A Preview of Calculus: The Limit, Derivative, and Integral of a Function

Two hundred years ago the Rev. Thomas Robert Malthus, an English economist and mathematician, anonymously published an essay predicting that the world's burgeoning population would overwhelm Earth's capacity to sustain it.

Malthus's gloomy forecast was condemned by Karl Marx, Friedrich Engels, and many other theorists, and it was still striking sparks last week at a meeting in Philadelphia of the American Anthropological Society. Despite continuing controversy, it was clear that Malthus's conjectures are far from dead.

Among the scores of special conferences organized for the 5,000 participating anthropologists, many touched directly and indirectly on the Malthusian dilemma: Although global food supplies increase arithmetically, the population increases geometrically—a vastly faster rate.

"Will Humans Overwhelm the Earth? The Debate Continues," Malcolm W. Browne, *New York Times*, December 8, 1998.

—See Chapter Project 1.

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A LOOK BACK In this book we have discussed a variety of functions: polynomial functions (including linear and quadratic functions), rational functions, exponential and logarithmic functions, trigonometric functions, and the inverse trigonometric functions. For each of these, we found their domain and range, intercepts, symmetry, if any, and asymptotes, if any, and we graphed each. We also discussed whether these functions were even, odd, or neither and determined on what intervals they were increasing and decreasing. We also discussed the idea of average rates of change.

A LOOK AHEAD In calculus, other properties are discussed, such as finding limits of functions, determining where functions are continuous, finding the derivative of functions, and finding the integral of functions. In this chapter, we give an introduction to these properties. By completing this chapter you will be well prepared for a first course in calculus.

OUTLINE

13.1 Finding Limits Using Tables and Graphs

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Piecewise–defined Functions (Section 2.5, pp. 112–114)

Now work the 'Are You Prepared?' problems on page 914.

OBJECTIVES 1 Find a Limit Using a Table
2 Find a Limit Using a Graph

1 Find a Limit Using a Table

The idea of the limit of a function is what connects algebra and geometry to the mathematics of calculus. In working with the limit of a function, we encounter notation of the form

$$\lim_{x \to c} f(x) = N$$

This is read as "the limit of $f(x)$ as x approaches c equals the number N." Here f is a function defined on some open interval containing the number c; f need not be defined at c, however.

We may describe the meaning of $\lim_{x \to c} f(x) = N$ as follows:

For all x approximately equal to c, with $x \neq c$, the corresponding value $f(x)$ is approximately equal to N.

Another description of $\lim_{x \to c} f(x) = N$ is

As x gets closer to c, but remains unequal to c, the corresponding value of $f(x)$ gets closer to N.

Tables generated with the help of a calculator are useful for finding limits.

EXAMPLE 1 Finding a Limit Using a Table

Find: $\lim_{x \to 3}(5x^2)$

Solution Here $f(x) = 5x^2$ and $c = 3$. We choose values of x close to 3, arbitrarily starting with 2.99. Then we select additional numbers that get closer to 3, but remain less than 3. Next we choose values of x greater than 3, starting with 3.01, that get closer to 3. Table 1 shows the value of f at each choice.

Table 1

X	Y1
2.99	44.701
2.999	44.97
2.9999	44.997
3.0001	45.003
3.001	45.03
3.01	45.301

Y1 = 5X²

From Table 1, we infer that as x gets closer to 3 the value of $f(x) = 5x^2$ gets closer to 45. That is,

$$\lim_{x \to 3}(5x^2) = 45$$

When choosing the values of x in a table, the number to start with and the subsequent entries are arbitrary. However, the entries should be chosen so that the table makes it clear what the corresponding values of f are getting close to.

NOW WORK PROBLEM 7.

EXAMPLE 2 Finding a Limit Using a Table

Find: (a) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ (b) $\lim_{x \to 2} (x + 2)$

Solution (a) Here $f(x) = \dfrac{x^2 - 4}{x - 2}$ and $c = 2$. Notice that the domain of f is $\{x | x \neq 2\}$, so f is not defined at 2. We proceed to choose values of x close to 2 and evaluate f at each choice, as shown in Table 2. We infer that as x gets closer to 2 the value of $f(x) = \dfrac{x^2 - 4}{x - 2}$ gets closer to 4. That is,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

(b) Here $g(x) = x + 2$ and $c = 2$. The domain of g is all real numbers. See Table 3. We infer that as x gets closer to 2 the value of $g(x)$ gets closer to 4. That is,

$$\lim_{x \to 2} (x + 2) = 4$$

Table 2

X	Y1
1.99	3.99
1.999	3.999
1.9999	3.9999
2.0001	4.0001
2.001	4.001
2.01	4.01

Y1=(X²−4)/(X−2)

Table 3

X	Y1
1.99	3.99
1.999	3.999
1.9999	3.9999
2.0001	4.0001
2.001	4.001
2.01	4.01

Y1=X+2

◀

The conclusion that $\lim_{x \to 2} (x + 2) = 4$ could have been obtained without the use of Table 3; as x gets closer to 2, it follows that $x + 2$ will get closer to $2 + 2 = 4$.

Also, for part (a), you are right if you make the observation that, since $x \neq 2$, then

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}} = x + 2, \qquad x \neq 2$$

Now it is easy to conclude that

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

Let's look at an example for which the factoring technique used above does not work.

EXAMPLE 3 Finding a Limit Using a Table

Table 4

X	Y1
-.05	.99958
-.02	.99993
-.01	.99998
.01	.99998
.02	.99993
.05	.99958

Y1=sin(X)/X

Find: $\lim_{x \to 0} \dfrac{\sin x}{x}$

Solution First, we observe that the domain of the function $f(x) = \dfrac{\sin x}{x}$ is $\{x | x \neq 0\}$. We create Table 4, where x is measured in radians. We infer from Table 4 that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

◀

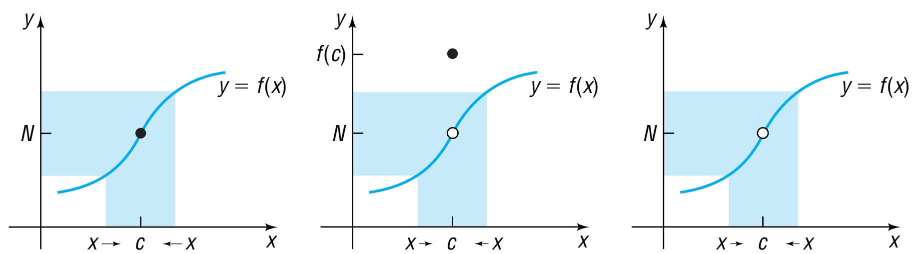
2 Find a Limit Using a Graph

The graph of a function f can also be of help in finding limits. See Figure 1. In each graph, notice that, as x gets closer to c, the value of f gets closer to the number N. We conclude that

$$\lim_{x \to c} f(x) = N$$

This is the conclusion regardless of the value of f at c. In Figure 1(a), $f(c) = N$, and in Figure 1(b), $f(c) \neq N$. Figure 1(c) illustrates that $\lim_{x \to c} f(x) = N$, even if f is not defined at c.

Figure 1



$f(c) = N$; $\lim_{x \to c} f(x) = N$

(a)

$f(c) \neq N$; $\lim_{x \to c} f(x) = N$

(b)

$f(c)$ not defined; $\lim_{x \to c} f(x) = N$

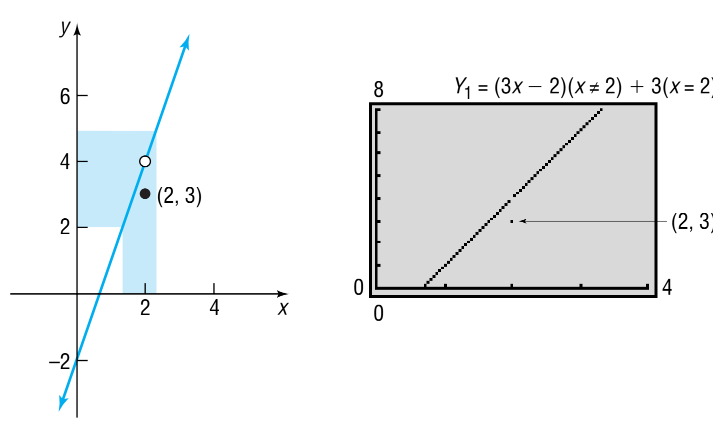
(c)

EXAMPLE 4 Finding a Limit by Graphing

Find: $\lim_{x \to 2} f(x)$ if $f(x) = \begin{cases} 3x - 2 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

Solution The function f is a piecewise-defined function. Its graph is shown in Figure 2(a) drawn by hand. Figure 2(b) shows the graph using a graphing utility.

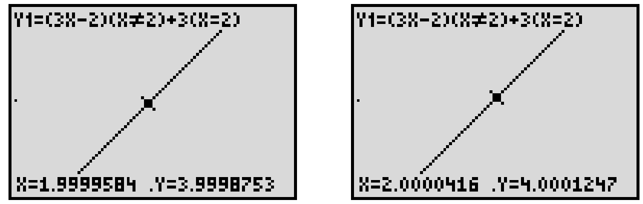
Figure 2



(a) **(b)**

See Figure 3. By ZOOMing in around $x = 2$ and TRACEing, we conclude from the graph that $\lim_{x \to 2} f(x) = 4$.

Figure 3



Notice in Example 4 that the value of f at 2, that is, $f(2) = 3$, plays no role in the conclusion that $\lim_{x\to 2} f(x) = 4$. In fact, even if f were undefined at 2, it would still happen that $\lim_{x\to 2} f(x) = 4$.

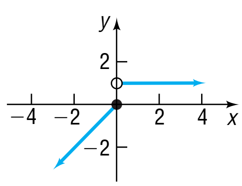
NOW WORK PROBLEM 23.

Sometimes there is no *single* number that the values of f get closer to as x gets closer to c. In this case, we say that **f has no limit as x approaches c** or that $\lim_{x\to c} f(x)$ **does not exist**.

EXAMPLE 5 A Function That Has No Limit at 0

Find: $\lim_{x\to 0} f(x)$ if $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$

Figure 4



Solution See Figure 4. As x gets closer to 0, but remains negative, the value of f also gets closer to 0. As x gets closer to 0, but remains positive, the value of f always equals 1. Since there is no single number that the values of f are close to when x is close to 0, we conclude that $\lim_{x\to 0} f(x)$ does not exist.

NOW WORK PROBLEMS 17 AND 37.

EXAMPLE 6 Using a Graphing Utility to Find a Limit

Find: $\lim_{x\to 2} \dfrac{x^3 - 2x^2 + 4x - 8}{x^4 - 2x^3 + x - 2}$

Solution Table 5 shows the solution, from which we conclude that

$$\lim_{x\to 2} \frac{x^3 - 2x^2 + 4x - 8}{x^4 - 2x^3 + x - 2} = 0.889$$

rounded to three decimal places.

Table 5

X	Y1
1	2.5
1.5	1.4286
1.8	1.0597
1.9	.96832
1.99	.89635
1.999	.88963
1.9999	.88896

Y1■(X^3−2X²+4X−...

X	Y1
3	.46429
2.5	.61654
2.3	.70555
2.1	.81961
2.01	.88153
2.001	.88815
2.0001	.88881

Y1■(X^3−2X²+4X−...

NOW WORK PROBLEM 43.

In the next section, we will see how algebra can be used to obtain exact solutions to limits like the one in Example 6.

13.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Graph $f(x) = \begin{cases} 3x - 2 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$ (pp. 112–113)

2. If $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ what is $f(0)$? (pp. 112–113)

Concepts and Vocabulary

3. The limit of a function $f(x)$ as x approaches c is denoted by the symbol ______.

4. If a function f has no limit as x approaches c, then we say that $\lim\limits_{x \to c} f(x)$ ______ ______ ______.

5. *True or False:* $\lim\limits_{x \to c} f(x) = N$ may be described by saying that the value of $f(x)$ gets closer to N as x gets closer to c, but remains unequal to c.

6. *True or False:* $\lim\limits_{x \to c} f(x)$ exists and equals some number for any function f as long as c is in the domain of f.

Skill Building

In Problems 7–16, use a table to find the indicated limit.

7. $\lim\limits_{x \to 2} (4x^3)$

8. $\lim\limits_{x \to 3} (2x^2 + 1)$

9. $\lim\limits_{x \to 0} \dfrac{x + 1}{x^2 + 1}$

10. $\lim\limits_{x \to 0} \dfrac{2 - x}{x^2 + 4}$

11. $\lim\limits_{x \to 4} \dfrac{x^2 - 4x}{x - 4}$

12. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 - 3x}$

13. $\lim\limits_{x \to 0} (e^x + 1)$

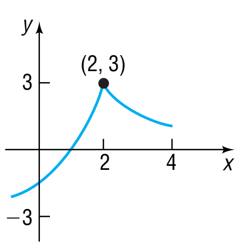
14. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{2}$

15. $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x}$, x in radians

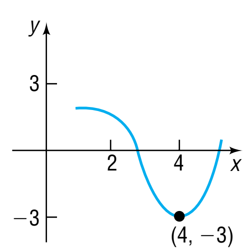
16. $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$, x in radians

In Problems 17–22, use the graph shown to determine if the limit exists. If it does, find its value.

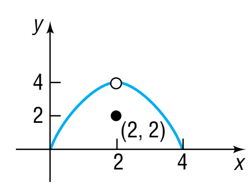
17. $\lim\limits_{x \to 2} f(x)$



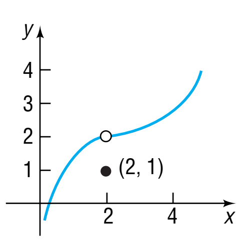
18. $\lim\limits_{x \to 4} f(x)$



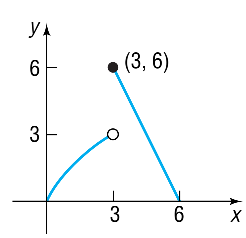
19. $\lim\limits_{x \to 2} f(x)$



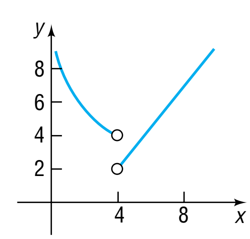
20. $\lim\limits_{x \to 2} f(x)$



21. $\lim\limits_{x \to 3} f(x)$



22. $\lim\limits_{x \to 4} f(x)$



In Problems 23–42, graph each function. Use the graph to find the indicated limit, if it exists.

23. $\lim\limits_{x \to 4} f(x)$, $f(x) = 3x + 1$

24. $\lim\limits_{x \to -1} f(x)$, $f(x) = 2x - 1$

25. $\lim\limits_{x \to 2} f(x)$, $f(x) = 1 - x^2$

26. $\lim\limits_{x \to -1} f(x)$, $f(x) = x^3 - 1$

27. $\lim\limits_{x \to -3} f(x)$, $f(x) = |2x|$

28. $\lim\limits_{x \to 4} f(x)$, $f(x) = 3\sqrt{x}$

29. $\lim\limits_{x \to \pi/2} f(x)$, $f(x) = \sin x$

30. $\lim\limits_{x \to \pi} f(x)$, $f(x) = \cos x$

31. $\lim\limits_{x \to 0} f(x)$, $f(x) = e^x$

32. $\lim\limits_{x \to 1} f(x)$, $f(x) = \ln x$

33. $\lim\limits_{x \to -1} f(x)$, $f(x) = \dfrac{1}{x}$

34. $\lim\limits_{x \to 2} f(x)$, $f(x) = \dfrac{1}{x^2}$

35. $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} x^2 & x \geq 0 \\ 2x & x < 0 \end{cases}$

36. $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} x - 1 & x < 0 \\ 3x - 1 & x \geq 0 \end{cases}$

37. $\lim_{x \to 1} f(x)$, $f(x) = \begin{cases} 3x & x \leq 1 \\ x + 1 & x > 1 \end{cases}$

38. $\lim_{x \to 2} f(x)$, $f(x) = \begin{cases} x^2 & x \leq 2 \\ 2x - 1 & x > 2 \end{cases}$

39. $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} x & x < 0 \\ 1 & x = 0 \\ 3x & x > 0 \end{cases}$

40. $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases}$

41. $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} \sin x & x \leq 0 \\ x^2 & x > 0 \end{cases}$

42. $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} e^x & x > 0 \\ 1 - x & x \leq 0 \end{cases}$

In Problems 43–48, use a graphing utility to find the indicated limit rounded to two decimal places.

43. $\lim_{x \to 1} \dfrac{x^3 - x^2 + x - 1}{x^4 - x^3 + 2x - 2}$

44. $\lim_{x \to -1} \dfrac{x^3 + x^2 + 3x + 3}{x^4 + x^3 + 2x + 2}$

45. $\lim_{x \to 2} \dfrac{x^3 - 2x^2 + 4x - 8}{x^2 + x - 6}$

46. $\lim_{x \to 1} \dfrac{x^3 - x^2 + 3x - 3}{x^2 + 3x - 4}$

47. $\lim_{x \to -1} \dfrac{x^3 + 2x^2 + x}{x^4 + x^3 + 2x + 2}$

48. $\lim_{x \to 3} \dfrac{x^3 - 3x^2 + 4x - 12}{x^4 - 3x^3 + x - 3}$

'Are You Prepared?' Answers

1. See Figure 2(a) on page 912. **2.** $f(0) = 0$

13.2 Algebra Techniques for Finding Limits

OBJECTIVES
1. Find the Limit of a Sum, a Difference, and a Product
2. Find the Limit of a Polynomial
3. Find the Limit of a Power or a Root
4. Find the Limit of a Quotient
5. Find the Limit of an Average Rate of Change

We mentioned in the previous section that algebra can sometimes be used to find the exact value of a limit. This is accomplished by developing two formulas involving limits and several properties of limits.

Theorem

Two Formulas: $\lim_{x \to c} b$ and $\lim_{x \to c} x$

Limit of a Constant

In Words
The limit of a constant is the constant.

For the constant function $f(x) = b$,

$$\lim_{x \to c} f(x) = \lim_{x \to c} b = b \tag{1}$$

where c is any number.

Limit of x

In Words
The limit of x as x approaches c is c.

For the identity function $f(x) = x$,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c \tag{2}$$

where c is any number.

We use graphs to establish formulas (1) and (2). Since the graph of a constant function is a horizontal line, it follows that, no matter how close x is to c, the corresponding value of $f(x)$ equals b. That is, $\lim_{x \to c} b = b$. See Figure 5.

See Figure 6. For any choice of c, as x gets closer to c, the corresponding value of $f(x)$ is x, which is just as close to c. That is, $\lim_{x \to c} x = c$.

Figure 5

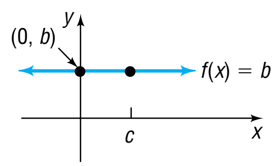
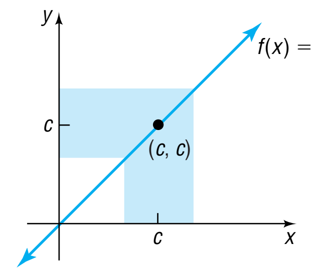


Figure 6



EXAMPLE 1 Using Formulas (1) and (2)

(a) $\lim_{x \to 3} 5 = 5$ (b) $\lim_{x \to 3} x = 3$ (c) $\lim_{x \to 0} (-8) = -8$ (d) $\lim_{x \to -1/2} x = -\frac{1}{2}$ ◀

NOW WORK PROBLEM 7.

Formulas (1) and (2), when used with the properties that follow, enable us to evaluate limits of more complicated functions.

1 Find the Limit of a Sum, a Difference, and a Product

In the following properties, we assume that f and g are two functions for which both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

Theorem

In Words
The limit of the sum of two functions equals the sum of their limits.

Limit of a Sum

$$\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \qquad \textbf{(3)}$$

EXAMPLE 2 Finding the Limit of a Sum

Find: $\lim_{x \to -3}(x + 4)$

Solution The limit we seek is the sum of two functions $f(x) = x$ and $g(x) = 4$. From formulas (1) and (2), we know that

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} x = -3 \quad \text{and} \quad \lim_{x \to -3} g(x) = \lim_{x \to -3} 4 = 4$$

From formula (3), it follows that

$$\lim_{x \to -3}(x + 4) = \lim_{x \to -3} x + \lim_{x \to -3} 4 = -3 + 4 = 1$$ ◀

Theorem

In Words
The limit of the difference of two functions equals the difference of their limits.

Limit of a Difference

$$\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) \qquad \textbf{(4)}$$

EXAMPLE 3 Finding the Limit of a Difference

Find: $\lim_{x\to 4}(6 - x)$

Solution The limit we seek is the difference of two functions $f(x) = 6$ and $g(x) = x$. From formulas (1) and (2), we know that

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} 6 = 6 \quad \text{and} \quad \lim_{x\to 4} g(x) = \lim_{x\to 4} x = 4$$

From formula (4), it follows that

$$\lim_{x\to 4}(6 - x) = \lim_{x\to 4} 6 - \lim_{x\to 4} x = 6 - 4 = 2$$

◀

Theorem

In Words
The limit of the product of two functions equals the product of their limits.

Limit of a Product

$$\lim_{x\to c}[f(x)\cdot g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)] \qquad \textbf{(5)}$$

EXAMPLE 4 Finding the Limit of a Product

Find: $\lim_{x\to -5}(-4x)$

Solution The limit we seek is the product of two functions $f(x) = -4$ and $g(x) = x$. From formulas (1) and (2), we know that

$$\lim_{x\to -5} f(x) = \lim_{x\to -5}(-4) = -4 \quad \text{and} \quad \lim_{x\to -5} g(x) = \lim_{x\to -5} x = -5$$

From formula (5), it follows that

$$\lim_{x\to -5}(-4x) = [\lim_{x\to -5} -4][\lim_{x\to -5} x] = (-4)(-5) = 20$$

◀

EXAMPLE 5 Finding Limits Using Algebraic Properties

Find: (a) $\lim_{x\to -2}(3x - 5)$ (b) $\lim_{x\to 2}(5x^2)$

Solution (a) $\lim_{x\to -2}(3x - 5) = \lim_{x\to -2}(3x) - \lim_{x\to -2} 5 = [\lim_{x\to -2} 3][\lim_{x\to -2} x] - \lim_{x\to -2} 5$

$$= (3)(-2) - 5 = -6 - 5 = -11$$

(b) $\lim_{x\to 2}(5x^2) = [\lim_{x\to 2} 5][\lim_{x\to 2} x^2] = 5\lim_{x\to 2}(x\cdot x) = 5[\lim_{x\to 2} x][\lim_{x\to 2} x]$

$$= 5\cdot 2\cdot 2 = 20$$

◀

NOW WORK PROBLEM 11.

Notice in the solution to part (b) that $\lim_{x \to 2}(5x^2) = 5 \cdot 2^2$.

Theorem

Limit of a Monomial

If $n \geq 1$ is a positive integer and a is a constant, then

$$\boxed{\lim_{x \to c}(ax^n) = ac^n \qquad \textbf{(6)}}$$

for any number c.

Proof

$$\lim_{x \to c}(ax^n) = [\lim_{x \to c} a][\lim_{x \to c} x^n] = a[\lim_{x \to c} \underbrace{(x \cdot x \cdot x \cdot \ldots \cdot x)}_{n \text{ factors}}]$$
$$= a\underbrace{[\lim_{x \to c} x][\lim_{x \to c} x][\lim_{x \to c} x] \ldots [\lim_{x \to c} x]}_{n \text{ factors}}$$
$$= a \cdot \underbrace{c \cdot c \cdot c \cdot \ldots \cdot c}_{n \text{ factors}} = ac^n$$

■

EXAMPLE 6 Finding the Limit of a Monomial

Find: $\lim_{x \to 2}(-4x^3)$

Solution $\lim_{x \to 2}(-4x^3) = -4 \cdot 2^3 = -4 \cdot 8 = -32$ ◀

2 Find the Limit of a Polynomial

Since a polynomial is a sum of monomials, we can use formula (6) and repeated use of formula (3) to obtain the following result:

Theorem

Limit of a Polynomial

If P is a polynomial function, then

$$\boxed{\lim_{x \to c} P(x) = P(c) \qquad \textbf{(7)}}$$

for any number c.

Proof If P is a polynomial function, that is, if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

then

$$\begin{aligned}\lim_{x \to c} P(x) &= \lim_{x \to c}[a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0] \\ &= \lim_{x \to c}(a_n x^n) + \lim_{x \to c}(a_{n-1}x^{n-1}) + \cdots + \lim_{x \to c}(a_1 x) + \lim_{x \to c} a_0 \\ &= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 \\ &= P(c)\end{aligned}$$

■

Formula (7) states that to find the limit of a polynomial as x approaches c all we need to do is to evaluate the polynomial at c.

EXAMPLE 7 Finding the Limit of a Polynomial

Find: $\lim_{x\to 2}[5x^4 - 6x^3 + 3x^2 + 4x - 2]$

Solution

$$\begin{aligned}\lim_{x\to 2}[5x^4 - 6x^3 + 3x^2 + 4x - 2] &= 5\cdot 2^4 - 6\cdot 2^3 + 3\cdot 2^2 + 4\cdot 2 - 2\\ &= 5\cdot 16 - 6\cdot 8 + 3\cdot 4 + 8 - 2\\ &= 80 - 48 + 12 + 6 = 50\end{aligned}$$

NOW WORK PROBLEM 13.

3 Find the Limit of a Power or a Root

Theorem

Limit of a Power or Root

If $\lim_{x\to c} f(x)$ exists and if $n \geq 2$ is a positive integer, then

$$\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n \tag{8}$$

and

$$\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} \tag{9}$$

In formula (9), we require that both $\sqrt[n]{f(x)}$ and $\sqrt[n]{\lim_{x\to c} f(x)}$ be defined.

Look carefully at equations (8) and (9) and compare each side.

EXAMPLE 8 Finding the Limit of a Power or a Root

Find: (a) $\lim_{x\to 1}(3x - 5)^4$ (b) $\lim_{x\to 0}\sqrt{5x^2 + 8}$ (c) $\lim_{x\to -1}(5x^3 - x + 3)^{4/3}$

Solution

(a) $\lim_{x\to 1}(3x - 5)^4 = [\lim_{x\to 1}(3x - 5)]^4 = (-2)^4 = 16$

(b) $\lim_{x\to 0}\sqrt{5x^2 + 8} = \sqrt{\lim_{x\to 0}(5x^2 + 8)} = \sqrt{8} = 2\sqrt{2}$

(c) $$\begin{aligned}\lim_{x\to -1}(5x^3 - x + 3)^{4/3} &= \sqrt[3]{\lim_{x\to -1}(5x^3 - x + 3)^4}\\ &= \sqrt[3]{[\lim_{x\to -1}(5x^3 - x + 3)]^4} = \sqrt[3]{(-1)^4} = \sqrt[3]{1} = 1\end{aligned}$$

NOW WORK PROBLEM 23.

4 Find the Limit of a Quotient

Theorem

In Words
The limit of the quotient of two functions equals the quotient of their limits, provided that the limit of the denominator is not zero.

Limit of a Quotient

$$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \qquad \textbf{(10)}$$

provided that $\lim_{x\to c} g(x) \neq 0$.

EXAMPLE 9 Finding the Limit of a Quotient

Find: $\lim_{x\to 1} \dfrac{5x^3 - x + 2}{3x + 4}$

Solution The limit we seek is the quotient of two functions: $f(x) = 5x^3 - x + 2$ and $g(x) = 3x + 4$. First, we find the limit of the denominator $g(x)$.

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}(3x + 4) = 7$$

Since the limit of the denominator is not zero, we can proceed to use formula (10).

$$\lim_{x\to 1}\frac{5x^3 - x + 2}{3x + 4} = \frac{\lim_{x\to 1}(5x^3 - x + 2)}{\lim_{x\to 1}(3x + 4)} = \frac{6}{7}$$

◀

NOW WORK PROBLEM 21.

When the limit of the denominator is zero, formula (10) cannot be used. In such cases, other strategies need to be used. Let's look at two examples.

EXAMPLE 10 Finding the Limit of a Quotient

Find: (a) $\lim_{x\to 3}\dfrac{x^2 - x - 6}{x^2 - 9}$ (b) $\lim_{x\to 0}\dfrac{5x - \sin x}{x}$

Solution (a) The limit of the denominator equals zero, so formula (10) cannot be used. Instead, we notice that the expression can be factored as

$$\frac{x^2 - x - 6}{x^2 - 9} = \frac{(x-3)(x+2)}{(x-3)(x+3)}$$

When we compute a limit as x approaches 3, we are interested in the values of the function when x is close to 3, but unequal to 3. Since $x \neq 3$, we can cancel the $(x - 3)$'s. Formula (10) can then be used.

$$\lim_{x\to 3}\frac{x^2 - x - 6}{x^2 - 9} = \lim_{x\to 3}\frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x+3)} = \frac{\lim_{x\to 3}(x+2)}{\lim_{x\to 3}(x+3)} = \frac{5}{6}$$

(b) Again, the limit of the denominator is zero. In this situation, we perform the indicated operation and divide by x.

$$\lim_{x\to 0}\frac{5x - \sin x}{x} = \lim_{x\to 0}\left[\frac{5x}{x} - \frac{\sin x}{x}\right] \underset{\substack{\uparrow \\ \text{Limit of} \\ \text{a Difference}}}{=} \lim_{x\to 0}\frac{5\cancel{x}}{\cancel{x}} - \lim_{x\to 0}\frac{\sin x}{x} \underset{\substack{\uparrow \\ \text{Refer to Example 3,} \\ \text{Section 13.1}}}{=} 5 - 1 = 4$$

◀

EXAMPLE 11 **Finding Limits Using Algebraic Properties**

Find: $\lim\limits_{x\to 2} \dfrac{x^3 - 2x^2 + 4x - 8}{x^4 - 2x^3 + x - 2}$

Solution The limit of the denominator is zero, so formula (10) cannot be used. We factor the expression.

$$\frac{x^3 - 2x^2 + 4x - 8}{x^4 - 2x^3 + x - 2} \underset{\substack{\uparrow \\ \text{Factor by grouping}}}{=} \frac{x^2(x-2) + 4(x-2)}{x^3(x-2) + 1(x-2)} = \frac{(x^2+4)(x-2)}{(x^3+1)(x-2)}$$

Then

$$\lim_{x\to 2} \frac{x^3 - 2x^2 + 4x - 8}{x^4 - 2x^3 + x - 2} = \lim_{x\to 2} \frac{(x^2+4)\cancel{(x-2)}}{(x^3+1)\cancel{(x-2)}} = \frac{8}{9}$$

which is exact. ◀

Compare the exact solution above with the approximate solution found in Example 6 of Section 13.1.

5 Find the Limit of an Average Rate of Change

EXAMPLE 12 **Finding the Limit of an Average Rate of Change**

Find the limit as x approaches 2 of the average rate of change of the function

$$f(x) = x^2 + 3x$$

from 2 to x.

Solution The average rate of change of f from 2 to x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(2)}{x - 2} = \frac{(x^2 + 3x) - 10}{x - 2} = \frac{(x+5)(x-2)}{x-2}$$

The limit of the average rate of change is

$$\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{(x^2 + 3x) - 10}{x - 2} = \lim_{x\to 2} \frac{(x+5)\cancel{(x-2)}}{\cancel{x-2}} = 7$$ ◀

Summary

To find exact values for $\lim\limits_{x\to c} f(x)$, try the following:

1. If f is a polynomial function, then $\lim\limits_{x\to c} f(x) = f(c)$ (formula 7).
2. If f is a polynomial raised to a power or is the root of a polynomial, use formulas (8) and (9) with formula (7).
3. If f is a quotient and the limit of the denominator is not zero, use the fact that the limit of a quotient is the quotient of the limits.
4. If f is a quotient and the limit of the denominator is zero, use other techniques, such as factoring.

13.2 Assess Your Understanding

Concepts and Vocabulary

1. The limit of the product of two functions equals the _____ of their limits.
2. $\lim_{x\to 0} 5 =$ _____.
3. $\lim_{x\to 1} x =$ _____.
4. *True or False:* The limit of a polynomial function as x approaches 5 equals the value of the polynomial at 5.
5. *True or False:* The limit of a rational function at 5 equals the value of the rational function at 5.
6. *True or False:* The limit of a quotient equals the quotient of the limits.

Skill Building

In Problems 7–38, find each limit algebraically.

7. $\lim_{x\to 1} 5$
8. $\lim_{x\to 1}(-3)$
9. $\lim_{x\to 4} x$
10. $\lim_{x\to -3} x$
11. $\lim_{x\to 2}(3x+2)$
12. $\lim_{x\to 3}(2-5x)$
13. $\lim_{x\to -1}(3x^2-5x)$
14. $\lim_{x\to 2}(8x^2-4)$
15. $\lim_{x\to 1}(5x^4-3x^2+6x-9)$
16. $\lim_{x\to -1}(8x^5-7x^3+8x^2+x-4)$
17. $\lim_{x\to 1}(x^2+1)^3$
18. $\lim_{x\to 2}(3x-4)^2$
19. $\lim_{x\to 1}\sqrt{5x+4}$
20. $\lim_{x\to 0}\sqrt{1-2x}$
21. $\lim_{x\to 0}\dfrac{x^2-4}{x^2+4}$
22. $\lim_{x\to 2}\dfrac{3x+4}{x^2+x}$
23. $\lim_{x\to 2}(3x-2)^{5/2}$
24. $\lim_{x\to -1}(2x+1)^{5/3}$
25. $\lim_{x\to 2}\dfrac{x^2-4}{x^2-2x}$
26. $\lim_{x\to -1}\dfrac{x^2+x}{x^2-1}$
27. $\lim_{x\to -3}\dfrac{x^2-x-12}{x^2-9}$
28. $\lim_{x\to -3}\dfrac{x^2+x-6}{x^2+2x-3}$
29. $\lim_{x\to 1}\dfrac{x^3-1}{x-1}$
30. $\lim_{x\to 1}\dfrac{x^4-1}{x-1}$
31. $\lim_{x\to -1}\dfrac{(x+1)^2}{x^2-1}$
32. $\lim_{x\to 2}\dfrac{x^3-8}{x^2-4}$
33. $\lim_{x\to 1}\dfrac{x^3-x^2+x-1}{x^4-x^3+2x-2}$
34. $\lim_{x\to -1}\dfrac{x^3+x^2+3x+3}{x^4+x^3+2x+2}$
35. $\lim_{x\to 2}\dfrac{x^3-2x^2+4x-8}{x^2+x-6}$
36. $\lim_{x\to 1}\dfrac{x^3-x^2+3x-3}{x^2+3x-4}$
37. $\lim_{x\to -1}\dfrac{x^3+2x^2+x}{x^4+x^3+2x+2}$
38. $\lim_{x\to 3}\dfrac{x^3-3x^2+4x-12}{x^4-3x^3+x-3}$

In Problems 39–48, find the limit as x approaches c of the average rate of change of each function from c to x.

39. $c=2;\quad f(x)=5x-3$
40. $c=-2;\quad f(x)=4-3x$
41. $c=3;\quad f(x)=x^2$
42. $c=3;\quad f(x)=x^3$
43. $c=-1;\quad f(x)=x^2+2x$
44. $c=-1;\quad f(x)=2x^2-3x$
45. $c=0;\quad f(x)=3x^3-2x^2+4$
46. $c=0;\quad f(x)=4x^3-5x+8$
47. $c=1;\quad f(x)=\dfrac{1}{x}$
48. $c=1;\quad f(x)=\dfrac{1}{x^2}$

In Problems 49–52, use the properties of limits and the facts that

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\cos x-1}{x}=0 \qquad \lim_{x\to 0}\sin x=0 \qquad \lim_{x\to 0}\cos x=1$$

where x is in radians, to find each limit.

49. $\lim_{x\to 0}\dfrac{\tan x}{x}$
50. $\lim_{x\to 0}\dfrac{\sin(2x)}{x}$
   [Hint: Use a Double-angle Formula.]
51. $\lim_{x\to 0}\dfrac{3\sin x+\cos x-1}{4x}$
52. $\lim_{x\to 0}\dfrac{\sin^2 x+\sin x(\cos x-1)}{x^2}$

13.3 One-sided Limits; Continuous Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Piecewise-defined Functions (Section 2.5, pp. 112–114)
- Library of Functions (Section 2.5, pp. 107–112)
- Polynomial Functions (Section 3.2, pp. 170–178)
- Properties of Rational Functions (Section 3.3, pp. 186–195)
- The Graph of a Rational Function (Section 3.4, pp. 198–205)
- Properties of the Exponential Function (Section 4.3, pp. 276 and 277)
- Properties of the Logarithmic Function (Section 4.4, p. 291)
- Properties of the Trigonometric Functions (Section 5.3, pp. 388–390 and Section 5.4, pp. 404 and 406)

Now work the 'Are You Prepared?' problems on page 928.

OBJECTIVES 1 Find the One-sided Limits of a Function
2 Determine Whether a Function Is Continuous

1 Find The One-sided Limits of a Function

Earlier we described $\lim_{x \to c} f(x) = N$ by saying that as x gets closer to c, but remains unequal to c, the corresponding values of $f(x)$ get closer to N. Whether we use a numerical argument or the graph of the function f, the variable x can get closer to c in only two ways: either by approaching c from the left, through numbers less than c, or by approaching c from the right, through numbers greater than c.

If we only approach c from one side, we have a **one-sided limit**. The notation

$$\lim_{x \to c^-} f(x) = L$$

sometimes called the **left limit**, read as "the limit of $f(x)$ as x approaches c from the left equals L," may be described by the following statement:

As x gets closer to c, but remains less than c, the corresponding value of $f(x)$ gets closer to L.

In Words
$x \to c^-$ means $x < c$.

The notation $x \to c^-$ is used to remind us that x is less than c.

The notation

$$\lim_{x \to c^+} f(x) = R$$

sometimes called the **right limit**, read as "the limit of $f(x)$ as x approaches c from the right equals R," may be described by the following statement:

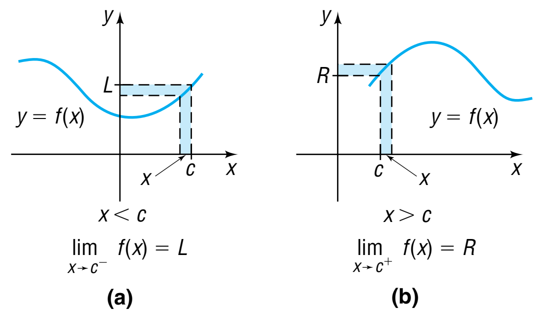
As x gets closer to c, but remains greater than c, the corresponding value of $f(x)$ gets closer to R.

In Words
$x \to c^+$ means $x > c$.

The notation $x \to c^+$ is used to remind us that x is greater than c.

Figure 7 illustrates left and right limits.

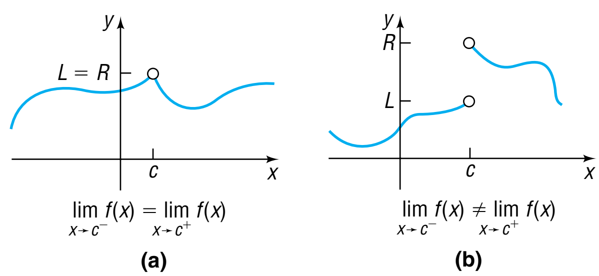
Figure 7



The left and right limits can be used to determine whether $\lim_{x\to c} f(x)$ exists. See Figure 8.

As Figure 8(a) illustrates, $\lim_{x\to c} f(x)$ exists and equals the common value of the left limit and the right limit $(L = R)$. In Figure 8(b), we see that $\lim_{x\to c} f(x)$ does not exist because $L \neq R$. This leads us to the following result:

Figure 8



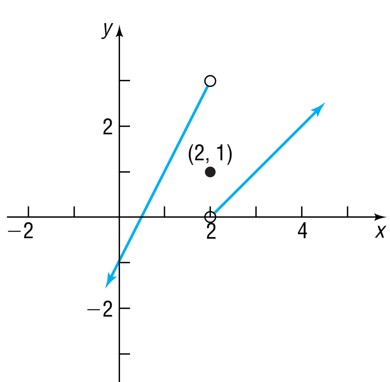
Theorem

Suppose that $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = R$. Then $\lim_{x\to c} f(x)$ exists if and only if $L = R$. Furthermore, if $L = R$, then $\lim_{x\to c} f(x) = L(=R)$.

Collectively, the left and right limits of a function are called **one-sided limits** of the function.

EXAMPLE 1 Finding One-sided Limits of a Function

Figure 9



For the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < 2 \\ 1 & \text{if} x = 2 \\ x - 2 & \text{if } x > 2 \end{cases}$$

find: (a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$

Solution Figure 9 shows the graph of f.

(a) To find $\lim_{x\to 2^-} f(x)$, we look at the values of f when x is close to 2, but less than 2. Since $f(x) = 2x - 1$ for such numbers, we conclude that

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x - 1) = 3$$

(b) To find $\lim_{x \to 2^+} f(x)$, we look at the values of f when x is close to 2, but greater than 2. Since $f(x) = x - 2$ for such numbers, we conclude that

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 2) = 0$$

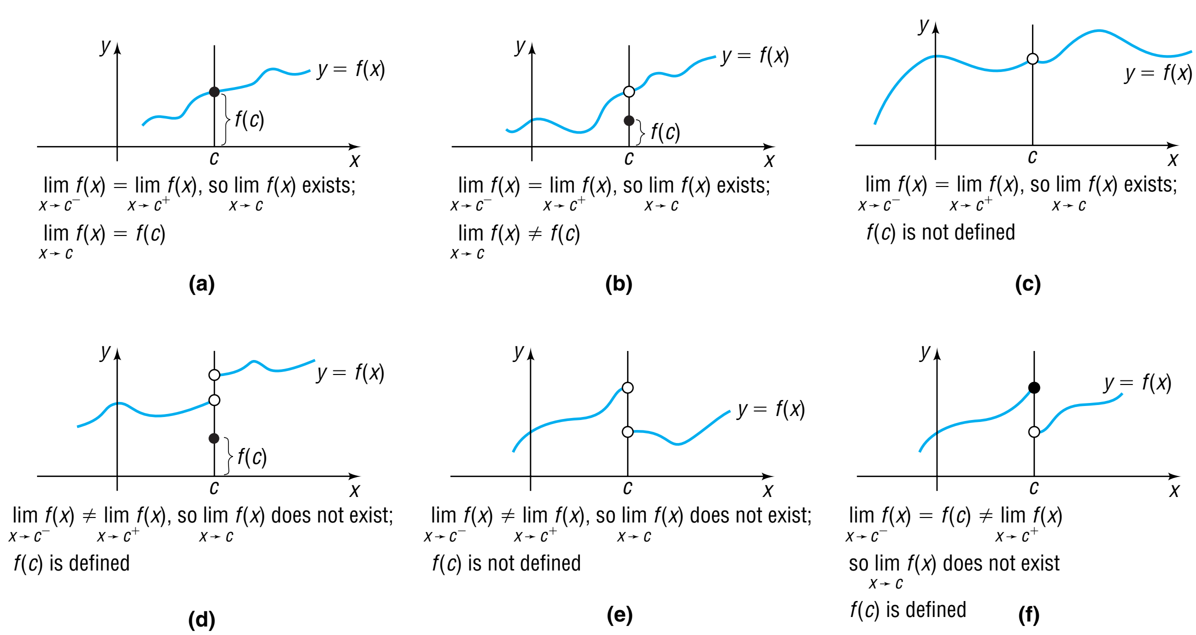
(c) Since the left and right limits are unequal, $\lim_{x \to 2} f(x)$ does not exist.

NOW WORK PROBLEMS 21 AND 35.

2 Determine Whether a Function Is Continuous

We have observed that $f(c)$, the value of the function f at c, plays no role in determining the one-sided limits of f at c. What is the role of the value of a function at c and its one-sided limits at c? Let's look at some of the possibilities. See Figure 10.

Figure 10



Much earlier in this book, we said that a function f was *continuous* if its graph could be drawn without lifting pencil from paper. In looking at Figure 10, the only graph that has this characteristic is the graph in Figure 10(a), for which the one-sided limits at c each exist and are equal to the value of f at c. This leads us to the following definition:

A function f is **continuous** at c if:

1. f is defined at c; that is, c is in the domain of f so that $f(c)$ equals a number.

2. $\lim_{x \to c^-} f(x) = f(c)$

3. $\lim_{x \to c^+} f(x) = f(c)$

In other words, a function f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

If f is not continuous at c, we say that f is **discontinuous at c**. Each of the functions whose graphs appear in Figures 10(b) to 10(f) is discontinuous at c.

Look again at formula (7) on page 918. Based on (7), we conclude that a polynomial function is continuous at every number.

Look at formula (10) on page 920. We conclude that a rational function is continuous at every number, except any at which it is not defined. At numbers where a rational function is not defined, either a hole appears in the graph or else an asymptote appears.

NOW WORK PROBLEM 27.

EXAMPLE 2

Determining the Numbers at Which a Rational Function Is Continuous

(a) Determine the numbers at which the rational function

$$R(x) = \frac{x-2}{x^2-6x+8}$$

is continuous.

(b) Use limits to analyze the graph of R near 2 and near 4.

(c) Graph R.

Solution

(a) Since $R(x) = \dfrac{x-2}{(x-2)(x-4)}$, the domain of R is $\{x \mid x \neq 2, x \neq 4\}$. We conclude that R is discontinuous at both 2 and 4. (Condition 1 of the definition is violated.) Based on formula (10) (page 920), R is continuous at every number except 2 and 4.

(b) To determine the behavior of the graph near 2 and near 4, we look at $\lim\limits_{x \to 2} R(x)$ and $\lim\limits_{x \to 4} R(x)$.

For $\lim\limits_{x \to 2} R(x)$, we have

$$\lim_{x \to 2} R(x) = \lim_{x \to 2} \frac{\cancel{x-2}}{\cancel{(x-2)}(x-4)} = \lim_{x \to 2} \frac{1}{x-4} = -\frac{1}{2}$$

As x gets closer to 2, the graph of R gets closer to $-\dfrac{1}{2}$. Since R is not defined at 2, the graph will have a hole at $\left(2, -\dfrac{1}{2}\right)$.

For $\lim\limits_{x \to 4} R(x)$, we have

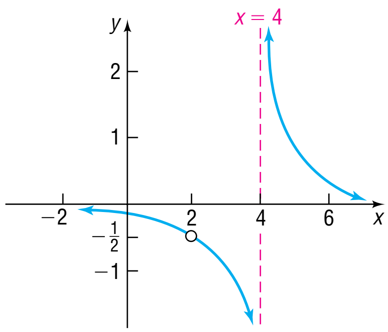
$$\lim_{x \to 4} R(x) = \lim_{x \to 4} \frac{\cancel{x-2}}{\cancel{(x-2)}(x-4)} = \lim_{x \to 4} \frac{1}{x-4}$$

If $x < 4$ and x is getting closer to 4, the value of $\dfrac{1}{x-4}$ is negative and is becoming unbounded; that is, $\lim\limits_{x \to 4^-} R(x) = -\infty$.

If $x > 4$ and x is getting closer to 4, the value of $\dfrac{1}{x-4}$ is positive and is becoming unbounded; that is, $\lim\limits_{x \to 4^+} R(x) = \infty$.

Since $|R(x)| \to \infty$ for x close to 4, the graph of R will have a vertical asymptote at $x = 4$.

Figure 11



(c) It is easiest to graph R by observing that

$$\text{if } x \neq 2, \qquad \text{then } R(x) = \frac{\cancel{x-2}}{\cancel{(x-2)}(x-4)} = \frac{1}{x-4}$$

So the graph of R is the graph of $y = \frac{1}{x}$ shifted to the right 4 units with a hole at $\left(2, -\frac{1}{2}\right)$. See Figure 11. ◀

NOW WORK PROBLEM 73.

The exponential, logarithmic, sine, and cosine functions are continuous at every number in their domain. The tangent, cotangent, secant, and cosecant functions are continuous except at numbers for which they are not defined, where asymptotes occur. The square root function and absolute value function are continuous at every number in their domain. The function $f(x) = \text{int}(x)$ is continuous except for $x =$ an integer, where a jump occurs in the graph.

Piecewise-defined functions require special attention.

EXAMPLE 3 Determining Where a Piecewise-defined Function Is Continuous

Determine the numbers at which the following function is continuous.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x + 1 & \text{if } 0 < x < 2 \\ 5 - x & \text{if } 2 \leq x \leq 5 \end{cases}$$

Solution The "pieces" of f, that is, $y = x^2$, $y = x + 1$, and $y = 5 - x$, are each continuous for every number since they are polynomials. In other words, when we graph the pieces, we will not lift our pencil. When we graph the function f, however, we have to be careful, because the pieces change at $x = 0$ and at $x = 2$. So the numbers we need to investigate for f are $x = 0$ and $x = 2$.

$$\text{At } x = 0: \qquad f(0) = 0^2 = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1) = 1$$

Since $\lim_{x \to 0^+} f(x) \neq f(0)$, we conclude that f is not continuous at $x = 0$.

$$\text{At } x = 2: \qquad f(2) = 5 - 2 = 3$$

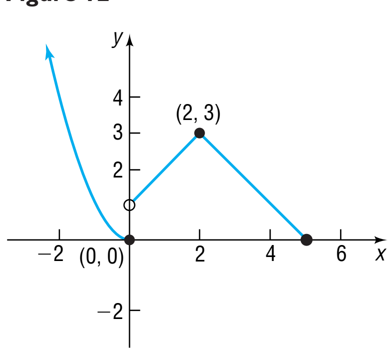
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 1) = 3$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5 - x) = 3$$

We conclude that f is continuous at $x = 2$.

The graph of f, given in Figure 12, demonstrates the conclusions drawn above. ◀

Figure 12



NOW WORK PROBLEMS 53 AND 61.

Summary

Library of Functions: Continuity Properties

Function	Domain	Property
Polynomial function	All real numbers	Continuous at every number in the domain
Rational function $R(x) = \dfrac{P(x)}{Q(x)}$, P, Q are polynomials	$\{x \mid Q(x) \neq 0\}$	Continuous at every number in the domain Hole or vertical asymptote where R is undefined
Exponential function	All real numbers	Continuous at every number in the domain
Logarithmic function	Positive real numbers	Continuous at every number in the domain
Sine and cosine functions	All real numbers	Continuous at every number in the domain
Tangent and secant functions	All real numbers, except odd multiples of $\dfrac{\pi}{2}$	Continuous at every number in the domain Vertical asymptotes at odd multiples of $\dfrac{\pi}{2}$
Cotangent and cosecant functions	All real numbers, except multiples of π	Continuous at every number in the domain Vertical asymptotes at multiples of π

13.3 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. For the function $f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ x+1 & \text{if } 0 < x < 2, \\ 5-x & \text{if } 2 \le x \le 5 \end{cases}$ find $f(0)$ and $f(2)$. (pp. 112–113)

2. What is the domain and range of $f(x) = \ln x$? (p. 291)

3. *True or False:* The exponential function $f(x) = e^x$ is increasing on the interval $(-\infty, \infty)$. (p. 276)

4. Name the trigonometric functions that have asymptotes. (pp. 388–399)

5. *True or False:* Some rational functions have holes in their graph. (pp. 202–203)

6. *True or False:* Every polynomial function has a graph that can be traced without lifting pencil from paper. (pp. 170–178)

Concepts and Vocabulary

7. If we only approach c from one side, then we have a(n) _____ limit.

8. The notation _____ is used to describe the fact that as x gets closer to c, but remains greater than c, the value of $f(x)$ gets closer to R.

9. If $\lim_{x \to c} f(x) = f(c)$, then f is _____ at _____.

10. *True or False:* For any function f, $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

11. *True or False:* If f is continuous at c, then $\lim_{x \to c^+} f(x) = f(c)$.

12. *True or False:* Every polynomial function is continuous at every real number.

Skill Building

In Problems 13–32, use the accompanying graph of $y = f(x)$.

13. What is the domain of f?

14. What is the range of f?

15. Find the x-intercept(s), if any, of f.

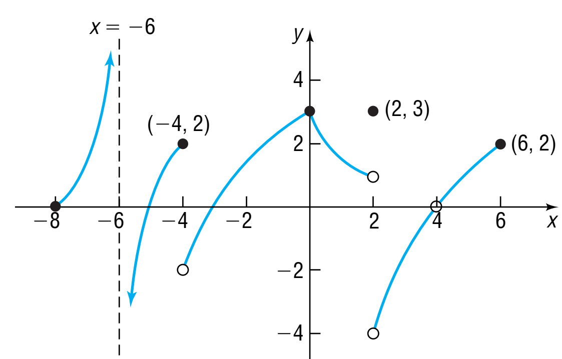
16. Find the y-intercept(s), if any, of f.

17. Find $f(-8)$ and $f(-4)$.

18. Find $f(2)$ and $f(6)$.

19. Find $\lim_{x \to -6^-} f(x)$.

20. Find $\lim_{x \to -6^+} f(x)$.



21. Find $\lim_{x\to -4^-} f(x)$.

22. Find $\lim_{x\to -4^+} f(x)$.

23. Find $\lim_{x\to 2^-} f(x)$.

24. Find $\lim_{x\to 2^+} f(x)$.

25. Does $\lim_{x\to 4} f(x)$ exist? If it does, what is it?

26. Does $\lim_{x\to 0} f(x)$ exist? If it does, what is it?

27. Is f continuous at -6?

28. Is f continuous at -4?

29. Is f continuous at 0?

30. Is f continuous at 2?

31. Is f continuous at 4?

32. Is f continuous at 5?

In Problems 33–44, find the one-sided limit.

33. $\lim_{x\to 1^+}(2x+3)$

34. $\lim_{x\to 2^-}(4-2x)$

35. $\lim_{x\to 1^-}(2x^3+5x)$

36. $\lim_{x\to -2^+}(3x^2-8)$

37. $\lim_{x\to \pi/2^+}\sin x$

38. $\lim_{x\to \pi^-}(3\cos x)$

39. $\lim_{x\to 2^+}\dfrac{x^2-4}{x-2}$

40. $\lim_{x\to 1^-}\dfrac{x^3-x}{x-1}$

41. $\lim_{x\to -1^-}\dfrac{x^2-1}{x^3+1}$

42. $\lim_{x\to 0^+}\dfrac{x^3-x^2}{x^4+x^2}$

43. $\lim_{x\to -2^+}\dfrac{x^2+x-2}{x^2+2x}$

44. $\lim_{x\to -4^-}\dfrac{x^2+x-12}{x^2+4x}$

In Problems 45–60, determine whether f is continuous at c.

45. $f(x)=x^3-3x^2+2x-6,\quad c=2$

46. $f(x)=3x^2-6x+5,\quad c=-3$

47. $f(x)=\dfrac{x^2+5}{x-6},\quad c=3$

48. $f(x)=\dfrac{x^3-8}{x^2+4},\quad c=2$

49. $f(x)=\dfrac{x+3}{x-3},\quad c=3$

50. $f(x)=\dfrac{x-6}{x+6},\quad c=-6$

51. $f(x)=\dfrac{x^3+3x}{x^2-3x},\quad c=0$

52. $f(x)=\dfrac{x^2-6x}{x^2+6x},\quad c=0$

53. $f(x)=\begin{cases}\dfrac{x^3+3x}{x^2-3x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}\quad c=0$

54. $f(x)=\begin{cases}\dfrac{x^2-6x}{x^2+6x} & \text{if } x\neq 0\\ -2 & \text{if } x=0\end{cases}\quad c=0$

55. $f(x)=\begin{cases}\dfrac{x^3+3x}{x^2-3x} & \text{if } x\neq 0\\ -1 & \text{if } x=0\end{cases}\quad c=0$

56. $f(x)=\begin{cases}\dfrac{x^2-6x}{x^2+6x} & \text{if } x\neq 0\\ -1 & \text{if } x=0\end{cases}\quad c=0$

57. $f(x)=\begin{cases}\dfrac{x^3-1}{x^2-1} & \text{if } x<1\\ 2 & \text{if } x=1\\ \dfrac{3}{x+1} & \text{if } x>1\end{cases}\quad c=1$

58. $f(x)=\begin{cases}\dfrac{x^2-2x}{x-2} & \text{if } x<2\\ 2 & \text{if } x=2\\ \dfrac{x-4}{x-1} & \text{if } x>2\end{cases}\quad c=2$

59. $f(x)=\begin{cases}2e^x & \text{if } x<0\\ 2 & \text{if } x=0\\ \dfrac{x^3+2x^2}{x^2} & \text{if } x>0\end{cases}\quad c=0$

60. $f(x)=\begin{cases}3\cos x & \text{if } x<0\\ 3 & \text{if } x=0\\ \dfrac{x^3+3x^2}{x^2} & \text{if } x>0\end{cases}\quad c=0$

In Problems 61–72, find the numbers at which f is continuous. At which numbers is f discontinuous?

61. $f(x) = 2x + 3$ **62.** $f(x) = 4 - 3x$ **63.** $f(x) = 3x^2 + x$ **64.** $f(x) = -3x^3 + 7$

65. $f(x) = 4 \sin x$ **66.** $f(x) = -2 \cos x$ **67.** $f(x) = 2 \tan x$ **68.** $f(x) = 4 \csc x$

69. $f(x) = \dfrac{2x + 5}{x^2 - 4}$ **70.** $f(x) = \dfrac{x^2 - 4}{x^2 - 9}$ **71.** $f(x) = \dfrac{x - 3}{\ln x}$ **72.** $f(x) = \dfrac{\ln x}{x - 3}$

In Problems 73–76, discuss whether R is continuous at c. Use limits to analyze the graph of R at c. Graph R.

73. $R(x) = \dfrac{x - 1}{x^2 - 1}$, $c = -1$ and $c = 1$ **74.** $R(x) = \dfrac{3x + 6}{x^2 - 4}$, $c = -2$ and $c = 2$

75. $R(x) = \dfrac{x^2 + x}{x^2 - 1}$, $c = -1$ and $c = 1$ **76.** $R(x) = \dfrac{x^2 + 4x}{x^2 - 16}$, $c = -4$ and $c = 4$

In Problems 77–82, determine where each rational function is undefined. Determine whether an asymptote or a hole appears at such numbers.

77. $R(x) = \dfrac{x^3 - x^2 + x - 1}{x^4 - x^3 + 2x - 2}$ **78.** $R(x) = \dfrac{x^3 + x^2 + 3x + 3}{x^4 + x^3 + 2x + 2}$

79. $R(x) = \dfrac{x^3 - 2x^2 + 4x - 8}{x^2 + x - 6}$ **80.** $R(x) = \dfrac{x^3 - x^2 + 3x - 3}{x^2 + 3x - 4}$

81. $R(x) = \dfrac{x^3 + 2x^2 + x}{x^4 + x^3 + 2x + 2}$ **82.** $R(x) = \dfrac{x^3 - 3x^2 + 4x - 12}{x^4 - 3x^3 + x - 3}$

For Problems 83–88, use a graphing utility to graph the functions R given in Problems 77–82. Verify the solutions found above.

Discussion and Writing

89. Name three functions that are continuous at every real number.

90. Create a function that is not continuous at the number 5.

'Are You Prepared?' Answers

1. $f(0) = 0; f(2) = 3$ **2.** Domain: $\{x | x > 0\}$; range $\{y | -\infty < y < \infty\}$ **3.** True

4. Secant, cosecant, tangent, cotangent **5.** True **6.** True

13.4 The Tangent Problem; The Derivative

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Point-slope Form of a Line (Section 1.4, p. 32)
- Average Rate of Change (Section 2.3, pp. 85–88)

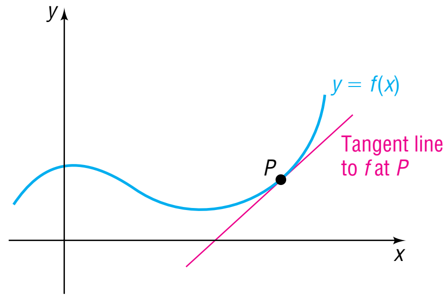
Now work the 'Are You Prepared?' problems on page 936.

OBJECTIVES
1. Find an Equation of the Tangent Line to the Graph of a Function
2. Find the Derivative of a Function
3. Find Instantaneous Rates of Change
4. Find the Instantaneous Speed of a Particle

Tangent Problem

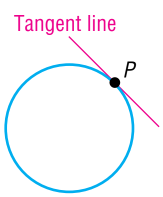
One question that motivated the development of calculus was a geometry problem, the **tangent problem**. This problem asks, "What is the slope of the tangent line to the graph of a function $y = f(x)$ at a point P on its graph?" See Figure 13.

Figure 13



We first need to define what we mean by a *tangent* line. In high school geometry, the tangent line to a circle is defined as the line that intersects the graph in exactly one point. Look at Figure 14. Notice that the tangent line just touches the graph of the circle.

Figure 14



This definition, however, does not work in general. Look at Figure 15. The lines L_1 and L_2 only intersect the graph in one point P, but neither touches the graph at P. Additionally, the tangent line L_T shown in Figure 16 touches the graph of f at P, but also intersects the graph elsewhere. So how should we define the tangent line to the graph of f at a point P?

Figure 15

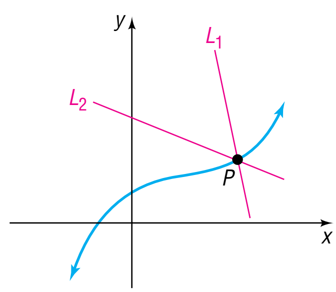
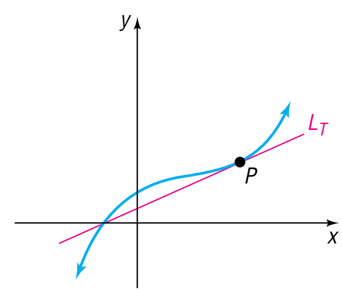


Figure 16



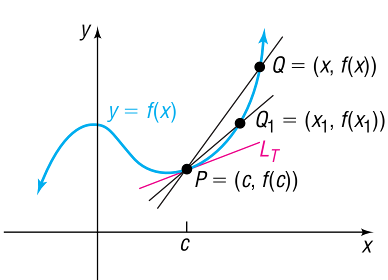
1 Find an Equation of the Tangent Line to the Graph of a Function

The tangent line L_T to the graph of a function $y = f(x)$ at a point P necessarily contains the point P. To find an equation for L_T using the point–slope form of the equation of a line, we need to find the slope $m_{\tan}$ of the tangent line.

Suppose that the coordinates of the point P are $(c, f(c))$. Locate another point $Q = (x, f(x))$ on the graph of f. The line containing P and Q is a secant line. (Refer to Section 2.3.) The slope m_{sec} of the secant line is

$$m_{\text{sec}} = \frac{f(x) - f(c)}{x - c}$$

Figure 17



Now look at Figure 17.

As we move along the graph of f from Q toward P, we obtain a succession of secant lines. The closer we get to P, the closer the secant line is to the tangent line L_T. The limiting position of these secant lines is the tangent line L_T. Therefore, the limiting value of the slopes of these secant lines equals the slope of the tangent line. But, as we move from Q toward P, the values of x get closer to c. Therefore,

$$m_{\tan} = \lim_{x \to c} m_{\text{sec}} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

The **tangent line** to the graph of a function $y = f(x)$ at a point $P = (c, f(c))$ on its graph is defined as the line containing the point P whose slope is

$$m_{\tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \textbf{(1)}$$

provided that this limit exists. If $m_{\tan}$ exists, an equation of the tangent line is

$$y - f(c) = m_{\tan}(x - c) \qquad \textbf{(2)}$$

EXAMPLE 1 Finding an Equation of the Tangent Line

Find an equation of the tangent line to the graph of $f(x) = \frac{x^2}{4}$ at the point $\left(1, \frac{1}{4}\right)$. Graph f and the tangent line.

Solution The tangent line contains the point $\left(1, \frac{1}{4}\right)$. The slope of the tangent line to the graph of $f(x) = \frac{x^2}{4}$ at $\left(1, \frac{1}{4}\right)$ is

$$m_{\tan} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{x^2}{4} - \frac{1}{4}}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x - 1)}(x + 1)}{4\cancel{(x - 1)}}$$

$$= \lim_{x \to 1} \frac{x + 1}{4} = \frac{1}{2}$$

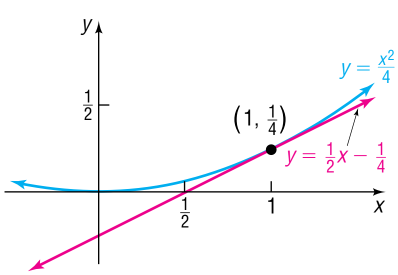
An equation of the tangent line is

$$y - \frac{1}{4} = \frac{1}{2}(x - 1) \qquad y - f(c) = m_{\tan}(x - c)$$

$$y = \frac{1}{2}x - \frac{1}{4}$$

Figure 18 shows the graph of $y = \frac{x^2}{4}$ and the tangent line at $\left(1, \frac{1}{4}\right)$. ◀

Figure 18



NOW WORK PROBLEM 11.

2 Find the Derivative of a Function

The limit in formula (1) has an important generalization: it is called the *derivative of f at c*.

Let $y = f(x)$ denote a function f. If c is a number in the domain of f, the **derivative of f at c**, denoted by $f'(c)$, read "f prime of c," is defined as

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \textbf{(3)}$$

provided that this limit exists.

EXAMPLE 2 **Finding the Derivative of a Function**

Find the derivative of $f(x) = 2x^2 - 5x$ at 2. That is, find $f'(2)$.

Solution Since $f(2) = 2(4) - 5(2) = -2$, we have

$$\frac{f(x) - f(2)}{x - 2} = \frac{(2x^2 - 5x) - (-2)}{x - 2} = \frac{2x^2 - 5x + 2}{x - 2} = \frac{(2x - 1)(x - 2)}{x - 2}$$

The derivative of f at 2 is

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(2x - 1)\cancel{(x - 2)}}{\cancel{x - 2}} = 3$$

◀

NOW WORK PROBLEM 21.

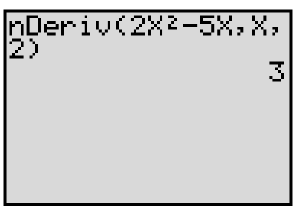
Example 2 provides a way of finding the derivative at 2 analytically. Graphing utilities have built-in procedures to approximate the derivative of a function at any number c. Consult your owner's manual for the appropriate keystrokes.

EXAMPLE 3 **Finding the Derivative of a Function Using a Graphing Utility**

Use a graphing utility to find the derivative of $f(x) = 2x^2 - 5x$ at 2. That is, find $f'(2)$.

Solution Figure 19 shows the solution using a TI-84 Plus graphing calculator. As shown, $f'(2) = 3$. ◀

Figure 19



NOW WORK PROBLEM 33.

EXAMPLE 4 **Finding the Derivative of a Function**

Find the derivative of $f(x) = x^2$ at c. That is, find $f'(c)$.

Solution Since $f(c) = c^2$, we have

$$\frac{f(x) - f(c)}{x - c} = \frac{x^2 - c^2}{x - c} = \frac{(x + c)(x - c)}{x - c}$$

The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{(x + c)\cancel{(x - c)}}{\cancel{x - c}} = 2c$$

◀

As Example 4 illustrates, the derivative of $f(x) = x^2$ exists and equals $2c$ for any number c. In other words, the derivative is itself a function and, using x for the independent variable, we can write $f'(x) = 2x$. The function f' is called the **derivative function of f** or the **derivative of f**. We also say that f is **differentiable**. The instruction "differentiate f" means "find the derivative of f."

3 Find Instantaneous Rates of Change

In Chapter 2 we defined the average rate of change of a function f from c to x as

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(c)}{x - c}$$

The limit as x approaches c of the average rate of change of f, based on formula (3), is the derivative of f at c. As a result, we call the derivative of f at c the **instantaneous rate of change of f with respect to x at c.** That is,

$$\left(\begin{array}{c}\text{Instantaneous rate of}\\ \text{change of } f \text{ with respect to } x \text{ at } c\end{array}\right) = f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{4}$$

EXAMPLE 5 Finding the Instantaneous Rate of Change

The volume V of a right circular cone of height $h = 6$ feet and radius r feet is $V = V(r) = \frac{1}{3}\pi r^2 h = 2\pi r^2$. If r is changing, find the instantaneous rate of change of the volume V with respect to the radius r at $r = 3$.

Solution The instantaneous rate of change of V with respect to r at $r = 3$ is the derivative $V'(3)$.

$$V'(3) = \lim_{r \to 3} \frac{V(r) - V(3)}{r - 3} = \lim_{r \to 3} \frac{2\pi r^2 - 18\pi}{r - 3} = \lim_{r \to 3} \frac{2\pi(r^2 - 9)}{r - 3}$$
$$= \lim_{r \to 3} [2\pi(r + 3)] = 12\pi$$

At the instant $r = 3$ feet, the volume of the cone is increasing with respect to r at a rate of $12\pi \approx 37.699$ cubic feet per 1-foot change in the radius. ◀

NOW WORK PROBLEM 43.

4 Find the Instantaneous Speed of a Particle

If $s = f(t)$ denotes the position of a particle at time t, then the average speed of the particle from c to t is

$$\frac{\text{Change in position}}{\text{Change in time}} = \frac{\Delta s}{\Delta t} = \frac{f(t) - f(c)}{t - c} \tag{5}$$

The limit as t approaches c of the expression in formula (5) is the **instantaneous speed of the particle at c** or the **velocity of the particle at c.** That is,

$$\left(\begin{array}{c}\text{Instantaneous speed of}\\ \text{a particle at time } c\end{array}\right) = f'(c) = \lim_{t \to c} \frac{f(t) - f(c)}{t - c} \tag{6}$$

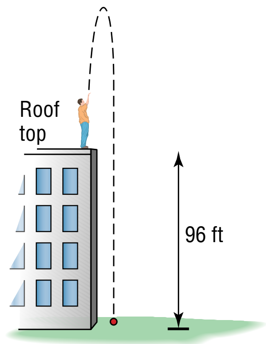
EXAMPLE 6 Finding the Instantaneous Speed of a Particle

In physics it is shown that the height s of a ball thrown straight up with an initial speed of 80 feet per second (ft/sec) from a rooftop 96 feet high is

$$s = s(t) = -16t^2 + 80t + 96$$

where t is the elapsed time that the ball is in the air. The ball misses the rooftop on

Figure 20



its way down and eventually strikes the ground. See Figure 20.

(a) When does the ball strike the ground? That is, how long is the ball in the air?
(b) At what time t will the ball pass the rooftop on its way down?
(c) What is the average speed of the ball from $t = 0$ to $t = 2$?
(d) What is the instantaneous speed of the ball at time t_0?
(e) What is the instantaneous speed of the ball at $t = 2$?
(f) When is the instantaneous speed of the ball equal to zero?
(g) What is the instantaneous speed of the ball as it passes the rooftop on the way down?
(h) What is the instantaneous speed of the ball when it strikes the ground?

Solution

(a) The ball strikes the ground when $s = s(t) = 0$.

$$\begin{aligned} -16t^2 + 80t + 96 &= 0 \\ t^2 - 5t - 6 &= 0 \\ (t-6)(t+1) &= 0 \\ t = 6 \quad \text{or} \quad t &= -1 \end{aligned}$$

We discard the solution $t = -1$. The ball strikes the ground after 6 sec.

(b) The ball passes the rooftop when $s = s(t) = 96$.

$$\begin{aligned} -16t^2 + 80t + 96 &= 96 \\ t^2 - 5t &= 0 \\ t(t-5) &= 0 \\ t = 0 \quad \text{or} \quad t &= 5 \end{aligned}$$

We discard the solution $t = 0$. The ball passes the rooftop on the way down after 5 sec.

(c) The average speed of the ball from $t = 0$ to $t = 2$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(2) - s(0)}{2 - 0} = \frac{192 - 96}{2} = 48 \text{ ft/sec}$$

(d) The instantaneous speed of the ball at time t_0 is the derivative $s'(t_0)$; that is,

$$\begin{aligned} s'(t_0) &= \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} \\ &= \lim_{t \to t_0} \frac{(-16t^2 + 80t + 96) - (-16t_0^2 + 80t_0 + 96)}{t - t_0} \\ &= \lim_{t \to t_0} \frac{-16[t^2 - t_0^2 - 5t + 5t_0]}{t - t_0} \\ &= \lim_{t \to t_0} \frac{-16[(t + t_0)(t - t_0) - 5(t - t_0)]}{t - t_0} \\ &= \lim_{t \to t_0} \frac{-16[(t + t_0 - 5)\cancel{(t - t_0)}]}{\cancel{t - t_0}} = \lim_{t \to t_0}[-16(t + t_0 - 5)] \\ &= -16(2t_0 - 5) \quad \text{ft/sec} \end{aligned}$$

The instantaneous speed of the ball at time t is

$$s'(t) = -16(2t - 5) \text{ ft/sec}$$

(e) At $t = 2$ sec, the instantaneous speed of the ball is

$$s'(2) = -16(-1) = 16 \text{ ft/sec}$$

(f) The instantaneous speed of the ball is zero when

$$\begin{aligned} s'(t) &= 0 \\ -16(2t - 5) &= 0 \\ t &= \frac{5}{2} = 2.5 \text{ sec} \end{aligned}$$

(g) The ball passes the rooftop on the way down when $t = 5$. The instantaneous speed at $t = 5$ is

$$s'(5) = -16(10 - 5) = -80 \text{ ft/sec}$$

At $t = 5$ sec, the ball is traveling -80 ft/sec. When the instantaneous rate of change is negative, it means that the direction of the object is downward. The ball is traveling 80 ft/sec in the downward direction when $t = 5$ sec.

(h) The ball strikes the ground when $t = 6$. The instantaneous speed when $t = 6$ is

$$s'(6) = -16(12 - 5) = -112 \text{ ft/sec}$$

The speed of the ball at $t = 6$ sec is -112 ft/sec. Again, the negative value implies that the ball is traveling downward. ◀

Exploration

Determine the vertex of the quadratic function given in Example 6. What do you conclude about the velocity when $s(t)$ is a maximum?

Summary

The derivative of a function $y = f(x)$ at c is defined as

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

In geometry, $f'(c)$ equals the slope of the tangent line to the graph of f at the point $(c, f(c))$.

In physics, $f'(c)$ equals the instantaneous speed (velocity) of a particle at time c, where $s = f(t)$ is the position of the particle at time t.

In applications, if two variables are related by the function $y = f(x)$, then $f'(c)$ equals the instantaneous rate of change of f with respect to x at c.

13.4 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Find an equation of the line with slope 5 containing the point $(2, -4)$. (p. 32)

2. *True or False:* If c is in the domain of a function f, the average rate of change of f from c to x is

$$\frac{f(x) + f(c)}{x + c}$$ (pp. 85–88)

Concepts and Vocabulary

3. If $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists, it equals the slope of the _____ _____ to the graph of f at the point $(c, f(c))$.

4. If $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists, it is called the _____ of f at c.

5. If $s = f(t)$ denotes the position of a particle at time t, the derivative $f'(c)$ is the _____ of the particle at c.

6. *True or False:* The tangent line to a function is the limiting position of a secant line.

7. *True or False:* The slope of the tangent line to the graph of f at $(c, f(c))$ is the derivative of f at c.

8. *True or False:* The velocity of a particle whose position at time t is $s(t)$ is the derivative $s'(t)$.

Skill Building

In Problems 9–20, find the slope of the tangent line to the graph of f at the given point. Graph f and the tangent line.

9. $f(x) = 3x + 5$ at $(1, 8)$
10. $f(x) = -2x + 1$ at $(-1, 3)$
11. $f(x) = x^2 + 2$ at $(-1, 3)$
12. $f(x) = 3 - x^2$ at $(1, 2)$
13. $f(x) = 3x^2$ at $(2, 12)$
14. $f(x) = -4x^2$ at $(-2, -16)$
15. $f(x) = 2x^2 + x$ at $(1, 3)$
16. $f(x) = 3x^2 - x$ at $(0, 0)$
17. $f(x) = x^2 - 2x + 3$ at $(-1, 6)$
18. $f(x) = -2x^2 + x - 3$ at $(1, -4)$
19. $f(x) = x^3 + x$ at $(2, 10)$
20. $f(x) = x^3 - x^2$ at $(1, 0)$

In Problems 21–32, find the derivative of each function at the given number.

21. $f(x) = -4x + 5$ at 3
22. $f(x) = -4 + 3x$ at 1
23. $f(x) = x^2 - 3$ at 0
24. $f(x) = 2x^2 + 1$ at -1
25. $f(x) = 2x^2 + 3x$ at 1
26. $f(x) = 3x^2 - 4x$ at 2
27. $f(x) = x^3 + 4x$ at -1
28. $f(x) = 2x^3 - x^2$ at 2
29. $f(x) = x^3 + x^2 - 2x$ at 1
30. $f(x) = x^3 - 2x^2 + x$ at -1
31. $f(x) = \sin x$ at 0
32. $f(x) = \cos x$ at 0

In Problems 33–42, find the derivative of each function at the given number using a graphing utility.

33. $f(x) = 3x^3 - 6x^2 + 2$ at -2
34. $f(x) = -5x^4 + 6x^2 - 10$ at 5
35. $f(x) = \dfrac{-x^3 + 1}{x^2 + 5x + 7}$ at 8
36. $f(x) = \dfrac{-5x^4 + 9x + 3}{x^3 + 5x^2 - 6}$ at -3
37. $f(x) = x \sin x$ at $\dfrac{\pi}{3}$
38. $f(x) = x \sin x$ at $\dfrac{\pi}{4}$
39. $f(x) = x^2 \sin x$ at $\dfrac{\pi}{3}$
40. $f(x) = x^2 \sin x$ at $\dfrac{\pi}{4}$
41. $f(x) = e^x \sin x$ at 2
42. $f(x) = e^{-x} \sin x$ at 2

Applications and Extensions

43. **Instantaneous Rate of Change** The volume V of a right circular cylinder of height 3 feet and radius r feet is $V = V(r) = 3\pi r^2$. Find the instantaneous rate of change of the volume with respect to the radius r at $r = 3$.

44. **Instantaneous Rate of Change** The surface area S of a sphere of radius r feet is $S = S(r) = 4\pi r^2$. Find the instantaneous rate of change of the surface area with respect to the radius r at $r = 2$.

45. **Instantaneous Rate of Change** The volume V of a sphere of radius r feet is $V = V(r) = \dfrac{4}{3}\pi r^3$. Find the instantaneous rate of change of the volume with respect to the radius r at $r = 2$.

46. **Instantaneous Rate of Change** The volume V of a cube of side x meters in $V = V(x) = x^3$. Find the instantaneous rate of change of the volume with respect to the side x at $x = 3$.

47. **Instantaneous Speed of a Ball** In physics it is shown that the height s of a ball thrown straight up with an initial speed of 96 ft/sec from ground level is

$$s = s(t) = -16t^2 + 96t$$

where t is the elapsed time that the ball is in the air.
(a) When does the ball strike the ground? That is, how long is the ball in the air?
(b) What is the average speed of the ball from $t = 0$ to $t = 2$?
(c) What is the instantaneous speed of the ball at time t?
(d) What is the instantaneous speed of the ball at $t = 2$?
(e) When is the instantaneous speed of the ball equal to zero?
(f) How high is the ball when its instantaneous speed equals zero?
(g) What is the instantaneous speed of the ball when it strikes the ground?

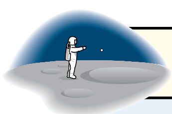
48. Instantaneous Speed of a Ball In physics it is shown that the height s of a ball thrown straight down with an initial speed of 48 ft/sec from a rooftop 160 feet high is

$$s = s(t) = -16t^2 - 48t + 160$$

where t is the elapsed time that the ball is in the air.

(a) When does the ball strike the ground? That is, how long is the ball in the air?
(b) What is the average speed of the ball from $t = 0$ to $t = 1$?
(c) What is the instantaneous speed of the ball at time t?
(d) What is the instantaneous speed of the ball at $t = 1$?
(e) What is the instantaneous speed of the ball when it strikes the ground?

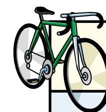
49. Instantaneous Speed on the Moon Neil Armstrong throws a ball down into a crater on the moon. The height s (in feet) of the ball from the bottom of the crater after t seconds is given in the following table:



Time, t (in Seconds)	Distance, s (in Feet)
0	1000
1	987
2	969
3	945
4	917
5	883
6	843
7	800
8	749

(a) Find the average speed from $t = 1$ to $t = 4$ seconds.
(b) Find the average speed from $t = 1$ to $t = 3$ seconds.
(c) Find the average speed from $t = 1$ to $t = 2$ seconds.
(d) Using a graphing utility, find the quadratic function of best fit.
(e) Using the function found in part (d), determine the instantaneous speed at $t = 1$ second.

50. Instantaneous Rate of Change The following data represent the total revenue R (in dollars) received from selling x bicycles at Tunney's Bicycle Shop.



Number of Bicycles, x	Total Revenue, R (in Dollars)
0	0
25	28,000
60	45,000
102	53,400
150	59,160
190	62,360
223	64,835
249	66,525

(a) Find the average rate of change in revenue from $x = 25$ to $x = 150$ bicycles.
(b) Find the average rate of change in revenue from $x = 25$ to $x = 102$ bicycles.
(c) Find the average rate of change in revenue from $x = 25$ to $x = 60$ bicycles.
(d) Using a graphing utility, find the quadratic function of best fit.
(e) Using the function found in part (d), determine the instantaneous rate of change of revenue at $x = 25$ bicycles.

'Are You Prepared?' Answers

1. $y = 5x - 14$ **2.** False

13.5 The Area Problem; The Integral

PREPARING FOR THIS SECTION *Before getting started, review the following:*

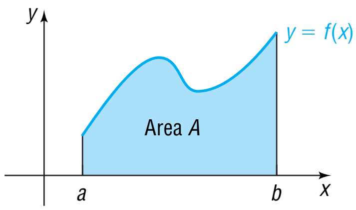
• Geometry Formulas (Appendix, Section A.2, pp. 963–964)
• Summation Notation (Section 11.1, pp. 836–838)

Now work the 'Are You Prepared?' problems on page 944.

OBJECTIVES 1 Approximate the Area Under the Graph of a Function
2 Approximate Integrals Using a Graphing Utility

The development of the integral, like that of the derivative, was originally motivated to a large extent by a problem in geometry: the *area problem*.

Figure 21



Area A = area under the graph of f from a to b

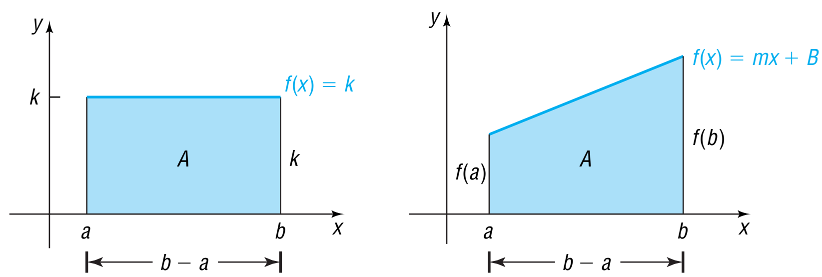
Area Problem

Suppose that $y = f(x)$ is a function whose domain is a closed interval $[a, b]$. We assume that $f(x) \geq 0$ for all x in $[a, b]$. Find the area enclosed by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.

Figure 21 illustrates the area problem. We refer to the area A shown in Figure 21 as the area under the graph of f from a to b.

For a constant function $f(x) = k$ and for a linear function $f(x) = mx + B$, we can solve the area problem using formulas from geometry. See Figures 22(a) and (b).

Figure 22



A = area of rectangle
= height × base
= $k(b - a)$

(a)

A = area of trapezoid
= $\frac{1}{2}[f(b) + f(a)](b - a)$

(b)

For most other functions, no formulas from geometry are available.

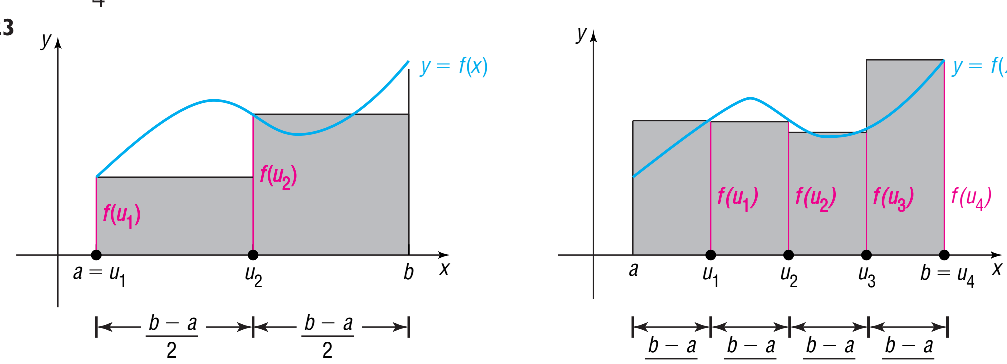
We begin by discussing a way to approximate the area under the graph of a function f from a to b.

1 Approximate the Area Under the Graph of a Function

We use rectangles to approximate the area under the graph of a function f. We do this by *partitioning* or dividing the interval $[a, b]$ into subintervals of equal length. On each subinterval, we form a rectangle whose base is the length of the subinterval and whose height is $f(u)$ for some number u in the subinterval. Look at Figure 23.

In Figure 23(a), the interval $[a, b]$ is partitioned into two subintervals, each of length $\dfrac{b-a}{2}$, and the number u is chosen as the left endpoint of each subinterval. In Figure 23(b), the interval $[a, b]$ is partitioned into four subintervals, each of length $\dfrac{b-a}{4}$, and the number u is chosen as the right endpoint of each subinterval.

Figure 23



2 subintervals
(a)

4 subintervals
(b)

We approximate the area A under f from a to b by adding the areas of the rectangles formed by the partition.

Using Figure 23(a),

$$\begin{aligned}\text{Area } A &\approx \text{area of first rectangle} + \text{area of second rectangle}\\ &= f(u_1)\frac{b-a}{2} + f(u_2)\frac{b-a}{2}\end{aligned}$$

Using Figure 23(b),

$$\begin{aligned}\text{Area } A \approx\ &\text{area of first rectangle} + \text{area of second rectangle}\\ &+ \text{area of third rectangle} + \text{area of fourth rectangle}\\ =\ &f(u_1)\frac{b-a}{4} + f(u_2)\frac{b-a}{4} + f(u_3)\frac{b-a}{4} + f(u_4)\frac{b-a}{4}\end{aligned}$$

In approximating the area under the graph of a function f from a to b, the choice of the number u in each subinterval is arbitrary. For convenience, we shall always pick u as either the left endpoint of each subinterval or the right endpoint. The choice of how many subintervals to use is also arbitrary. In general, the more subintervals used, the better the approximation will be. Let's look at a specific example.

EXAMPLE 1 Approximating the Area under the Graph of $f(x) = 2x$ from 0 to 1

Approximate the area A under the graph of $f(x) = 2x$ from 0 to 1 as follows:

(a) By partitioning $[0, 1]$ into two subintervals of equal length and choosing u as the left endpoint.

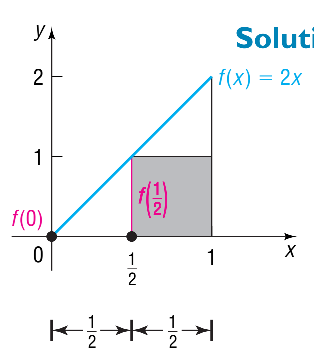
(b) By partitioning $[0, 1]$ into two subintervals of equal length and choosing u as the right endpoint.

(c) By partitioning $[0, 1]$ into four subintervals of equal length and choosing u as the left endpoint.

(d) By partitioning $[0, 1]$ into four subintervals of equal length and choosing u as the right endpoint.

(e) Compare the approximations found in parts (a)–(d) with the actual area.

Figure 24



2 subintervals; u's are left endpoints

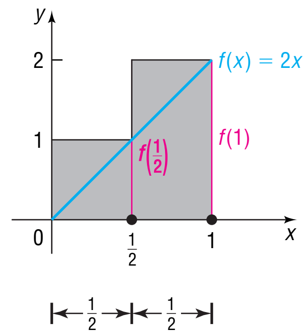
(a)

Solution

(a) We partition $[0, 1]$ into two subintervals, each of length $\frac{1}{2}$, and choose u as the left endpoint. See Figure 24(a). The area A is approximated as

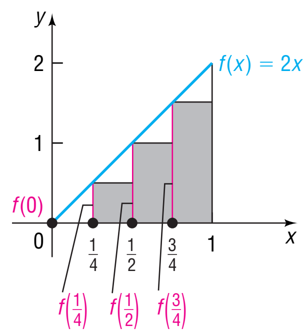
$$\begin{aligned}A &\approx f(0)\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\\ &= (0)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right)\\ &= \frac{1}{2} = 0.5\end{aligned}$$

Figure 24



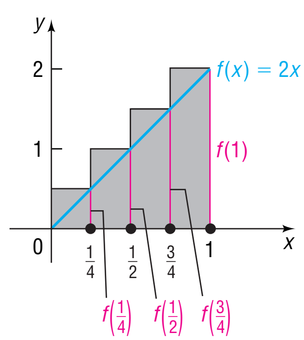
2 subintervals; u's are right endpoints

(b)



4 subintervals; u's are left endpoints

(c)



4 subintervals; u's are right endpoints

(d)

(b) We partition $[0, 1]$ into two subintervals, each of length $\frac{1}{2}$, and choose u as the right endpoint. See Figure 24(b). The area A is approximated as

$$A \approx f\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + f(1)\left(\frac{1}{2}\right)$$
$$= (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right)$$
$$= \frac{3}{2} = 1.5$$

(c) We partition $[0, 1]$ into four subintervals, each of length $\frac{1}{4}$, and choose u as the left endpoint. See Figure 24(c). The area A is approximated as

$$A \approx f(0)\left(\frac{1}{4}\right) + f\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$$
$$= (0)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)\left(\frac{1}{4}\right)$$
$$= \frac{3}{4} = 0.75$$

(d) We partition $[0, 1]$ into four subintervals, each of length $\frac{1}{4}$, and choose u as the right endpoint. See Figure 24(d). The area A is approximated as

$$A \approx f\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + f(1)\left(\frac{1}{4}\right)$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{4}\right)$$
$$= \frac{5}{4} = 1.25$$

(e) The actual area under the graph of $f(x) = 2x$ from 0 to 1 is the area of a right triangle whose base is of length 1 and whose height is 2. The actual area A is therefore

$$A = \frac{1}{2}\text{base} \times \text{height} = \left(\frac{1}{2}\right)(1)(2) = 1$$

◀

Now look at Table 6, which shows the approximations to the area under the graph of $f(x) = 2x$ from 0 to 1 for $n = 2, 4, 10,$ and 100 subintervals. Notice that the approximations to the actual area improve as the number of subintervals increases.

Table 6

Using left endpoints:	n	2	4	10	100
	Area	0.5	0.75	0.9	0.99
Using right endpoints:	n	2	4	10	100
	Area	1.5	1.25	1.1	1.01

You are asked to confirm the entries in Table 6 in Problem 31.

There is another useful observation about Example 1. Look again at Figures 24(a)–(d) and at Table 6. Since the graph of $f(x) = 2x$ is increasing on $[0, 1]$, the choice of u as left endpoint gives a lower estimate to the actual area, while choosing u as the right endpoint gives an upper estimate. Do you see why?

NOW WORK PROBLEM 9.

EXAMPLE 2 Approximating the Area Under the Graph of $f(x) = x^2$

Approximate the area under the graph of $f(x) = x^2$ from 1 to 5 as follows:

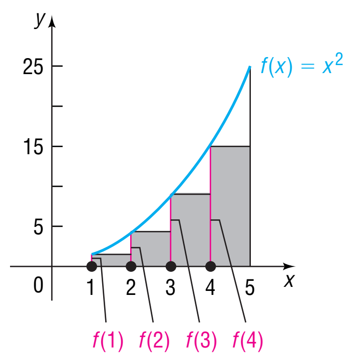
(a) Using four subintervals of equal length.

(b) Using eight subintervals of equal length.

In each case, choose the number u to be the left endpoint of each subinterval.

Solution

Figure 25



4 subintervals; each of length 1

(a) See Figure 25. Using four subintervals of equal length, the interval $[1, 5]$ is partitioned into subintervals of length $\frac{5-1}{4} = 1$ as follows:

$$[1, 2] \quad [2, 3] \quad [3, 4] \quad [4, 5]$$

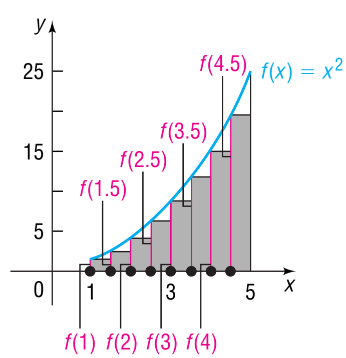
Each of these subintervals is of length 1. Choosing u as the left endpoint of each subinterval, the area A under the graph of $f(x) = x^2$ is approximated by

$$\begin{aligned}\text{Area } A &\approx f(1)(1) + f(2)(1) + f(3)(1) + f(4)(1)\\ &= 1 + 4 + 9 + 16 = 30\end{aligned}$$

(b) See Figure 26. Using eight subintervals of equal length, the interval $[1, 5]$ is partitioned into subintervals of length $\frac{5-1}{8} = 0.5$ as follows:

$$[1, 1.5] \quad [1.5, 2] \quad [2, 2.5] \quad [2.5, 3] \quad [3, 3.5] \quad [3.5, 4] \quad [4, 4.5] \quad [4.5, 5]$$

Figure 26



8 subintervals; each of length 1/2

Each of these subintervals is of length 0.5. Choosing u as the left endpoint of each subinterval, the area A under the graph of $f(x) = x^2$ is approximated by

$$\begin{aligned}\text{Area } A &\approx f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) + f(2.5)(0.5)\\ &\quad + f(3)(0.5) + f(3.5)(0.5) + f(4)(0.5) + f(4.5)(0.5)\\ &= [f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) + f(4.5)](0.5)\\ &= [1 + 2.25 + 4 + 6.25 + 9 + 12.25 + 16 + 20.25](0.5)\\ &= 35.5\end{aligned}$$

◀

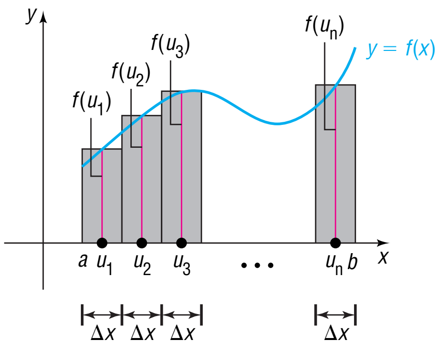
In general, we approximate the area under the graph of a function $y = f(x)$ from a to b as follows:

1. Partition the interval $[a, b]$ into n subintervals of equal length. The length Δx of each subinterval is then

$$\Delta x = \frac{b-a}{n}$$

2. In each of these subintervals, pick a number u and evaluate the function f at each u. This results in n numbers $u_1, u_2, \ldots, u_n$, and n functional values $f(u_1), f(u_2), \ldots, f(u_n)$.

Figure 27



3. Form n rectangles with base equal to Δx, the length of each subinterval, and with height equal to the functional value $f(u_i)$, $i = 1, 2, \dots, n$. See Figure 27.

4. Add up the areas of the n rectangles.

$$A_1 + A_2 + \cdots + A_n = f(u_1)\,\Delta x + f(u_2)\,\Delta x + \cdots + f(u_n)\,\Delta x = \sum_{i=1}^{n} f(u_i)\,\Delta x$$

This number is the approximation to the area under the graph of f from a to b.

Definition of Area

We have observed that the larger the number n of subintervals used, the better the approximation to the area. If we let n become unbounded, we obtain the exact area under the graph of f from a to b.

Area under a Graph

Let f denote a function whose domain is a closed interval $[a, b]$.

Partition $[a, b]$ into n subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. In each subinterval, pick a number u_i, $i = 1, 2, \dots, n$, and evaluate $f(u_i)$. Form the products $f(u_i)\,\Delta x$ and add them up obtaining the sum

$$\sum_{i=1}^{n} f(u_i)\,\Delta x$$

If the limit of this sum exists as $n \to \infty$, that is,

$$\text{if} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i)\,\Delta x \qquad \text{exists}$$

it is defined as the area under the graph of f from a to b. If this limit exists, it is denoted by the symbol

$$\boxed{\int_a^b f(x)\,dx}$$

read as "the integral from a to b of $f(x)$."

2 Approximate Integrals Using a Graphing Utility

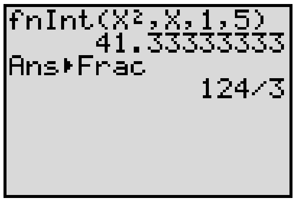
We can use a graphing utility to approximate integrals.

EXAMPLE 3 **Using a Graphing Utility to Approximate an Integral**

Use a graphing utility to approximate the area under the graph of $f(x) = x^2$ from 1 to 5. That is, evaluate the integral

$$\int_1^5 x^2\,dx$$

Figure 28



Solution Figure 28 shows the result using a TI-84 Plus calculator. Consult your owner's manual for the proper keystrokes. ◀

In calculus, techniques are given for evaluating integrals to obtain exact answers.

13.5 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The formula for the area A of a rectangle of length l and width w is _____. (p. 963)

2. $\sum_{k=1}^{4}(2k+1) =$ _____. (pp. 836–838)

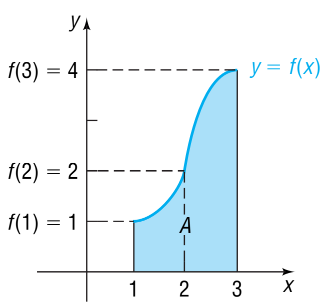
Concepts and Vocabulary

3. The integral from a to b of $f(x)$ is denoted by the symbol _____.

4. The area under the graph of f from a to b is denoted by the symbol _____.

Skill Building

In Problems 5 and 6, refer to the illustration. The interval $[1, 3]$ *is partitioned into two subintervals* $[1, 2]$ *and* $[2, 3]$.

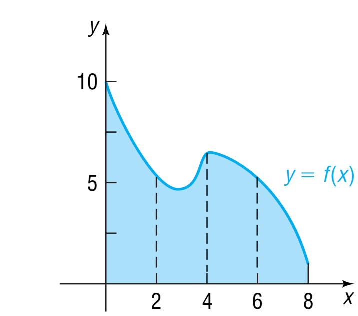


$f(1) = 1$, $f(2) = 2$, $f(3) = 4$

5. Approximate the area A choosing u as the left endpoint of each subinterval.

6. Approximate the area A choosing u as the right endpoint of each subinterval.

In Problems 7 and 8, refer to the illustration. The interval $[0, 8]$ *is partitioned into four subintervals* $[0, 2]$, $[2, 4]$, $[4, 6]$, *and* $[6, 8]$.



$f(0) = 10$, $f(2) = 6$, $f(4) = 7$, $f(6) = 5$, $f(8) = 1$

7. Approximate the area A choosing u as the left endpoint of each subinterval.

8. Approximate the area A choosing u as the right endpoint of each subinterval.

9. The function $f(x) = 3x$ is defined on the interval $[0, 6]$.
(a) Graph f.
In (b)–(e), approximate the area A under f from 0 to 6 as follows:
(b) By partitioning $[0, 6]$ into three subintervals of equal length and choosing u as the left endpoint of each subinterval.
(c) By partitioning $[0, 6]$ into three subintervals of equal length and choosing u as the right endpoint of each subinterval.
(d) By partitioning $[0, 6]$ into six subintervals of equal length and choosing u as the left endpoint of each subinterval.
(e) By partitioning $[0, 6]$ into six subintervals of equal length and choosing u as the right endpoint of each subinterval.
(f) What is the actual area A?

10. Repeat Problem 9 for $f(x) = 4x$.

11. The function $f(x) = -3x + 9$ is defined on the interval $[0, 3]$.
(a) Graph f.
In (b)–(e), approximate the area A under f from 0 to 3 as follows:
(b) By partitioning $[0, 3]$ into three subintervals of equal length and choosing u as the left endpoint of each subinterval.
(c) By partitioning $[0, 3]$ into three subintervals of equal length and choosing u as the right endpoint of each subinterval.
(d) By partitioning $[0, 3]$ into six subintervals of equal length and choosing u as the left endpoint of each subinterval.

(e) By partitioning $[0, 3]$ into six subintervals of equal length and choosing u as the right endpoint of each subinterval.

(f) What is the actual area A?

12. Repeat Problem 11 for $f(x) = -2x + 8$.

In Problems 13–22, a function f is defined over an interval $[a, b]$.

(a) *Graph f, indicating the area A under f from a to b.*

(b) *Approximate the area A by partitioning $[a, b]$ into four subintervals of equal length and choosing u as the left endpoint of each subinterval.*

(c) *Approximate the area A by partitioning $[a, b]$ into eight subintervals of equal length and choosing u as the left endpoint of each subinterval.*

(d) *Express the area A as an integral.*

(e) *Use a graphing utility to approximate the integral.*

13. $f(x) = x^2 + 2, \quad [0, 4]$ **14.** $f(x) = x^2 - 4, \quad [2, 6]$

15. $f(x) = x^3, \quad [0, 4]$ **16.** $f(x) = x^3, \quad [1, 5]$

17. $f(x) = \dfrac{1}{x}, \quad [1, 5]$ **18.** $f(x) = \sqrt{x}, \quad [0, 4]$

19. $f(x) = e^x, \quad [-1, 3]$ **20.** $f(x) = \ln x, \quad [3, 7]$

21. $f(x) = \sin x, \quad [0, \pi]$ **22.** $f(x) = \cos x, \quad \left[0, \dfrac{\pi}{2}\right]$

In Problems 23–30, an integral is given.

(a) *What area does the integral represent?*

(b) *Provide a graph that illustrates this area.*

(c) *Use a graphing utility to approximate this area.*

23. $\displaystyle\int_0^4 (3x + 1)\, dx$ **24.** $\displaystyle\int_1^3 (-2x + 7)\, dx$

25. $\displaystyle\int_2^5 (x^2 - 1)\, dx$ **26.** $\displaystyle\int_0^4 (16 - x^2)\, dx$

27. $\displaystyle\int_0^{\pi/2} \sin x\, dx$ **28.** $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx$

29. $\displaystyle\int_0^2 e^x\, dx$ **30.** $\displaystyle\int_e^{2e} \ln x\, dx$

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31. Confirm the entries in Table 6.

[Hint: Review the formula for the sum of an arithmetic sequence.]

32. Consider the function $f(x) = \sqrt{1 - x^2}$ whose domain is the interval $[-1, 1]$.

(a) Graph f.

(b) Approximate the area under the graph of f from -1 to 1 by dividing $[-1, 1]$ into five subintervals, each of equal length.

(c) Approximate the area under the graph of f from -1 to 1 by dividing $[-1, 1]$ into ten subintervals each of equal length.

(d) Express the area as an integral.

(e) Evaluate the integral using a graphing utility.

(f) What is the actual area?

'Are You Prepared?' Answers

1. $A = lw$ **2.** 24

Chapter Review

Things to Know

Limit (p. 910)

$\lim\limits_{x \to c} f(x) = N$	As x gets closer to c, $x \neq c$, the value of f gets closer to N.

Limit Formulas (p. 915)

$\lim\limits_{x \to c} b = b$	The limit of a constant is the constant.
$\lim\limits_{x \to c} x = c$	The limit of x as x approaches c is c.

Limit Properties

$\lim\limits_{x \to c}[f(x) + g(x)] = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$ (p. 916)	The limit of a sum equals the sum of the limits.
$\lim\limits_{x \to c}[f(x) - g(x)] = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} g(x)$ (p. 917)	The limit of a difference equals the difference of the limits.

$\lim_{x \to c}[f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$ (p. 917) — The limit of a product equals the product of the limits.

$\lim_{x \to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ (p. 920) provided that $\lim_{x \to c} g(x) \neq 0$ — The limit of a quotient equals the quotient of the limits, provided that the limit of the denominator is not zero.

Limit of a Polynomial (p. 918)

$\lim_{x \to c} P(x) = P(c)$, where P is a polynomial

Continuous Function (p. 925)

$\lim_{x \to c} f(x) = f(c)$

Derivative of a Function (p. 932)

$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, provided that the limit exists

Area under a Graph (p. 943)

$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(u_i)\,\Delta x$, provided that the limit exists

Objectives

Section		You should be able to . . .	Review Exercises
13.1	1	Find a limit using a table (p. 910)	1–22
	2	Find a limit using a graph (p. 912)	43, 44
13.2	1	Find the limit of a sum, a difference, and a product (p. 916)	1, 2
	2	Find the limit of a polynomial (p. 918)	1–4
	3	Find the limit of a power or a root (p. 919)	3–6, 9–10
	4	Find the limit of a quotient (p. 920)	11–22
	5	Find the limit of an average rate of change (p. 921)	55–60
13.3	1	Find the one-sided limits of a function (p. 923)	7, 8, 37–42
	2	Determine whether a function is continuous (p. 925)	23–30, 45–50, 51–54
13.4	1	Find an equation of the tangent line to the graph of a function (p. 931)	55–60
	2	Find the derivative of a function (p. 932)	61–70
	3	Find instantaneous rates of change (p. 933)	72, 73
	4	Find the instantaneous speed of a particle (p. 934)	71, 74
13.5	1	Approximate the area under the graph of a function (p. 939)	75–80
	2	Approximate integrals using a graphing utility (p. 943)	77(e)–80(e); 81(c)–84(c)

Review Exercises

In Problems 1–22, find the limit.

1. $\lim_{x \to 2}(3x^2 - 2x + 1)$
2. $\lim_{x \to 1}(-2x^3 + x + 4)$
3. $\lim_{x \to -2}(x^2 + 1)^2$
4. $\lim_{x \to -2}(x^3 + 1)^2$

5. $\lim_{x \to 3} \sqrt{x^2 + 7}$
6. $\lim_{x \to -2} \sqrt[3]{x + 10}$
7. $\lim_{x \to 1^-} \sqrt{1 - x^2}$
8. $\lim_{x \to 2^+} \sqrt{3x - 2}$

9. $\lim_{x \to 2}(5x + 6)^{3/2}$
10. $\lim_{x \to -3}(15 - 3x)^{-3/2}$
11. $\lim_{x \to -1} \frac{x^2 + x + 2}{x^2 - 9}$
12. $\lim_{x \to 3} \frac{3x + 4}{x^2 + 1}$

13. $\lim_{x \to 1} \frac{x - 1}{x^3 - 1}$
14. $\lim_{x \to -1} \frac{x^2 - 1}{x^2 + x}$
15. $\lim_{x \to -3} \frac{x^2 - 9}{x^2 - x - 12}$
16. $\lim_{x \to -3} \frac{x^2 + 2x - 3}{x^2 - 9}$

17. $\lim_{x \to -1^-} \frac{x^2 - 1}{x^3 - 1}$
18. $\lim_{x \to 2^+} \frac{x^2 - 4}{x^3 - 8}$
19. $\lim_{x \to 2} \frac{x^3 - 8}{x^3 - 2x^2 + 4x - 8}$
20. $\lim_{x \to 1} \frac{x^3 - 1}{x^3 - x^2 + 3x - 3}$

21. $\lim_{x \to 3} \frac{x^4 - 3x^3 + x - 3}{x^3 - 3x^2 + 2x - 6}$
22. $\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x^3 + x^2}$

In Problems 23–30, determine whether f is continuous at c.

23. $f(x) = 3x^4 - x^2 + 2, \quad c = 5$

24. $f(x) = \dfrac{x^2 - 9}{x + 10} \quad c = 2$

25. $f(x) = \dfrac{x^2 - 4}{x + 2} \quad c = -2$

26. $f(x) = \dfrac{x^2 + 6x}{x^2 - 6x} \quad c = 0$

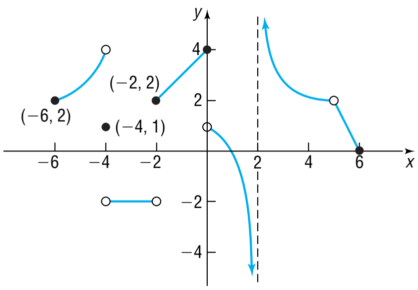
27. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x + 2} & \text{if } x \neq -2 \\ 4 & \text{if } x = -2 \end{cases} \quad c = -2$

28. $f(x) = \begin{cases} \dfrac{x^2 + 6x}{x^2 - 6x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \quad c = 0$

29. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x + 2} & \text{if } x \neq -2 \\ -4 & \text{if } x = -2 \end{cases} \quad c = -2$

30. $f(x) = \begin{cases} \dfrac{x^2 + 6x}{x^2 - 6x} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases} \quad c = 0$

In Problems 31–50, use the accompanying graph of $y = f(x)$.



31. What is the domain of f?

32. What is the range of f?

33. Find the x-intercept(s), if any, of f.

34. Find the y-intercept(s), if any, of f.

35. Find $f(-6)$ and $f(-4)$.

36. Find $f(-2)$ and $f(6)$.

37. Find $\lim\limits_{x \to -4^-} f(x)$.

38. Find $\lim\limits_{x \to -4^+} f(x)$.

39. Find $\lim\limits_{x \to -2^-} f(x)$.

40. Find $\lim\limits_{x \to -2^+} f(x)$.

41. Find $\lim\limits_{x \to 2^-} f(x)$.

42. Find $\lim\limits_{x \to 2^+} f(x)$.

43. Does $\lim\limits_{x \to 0} f(x)$ exist? If it does, what is it?

44. Does $\lim\limits_{x \to 2} f(x)$ exist? If it does, what is it?

45. Is f continuous at -2?

46. Is f continuous at -4?

47. Is f continuous at 0?

48. Is f continuous at 2?

49. Is f continuous at 4?

50. Is f continuous at 5?

In Problems 51 and 52, discuss whether R is continuous at c. Use limits to analyze the graph of R at c.

51. $R(x) = \dfrac{x + 4}{x^2 - 16} \quad \text{at } c = -4 \text{ and } c = 4$

52. $R(x) = \dfrac{3x^2 + 6x}{x^2 - 4} \quad \text{at } c = -2 \text{ and } c = 2$

In Problems 53 and 54, determine where each rational function is undefined. Determine whether an asymptote or a hole appears at such numbers.

53. $R(x) = \dfrac{x^3 - 2x^2 + 4x - 8}{x^2 - 11x + 18}$

54. $R(x) = \dfrac{x^3 + 3x^2 - 2x - 6}{x^2 + x - 6}$

In Problems 55–60, find the slope of the tangent line to the graph of f at the given point. Graph f and the tangent line.

55. $f(x) = 2x^2 + 8x \quad \text{at } (1, 10)$

56. $f(x) = 3x^2 - 6x \quad \text{at } (0, 0)$

57. $f(x) = x^2 + 2x - 3 \quad \text{at } (-1, -4)$

58. $f(x) = 2x^2 + 5x - 3 \quad \text{at } (1, 4)$

59. $f(x) = x^3 + x^2 \quad \text{at } (2, 12)$

60. $f(x) = x^3 - x^2 \quad \text{at } (1, 0)$

In Problems 61–66, find the derivative of each function at the number indicated.

61. $f(x) = -4x^2 + 5 \quad \text{at } 3$

62. $f(x) = -4 + 3x^2 \quad \text{at } 1$

63. $f(x) = x^2 - 3x \quad \text{at } 0$

64. $f(x) = 2x^2 + 4x \quad \text{at } -1$

65. $f(x) = 2x^2 + 3x + 2 \quad \text{at } 1$

66. $f(x) = 3x^2 - 4x + 1 \quad \text{at } 2$

In Problems 67–70, find the derivative of each function at the number indicated using a graphing utility.

67. $f(x) = 4x^4 - 3x^3 + 6x - 9 \quad \text{at } -2$

68. $f(x) = \dfrac{-6x^3 + 9x - 2}{8x^2 + 6x - 1} \quad \text{at } 5$

69. $f(x) = x^3 \tan x \quad \text{at } \dfrac{\pi}{6}$

70. $f(x) = x \sec x \quad \text{at } \dfrac{\pi}{6}$

71. Instantaneous Speed of a Ball In physics it is shown that the height s of a ball thrown straight up with an initial speed of 96 ft/sec from a rooftop 112 feet high is

$$s = s(t) = -16t^2 + 96t + 112$$

where t is the elapsed time that the ball is in the air. The ball misses the rooftop on its way down and eventually strikes the ground.

(a) When does the ball strike the ground? That is, how long is the ball in the air?
(b) At what time t will the ball pass the rooftop on its way down?
(c) What is the average speed of the ball from $t = 0$ to $t = 2$?
(d) What is the instantaneous speed of the ball at time t?
(e) What is the instantaneous speed of the ball at $t = 2$?
(f) When is the instantaneous speed of the ball equal to zero?
(g) What is the instantaneous speed of the ball as it passes the rooftop on the way down?
(h) What is the instantaneous speed of the ball when it strikes the ground?

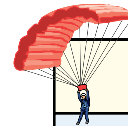
72. Finding an Instantaneous Rate of Change The area A of a circle is πr^2. Find the instantaneous rate of change of area with respect to r at $r = 2$ feet. What is the average rate of change from $r = 2$ to $r = 3$? What is the average rate of change from $r = 2$ to $r = 2.5$? From $r = 2$ to $r = 2.1$?

73. Instantaneous Rate of Change The following data represent the revenue R (in dollars) received from selling x wristwatches at Wilk's Watch Shop.

(a) Find the average rate of change of revenue from $x = 25$ to $x = 130$ wristwatches.
(b) Find the average rate of change of revenue from $x = 25$ to $x = 90$ wristwatches.
(c) Find the average rate of change of revenue from $x = 25$ to $x = 50$ wristwatches.
(d) Using a graphing utility, find the quadratic function of best fit.
(e) Using the function found in part (d), determine the instantaneous rate of change of revenue at $x = 25$ wristwatches.

Wristwatches, x	Revenue, R
0	0
25	2340
40	3600
50	4375
90	6975
130	8775
160	9600
200	10,000
220	9900
250	9375

74. Instantaneous Speed of a Parachutist The following data represent the distance s (in feet) that a parachutist has fallen over time t (in seconds).



Time, t (in Seconds)	Distance, s (in Feet)
1	16
2	64
3	144
4	256
5	400

(a) Find the average speed from $t = 1$ to $t = 4$ seconds.
(b) Find the average speed from $t = 1$ to $t = 3$ seconds.
(c) Find the average speed from $t = 1$ to $t = 2$ seconds.
(d) Using a graphing utility, find the power function of best fit.
(e) Using the function found in part (d), determine the instantaneous speed at $t = 1$ second.

75. The function $f(x) = 2x + 3$ is defined on the interval $[0, 4]$.

(a) Graph f.
In (b)–(e), approximate the area A under f from $x = 0$ to $x = 4$ as follows:
(b) By partitioning $[0, 4]$ into four subintervals of equal length and choosing u as the left endpoint of each subinterval.
(c) By partitioning $[0, 4]$ into four subintervals of equal length and choosing u as the right endpoint of each subinterval.
(d) By partitioning $[0, 4]$ into eight subintervals of equal length and choosing u as the left endpoint of each subinterval.
(e) By partitioning $[0, 4]$ into eight subintervals of equal length and choosing u as the right endpoint of each subinterval.
(f) What is the actual area A?

76. Repeat Problem 75 for $f(x) = -2x + 8$.

In Problems 77–80, a function f is defined over an interval $[a, b]$.
(a) *Graph f, indicating the area A under f from a to b.*
(b) *Approximate the area A by partitioning $[a, b]$ into three subintervals of equal length and choosing u as the left endpoint of each subinterval.*
(c) *Approximate the area A by partitioning $[a, b]$ into six subintervals of equal length and choosing u as the left endpoint of each subinterval.*
(d) *Express the area A as an integral.*
(e) *Use a graphing utility to approximate the integral.*

77. $f(x) = 4 - x^2, \quad [-1, 2]$

78. $f(x) = x^2 + 3, \quad [0, 6]$

79. $f(x) = \dfrac{1}{x^2}, \quad [1, 4]$

80. $f(x) = e^x, \quad [0, 6]$

In Problems 81–84, an integral is given.
(a) *What area does the integral represent?*
(b) *Provide a graph that illustrates this area.*
(c) *Use a graphing utility to approximate this area.*

81. $\int_{-1}^{3} (9 - x^2)\, dx$ **82.** $\int_{1}^{4} \sqrt{x}\, dx$ **83.** $\int_{-1}^{1} e^x\, dx$ **84.** $\int_{\pi/3}^{2\pi/3} \sin x\, dx$

Chapter Test

In Problems 1-6, find the limit. Use the TABLE feature of a graphing utility to verify your answer.

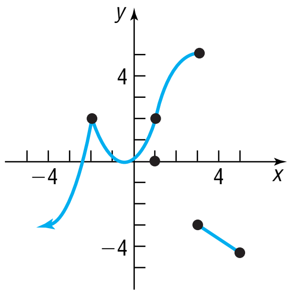
1. $\lim_{x \to 3} \left(-x^2 + 3x - 5\right)$ **2.** $\lim_{x \to 2^+} \frac{|x - 2|}{3x - 6}$ **3.** $\lim_{x \to -6} \sqrt{7 - 3x}$

4. $\lim_{x \to -1} \frac{x^2 - 4x - 5}{x^3 + 1}$ **5.** $\lim_{x \to 5} \left[(3x)(x - 2)^2\right]$ **6.** $\lim_{x \to \frac{\pi}{4}} \frac{\tan x}{1 + \cos^2 x}$

7. Determine the value for k that will make the function continuous at $c = 4$.

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3} & x \le 4 \\ kx + 5 & x > 4 \end{cases}$$

In Problems 8-12, use the accompanying graph of $y = f(x)$.



8. Find $\lim_{x \to 3^+} f(x)$

9. Find $\lim_{x \to 3^-} f(x)$

10. Find $\lim_{x \to -2} f(x)$

11. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, explain why not.

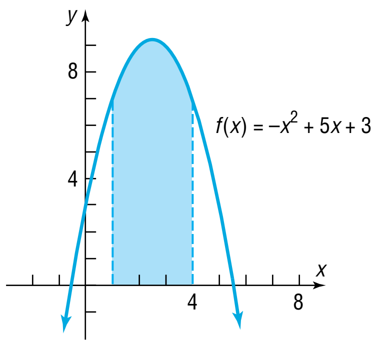
12. Determine whether f is continuous at each of the following numbers. If it is not, explain why not.
(a) $x = -2$ (b) $x = 1$
(c) $x = 3$ (d) $x = 4$

13. Determine where the rational function $R(x) = \dfrac{x^3 + 6x^2 - 4x - 24}{x^2 + 5x - 14}$ is undefined. Determine whether an asymptote or a hole appears at such numbers.

14. For the function $f(x) = 4x^2 - 11x - 3$:
(a) Find the derivative of f at $x = 2$.
(b) Find the equation of the tangent line to the graph of f at the point $(2, -9)$.
(c) Use a graphing utility to graph f and the tangent line in the same window.

15. The function $f(x) = \sqrt{16 - x^2}$ is defined on the interval $[0, 4]$.
(a) Graph f.
(b) Partition $[0, 4]$ into eight subintervals of equal length and choose u as the left endpoint of each subinterval. Use the partition to approximate the area under the graph of f and above the x-axis from $x = 0$ to $x = 4$.
(c) Find the exact area of the region and compare to the approximation in part (b).

16. Write the definite integral that represents the shaded area. Do not attempt to evaluate.



17. A particle is moving along a straight line according to some position function $s(t)$. The distance (in feet) of the particle, s, from its starting point after t seconds is given in the table.

t	s
0	0
1	2.5
2	14
3	31
4	49
5	89
6	137
7	173
8	240

(a) Find the average rate of change of distance from $t = 3$ to $t = 6$ seconds.
(b) Using a graphing utility, find the quadratic function of best fit.
(c) Using the function found in part (b), determine the instantaneous rate of change at $t = 3$ seconds.

Chapter Projects

1. World Population Thomas Malthus believed that "population, when unchecked, increases in a geometrical progression of such nature as to double itself every twenty-five years." However, the growth of population is limited because the resources available to us are limited in supply. If Malthus's conjecture were true, then geometric growth of the world's population would imply that

$$\frac{P_t}{P_{t-1}} = r + 1, \text{ where } r \text{ is the growth rate}$$

(a) Using *world population data* and a graphing utility, find the logistic growth function of best fit, treating the year as the independent variable. Let $t = 0$ represent 1950, $t = 1$ represent 1951, and so on, until you have entered all the years and the corresponding population up to the current year.
(b) Graph $Y_1 = f(t)$, where $f(t)$ represents the logistic growth function of best fit found in part (a).
(c) Determine the instantaneous rate of growth of population in 1960 using the numerical derivative function on your graphing utility.
(d) Use the result from part (c) to predict the population in 1961. What was the actual population in 1961?
(e) Determine the instantaneous growth of population in 1970, 1980, and 1990. What is happening to the instantaneous growth rate as time passes? Is Malthus's contention of a geometric growth rate accurate?
(f) Using the numerical derivative function on your graphing utility, graph $Y_2 = f'(t)$, where $f'(t)$ represents the derivative of $f(t)$ with respect to time. Y_2 is the growth rate of the population at any time t.
(g) Using the MAXIMUM function on your graphing utility, determine the year in which the growth rate of the population is largest. What is happening to the growth rate in the years following the maximum? Find this point on the graph of $Y_1 = f(t)$.
(h) Evaluate $\lim_{t \to \infty} f(t)$. This limiting value is the carrying capacity of Earth. What is the carrying capacity of Earth?
(i) What do you think will happen if the population of Earth exceeds the carrying capacity? Do you think that agricultural output will continue to increase at the same rate as population growth? What effect will urban sprawl have on agricultural output?

The following projects are available on the Instructor's Resource Center (IRC):

2. Project at Motorola ***Curing Scalar***

3. Finding the Profit-maximizing Level of Output