theta)$, with $v_0 = 34.8$ meters per second.

(d) Verify the results obtained in parts (a) and (b) using a graphing utility.

70. Projectile Motion Refer to Problem 69.

(a) If you can throw a baseball with an initial speed of 40 meters per second, at what angle of elevation θ should you direct the throw so that the ball travels a distance of 110 meters before striking the ground?

(b) Determine the maximum distance that you can throw the ball.

(c) Graph $R = R(\theta)$, with $v_0 = 40$ meters per second.

(d) Verify the results obtained in parts (a) and (b) using a graphing utility.

'Are You Prepared?' Answers

1. $\left\{-1, \frac{5}{4}\right\}$ **2.** $\left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$ **3.** $\left\{0, \frac{5}{2}\right\}$ **4.** $\{0.76\}$

Chapter Review

Things to Know

Definitions of the six inverse trigonometric functions

$y = \sin^{-1} x$ means $x = \sin y$ where $-1 \le x \le 1$, $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ (p. 449)

$y = \cos^{-1} x$ means $x = \cos y$ where $-1 \le x \le 1$, $0 \le y \le \pi$ (p. 453)

$y = \tan^{-1} x$ means $x = \tan y$ where $-\infty < x < \infty$, $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$ (p. 456)

$y = \sec^{-1} x$ means $x = \sec y$ where $|x| \ge 1$, $0 \le y \le \pi$, $y \ne \dfrac{\pi}{2}$ (p. 462)

$y = \csc^{-1} x$ means $x = \csc y$ where $|x| \ge 1$, $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$, $y \ne 0$ (p. 462)

$y = \cot^{-1} x$ means $x = \cot y$ where $-\infty < x < \infty$, $0 < y < \pi$ (p. 462)

Sum and Difference Formulas (pp. 473, 476, and 479)

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double-angle Formulas (pp. 484 and 485)

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta \qquad \tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \qquad \cos(2\theta) = 1 - 2\sin^2\theta$$

Half-angle Formulas (pp. 487 and 490)

$$\sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2} \qquad \tan^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{1 + \cos\alpha}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}} \qquad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} = \frac{1 - \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 + \cos\alpha}$$

where the $+$ or $-$ is determined by the quadrant of $\dfrac{\alpha}{2}$

Product-to-Sum Formulas (p. 493)

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Sum-to-Product Formulas (p. 494)

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2} \qquad \sin\alpha - \sin\beta = 2\sin\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2} \qquad \cos\alpha - \cos\beta = -2\sin\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2}$$

Objectives

Section		You should be able to . . .	Review Exercises
6.1	1	Find the exact value of the inverse sine, cosine, and tangent functions (p. 449)	1–6
	2	Find an approximate value of the inverse sine, cosine, and tangent functions (p. 451)	101–104
6.2	1	Find the exact value of expressions involving the inverse sine, cosine, and tangent functions (p. 460)	9–20
	2	Know the definition of the inverse secant, cosecant, and cotangent functions (p. 462)	7, 8
	3	Use a calculator to evaluate $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$ (p. 463)	105, 106
6.3	1	Use algebra to simplify trigonometric expressions (p. 466)	21–52
	2	Establish identities (p. 467)	21–38
6.4	1	Use sum and difference formulas to find exact values (p. 474)	53–60, 61–70(a)–(d)
	2	Use sum and difference formulas to establish identities (p. 475)	39–42
	3	Use sum and difference formulas involving inverse trigonometric functions (p. 480)	71–74
6.5	1	Use Double-angle Formulas to find exact values (p. 484)	61–70(e)–(f), 75, 76
	2	Use Double-angle Formulas to establish identities (p. 485)	43–47
	3	Use Half-angle Formulas to find exact values (p. 487)	61–70(g)–(h)
6.6	1	Express products as sums (p. 493)	48
	2	Express sums as products (p. 494)	49–52
6.7	1	Solve equations involving a single trigonometric function (p. 496)	77–86
6.8	1	Solve trigonometric equations quadratic in form (p. 503)	93, 94
	2	Solve trigonometric equations using identities (p. 504)	87–92, 95–98
	3	Solve trigonometric equations linear in sine and cosine (p. 506)	99–100
	4	Solve trigonometric equations using a graphing utility (p. 508)	107–112

Review Exercises

In Problems 1–20, find the exact value of each expression. Do not use a calculator.

1. $\sin^{-1} 1$ **2.** $\cos^{-1} 0$ **3.** $\tan^{-1} 1$ **4.** $\sin^{-1}\left(-\frac{1}{2}\right)$

5. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **6.** $\tan^{-1}\left(-\sqrt{3}\right)$ **7.** $\sec^{-1}\sqrt{2}$ **8.** $\cot^{-1}(-1)$

9. $\tan\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ **10.** $\tan\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ **11.** $\sec\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$ **12.** $\csc\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$

13. $\sin\left(\tan^{-1}\frac{3}{4}\right)$ **14.** $\cos\left(\sin^{-1}\frac{3}{5}\right)$ **15.** $\tan\left[\sin^{-1}\left(-\frac{4}{5}\right)\right]$ **16.** $\tan\left[\cos^{-1}\left(-\frac{3}{5}\right)\right]$

17. $\sin^{-1}\left(\cos\frac{2\pi}{3}\right)$ **18.** $\cos^{-1}\left(\tan\frac{3\pi}{4}\right)$ **19.** $\tan^{-1}\left(\tan\frac{7\pi}{4}\right)$ **20.** $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

In Problems 21–52, establish each identity.

21. $\tan\theta\cot\theta - \sin^2\theta = \cos^2\theta$ **22.** $\sin\theta\csc\theta - \sin^2\theta = \cos^2\theta$ **23.** $\cos^2\theta(1 + \tan^2\theta) = 1$

24. $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$ **25.** $4\cos^2\theta + 3\sin^2\theta = 3 + \cos^2\theta$ **26.** $4\sin^2\theta + 2\cos^2\theta = 4 - 2\cos^2\theta$

27. $\frac{1 - \cos\theta}{\sin\theta} + \frac{\sin\theta}{1 - \cos\theta} = 2\csc\theta$ **28.** $\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = 2\csc\theta$ **29.** $\frac{\cos\theta}{\cos\theta - \sin\theta} = \frac{1}{1 - \tan\theta}$

30. $1 - \frac{\cos^2\theta}{1 + \sin\theta} = \sin\theta$ **31.** $\frac{\csc\theta}{1 + \csc\theta} = \frac{1 - \sin\theta}{\cos^2\theta}$ **32.** $\frac{1 + \sec\theta}{\sec\theta} = \frac{\sin^2\theta}{1 - \cos\theta}$

33. $\csc\theta - \sin\theta = \cos\theta\cot\theta$

34. $\dfrac{\csc\theta}{1-\cos\theta} = \dfrac{1+\cos\theta}{\sin^3\theta}$

35. $\dfrac{1-\sin\theta}{\sec\theta} = \dfrac{\cos^3\theta}{1+\sin\theta}$

36. $\dfrac{1-\cos\theta}{1+\cos\theta} = (\csc\theta - \cot\theta)^2$

37. $\dfrac{1-2\sin^2\theta}{\sin\theta\cos\theta} = \cot\theta - \tan\theta$

38. $\dfrac{(2\sin^2\theta - 1)^2}{\sin^4\theta - \cos^4\theta} = 1 - 2\cos^2\theta$

39. $\dfrac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta} = \cot\beta - \tan\alpha$

40. $\dfrac{\sin(\alpha-\beta)}{\sin\alpha\cos\beta} = 1 - \cot\alpha\tan\beta$

41. $\dfrac{\cos(\alpha-\beta)}{\cos\alpha\cos\beta} = 1 + \tan\alpha\tan\beta$

42. $\dfrac{\cos(\alpha+\beta)}{\sin\alpha\cos\beta} = \cot\alpha - \tan\beta$

43. $(1+\cos\theta)\tan\dfrac{\theta}{2} = \sin\theta$

44. $\sin\theta\tan\dfrac{\theta}{2} = 1 - \cos\theta$

45. $2\cot\theta\cot(2\theta) = \cot^2\theta - 1$

46. $2\sin(2\theta)(1 - 2\sin^2\theta) = \sin(4\theta)$

47. $1 - 8\sin^2\theta\cos^2\theta = \cos(4\theta)$

48. $\dfrac{\sin(3\theta)\cos\theta - \sin\theta\cos(3\theta)}{\sin(2\theta)} = 1$

49. $\dfrac{\sin(2\theta)+\sin(4\theta)}{\cos(2\theta)+\cos(4\theta)} = \tan(3\theta)$

50. $\dfrac{\sin(2\theta)+\sin(4\theta)}{\sin(2\theta)-\sin(4\theta)} + \dfrac{\tan(3\theta)}{\tan\theta} = 0$

51. $\dfrac{\cos(2\theta)-\cos(4\theta)}{\cos(2\theta)+\cos(4\theta)} - \tan\theta\tan(3\theta) = 0$

52. $\cos(2\theta) - \cos(10\theta) = \tan(4\theta)[\sin(2\theta) + \sin(10\theta)]$

In Problems 53–60, find the exact value of each expression.

53. $\sin 165^\circ$

54. $\tan 105^\circ$

55. $\cos\dfrac{5\pi}{12}$

56. $\sin\left(-\dfrac{\pi}{12}\right)$

57. $\cos 80^\circ\cos 20^\circ + \sin 80^\circ\sin 20^\circ$

58. $\sin 70^\circ\cos 40^\circ - \cos 70^\circ\sin 40^\circ$

59. $\tan\dfrac{\pi}{8}$

60. $\sin\dfrac{5\pi}{8}$

In Problems 61–70, use the information given about the angles α and β to find the exact value of:

(a) $\sin(\alpha+\beta)$ (b) $\cos(\alpha+\beta)$ (c) $\sin(\alpha-\beta)$ (d) $\tan(\alpha+\beta)$

(e) $\sin(2\alpha)$ (f) $\cos(2\beta)$ (g) $\sin\dfrac{\beta}{2}$ (h) $\cos\dfrac{\alpha}{2}$

61. $\sin\alpha = \dfrac{4}{5}, 0 < \alpha < \dfrac{\pi}{2}; \sin\beta = \dfrac{5}{13}, \dfrac{\pi}{2} < \beta < \pi$

62. $\cos\alpha = \dfrac{4}{5}, 0 < \alpha < \dfrac{\pi}{2}; \cos\beta = \dfrac{5}{13}, -\dfrac{\pi}{2} < \beta < 0$

63. $\sin\alpha = -\dfrac{3}{5}, \pi < \alpha < \dfrac{3\pi}{2}; \cos\beta = \dfrac{12}{13}, \dfrac{3\pi}{2} < \beta < 2\pi$

64. $\sin\alpha = -\dfrac{4}{5}, -\dfrac{\pi}{2} < \alpha < 0; \cos\beta = -\dfrac{5}{13}, \dfrac{\pi}{2} < \beta < \pi$

65. $\tan\alpha = \dfrac{3}{4}, \pi < \alpha < \dfrac{3\pi}{2}; \tan\beta = \dfrac{12}{5}, 0 < \beta < \dfrac{\pi}{2}$

66. $\tan\alpha = -\dfrac{4}{3}, \dfrac{\pi}{2} < \alpha < \pi; \cot\beta = \dfrac{12}{5}, \pi < \beta < \dfrac{3\pi}{2}$

67. $\sec\alpha = 2, -\dfrac{\pi}{2} < \alpha < 0; \sec\beta = 3, \dfrac{3\pi}{2} < \beta < 2\pi$

68. $\csc\alpha = 2, \dfrac{\pi}{2} < \alpha < \pi; \sec\beta = -3, \dfrac{\pi}{2} < \beta < \pi$

69. $\sin\alpha = -\dfrac{2}{3}, \pi < \alpha < \dfrac{3\pi}{2}; \cos\beta = -\dfrac{2}{3}, \pi < \beta < \dfrac{3\pi}{2}$

70. $\tan\alpha = -2, \dfrac{\pi}{2} < \alpha < \pi; \cot\beta = -2, \dfrac{\pi}{2} < \beta < \pi$

In Problems 71–76, find the exact value of each expression.

71. $\cos\left(\sin^{-1}\dfrac{3}{5} - \cos^{-1}\dfrac{1}{2}\right)$

72. $\sin\left(\cos^{-1}\dfrac{5}{13} - \cos^{-1}\dfrac{4}{5}\right)$

73. $\tan\left[\sin^{-1}\left(-\dfrac{1}{2}\right) - \tan^{-1}\dfrac{3}{4}\right]$

74. $\cos\left[\tan^{-1}(-1) + \cos^{-1}\left(-\dfrac{4}{5}\right)\right]$

75. $\sin\left[2\cos^{-1}\left(-\dfrac{3}{5}\right)\right]$

76. $\cos\left(2\tan^{-1}\dfrac{4}{3}\right)$

In Problems 77–100, solve each equation on the interval $0 \le \theta < 2\pi$.

77. $\cos\theta = \dfrac{1}{2}$

78. $\sin\theta = -\dfrac{\sqrt{3}}{2}$

79. $2\cos\theta + \sqrt{2} = 0$

80. $\tan\theta + \sqrt{3} = 0$

81. $\sin(2\theta) + 1 = 0$

82. $\cos(2\theta) = 0$

83. $\tan(2\theta) = 0$

84. $\sin(3\theta) = 1$

85. $\sec^2\theta = 4$

86. $\csc^2\theta = 1$

87. $\sin\theta = \tan\theta$

88. $\cos\theta = \sec\theta$

89. $\sin\theta + \sin(2\theta) = 0$

90. $\cos(2\theta) = \sin\theta$

91. $\sin(2\theta) - \cos\theta - 2\sin\theta + 1 = 0$

92. $\sin(2\theta) - \sin\theta - 2\cos\theta + 1 = 0$

93. $2\sin^2\theta - 3\sin\theta + 1 = 0$

94. $2\cos^2\theta + \cos\theta - 1 = 0$

95. $4\sin^2\theta = 1 + 4\cos\theta$
96. $8 - 12\sin^2\theta = 4\cos^2\theta$
97. $\sin(2\theta) = \sqrt{2}\cos\theta$
98. $1 + \sqrt{3}\cos\theta + \cos(2\theta) = 0$
99. $\sin\theta - \cos\theta = 1$
100. $\sin\theta - \sqrt{3}\cos\theta = 2$

In Problems 101–106, use a calculator to find an approximate value for each expression, rounded to two decimal places.

101. $\sin^{-1} 0.7$
102. $\cos^{-1}\dfrac{4}{5}$
103. $\tan^{-1}(-2)$
104. $\cos^{-1}(-0.2)$
105. $\sec^{-1} 3$
106. $\cot^{-1}(-4)$

In Problems 107–112, use a graphing utility to solve each equation on the interval $0 \le x \le 2\pi$. Approximate any solutions rounded to two decimal places.

107. $2x = 5\cos x$
108. $2x = 5\sin x$
109. $2\sin x + 3\cos x = 4x$
110. $3\cos x + x = \sin x$
111. $\sin x = \ln x$
112. $\sin x = e^{-x}$

113. Use a Half-angle formula to find the exact value of $\sin 15°$. Then use a difference formula to find the exact value of $\sin 15°$. Show that the answers found are the same.

114. If you are given the value of $\cos\theta$ and want the exact value of $\cos(2\theta)$, what form of the Double-angle Formula for $\cos(2\theta)$ is most efficient to use?

Chapter Test

In Problems 1–6, find the exact value of each expression. Express all angles in radians.

1. $\sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$
2. $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$
3. $\cos^{-1}\left(\sin\dfrac{11\pi}{6}\right)$
4. $\sin\left(\tan^{-1}\dfrac{7}{3}\right)$
5. $\cot\left(\csc^{-1}\sqrt{10}\right)$
6. $\sec\left(\cos^{-1}\left(-\dfrac{3}{4}\right)\right)$

In Problems 7–10, use a calculator to evaluate each espression. Express angles in radians.

7. $\sin^{-1} 0.382$
8. $\sec^{-1} 1.4$
9. $\tan^{-1} 3$
10. $\cot^{-1} 5$

In Problems 11–16 establish each identity.

11. $\dfrac{\csc\theta + \cot\theta}{\sec\theta + \tan\theta} = \dfrac{\sec\theta - \tan\theta}{\csc\theta - \cot\theta}$
12. $\sin\theta\tan\theta + \cos\theta = \sec\theta$
13. $\tan\theta + \cot\theta = 2\csc(2\theta)$
14. $\dfrac{\sin(\alpha + \beta)}{\tan\alpha + \tan\beta} = \cos\alpha\cos\beta$
15. $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$
16. $\dfrac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} = 1 - 2\cos^2\theta$

In Problems 17–24 use sum, difference, product or half-angle formulas to find the exact value of each expression.

17. $\cos 15°$
18. $\tan 75°$
19. $\sin\left(\dfrac{1}{2}\cos^{-1}\dfrac{3}{5}\right)$
20. $\tan\left(2\sin^{-1}\dfrac{6}{11}\right)$
21. $\cos\left(\sin^{-1}\dfrac{2}{3} + \tan^{-1}\dfrac{3}{2}\right)$
22. $\sin 75°\cos 15°$
23. $\sin 75° + \sin 15°$
24. $\cos 65°\cos 20° + \sin 65°\sin 20°$

In Problems 25–29, solve each equation on $0 \le \theta < 2\pi$.

25. $4\sin^2\theta - 3 = 0$
26. $-3\cos\left(\dfrac{\pi}{2} - \theta\right) = \tan\theta$
27. $\cos^2\theta + 2\sin\theta\cos\theta - \sin^2\theta = 0$
28. $\sin(\theta + 1) = \cos\theta$
29. $4\sin^2\theta + 7\sin\theta = 2$

30. Stage 16 of the 2004 Tour de France was a time trial from Bourg d'Oisans to L'Alpe d'Huez. The average grade (slope as a percent) for most of the 15 km mountainous trek was 7.9%. What was the change in elevation from the beginning to the end of the route?

Chapter Projects

1. Waves A stretched string that is attached at both ends, pulled in a direction perpendicular to the string, and released has motion that is described as wave motion. If we assume no friction and a length such that there are no "echoes" (that is, the wave doesn't bounce back), the transverse motion (motion perpendicular to the string) can be described by the equation

$$y = y_m \sin(kx - \omega t)$$

where y_m is the amplitude measured in meters and k and ω are constants. The height of the sound wave depends on the distance x from one endpoint of the string and on the time t, so a typical wave has horizontal and vertical motion over time.

(a) What is the amplitude of the wave $y = 0.00421 \sin(68.3x - 2.68t)$?

(b) The value of ω is the angular frequency measured in radians per second. What is the angular frequency of the wave given in part (a)?

(c) The frequency f is the number of vibrations per second (hertz) made by the wave as it passes a certain point. Its value is found using the formula $f = \dfrac{\omega}{2\pi}$. What is the frequency of the wave given in part (a)?

(d) The wavelength, λ, of a wave is the shortest distance at which the wave pattern repeats itself for a constant t. Thus, $\lambda = \dfrac{2\pi}{k}$. What is the wavelength of the wave given in part (a)?

(e) Graph the height of the string a distance $x = 1$ meter from an endpoint.

(f) If two waves travel simultaneously along the same stretched string, the vertical displacement of the string when both waves act is $y = y_1 + y_2$, where y_1 is the vertical displacement of the first wave and y_2 is the vertical displacement of the second wave. This result is called the Principle of Superposition and was analyzed by the French mathematician Jean Baptiste Fourier (1768–1830). When two waves travel along the same string, one wave will differ from the other wave by a phase constant ϕ. That is,

$$y_1 = y_m \sin(kx - \omega t)$$
$$y_2 = y_m \sin(kx - \omega t + \phi)$$

assuming that each wave has the same amplitude. Write $y_1 + y_2$ as a product using the Sum-to-Product Formulas.

(g) Suppose that two waves are moving in the same direction along a stretched string. The amplitude of each wave is 0.0045 meter, and the phase difference between them is 2.5 radians. The wavelength, λ, of each wave is 0.09 meter and the frequency, f, is 2.3 hertz. Find y_1, y_2, and $y_1 + y_2$.

(h) Using a graphing utility, graph y_1, y_2, and $y_1 + y_2$ on the same viewing window.

(i) Redo parts (g) and (h) with the phase difference between the waves being 0.4 radian.

(j) What effect does the phase difference have on the amplitude of $y_1 + y_2$?

The following projects are available on the Instructor's Resource Center (IRC):

2. Project at Motorola ***Sending Pictures Wirelessly***

3. Jacob's Field

4. Calculus of Differences

Cumulative Review

1. Find the real solutions, if any, of the equation $3x^2 + x - 1 = 0$.

2. Find an equation for the line containing the points $(-2, 5)$ and $(4, -1)$. What is the distance between these points? What is their midpoint?

3. Test the equation $3x + y^2 = 9$ for symmetry with respect to the x-axis, y-axis, and origin. List the intercepts.

4. Use transformations to graph the equation $y = |x - 3| + 2$.

5. Use transformations to graph the equation $y = 3e^x - 2$.

6. Use transformations to graph the equation $y = \cos\left(x - \frac{\pi}{2}\right) - 1$.

7. Sketch a graph of each of the following functions. Label at least three points on each graph. Name the inverse function of each and show its graph.

(a) $y = x^3$

(b) $y = e^x$

(c) $y = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(d) $y = \cos x, \quad 0 \le x \le \pi$

8. If $\sin\theta = -\frac{1}{3}$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact value of:

(a) $\cos\theta$ (b) $\tan\theta$ (c) $\sin(2\theta)$

(d) $\cos(2\theta)$ (e) $\sin\left(\frac{1}{2}\theta\right)$ (f) $\cos\left(\frac{1}{2}\theta\right)$

9. Find the exact value of $\cos(\tan^{-1} 2)$.

10. If $\sin\alpha = \frac{1}{3}, \frac{\pi}{2} < \alpha < \pi$, and $\cos\beta = -\frac{1}{3}, \pi < \beta < \frac{3\pi}{2}$, find the exact value of:

(a) $\cos\alpha$ (b) $\sin\beta$ (c) $\cos(2\alpha)$

(d) $\cos(\alpha + \beta)$ (e) $\sin\frac{\beta}{2}$

11. For the function

$$f(x) = 2x^5 - x^4 - 4x^3 + 2x^2 + 2x - 1:$$

(a) Find the real zeros and their multiplicity.

(b) Find the intercepts.

(c) Find the power function that the graph of f resembles for large $|x|$.

(d) Graph f using a graphing utility.

(e) Approximate the turning points, if any exist.

(f) Use the information obtained in parts (a)–(e) to sketch a graph of f by hand.

(g) Identify the intervals on which f is increasing, decreasing, or constant.

12. If $f(x) = 2x^2 + 3x + 1$ and $g(x) = x^2 + 3x + 2$, solve:

(a) $f(x) = 0$

(b) $f(x) = g(x)$

(c) $f(x) > 0$

(d) $f(x) \ge g(x)$